STOCHASTIC EARTHQUAKE ANALYSIS OF STRUCTURES WITH NON-PROPORTIONAL DAMPING

Ricardo T. Duarte [I)]

SUMMARY

This paper presents an analytical technique to compute maximum values of the non-stationary response of a structure with non-proportional damping acted by a stochastic non-stationary model of the ground shaking. The maximum value is calculated from the response three first time-dependent spectral moments. These spectral moments are computed, by filtering, from the generalized spectral moments, which describe the stationary dynamic response of the structure. The generalized spectral moments are moments of the product of the power and cross spectral density functions modeling the ground motion by the elementary transfer function for each modal coordinate.

INTRODUCTION

Dynamics of Structures

The general form of the equation of motion is:

$$1)\quad \mathbf{M}\,\ddot{\mathbf{q}}+\mathbf{C}\,\dot{\mathbf{q}}+\mathbf{K}\,\mathbf{q}=\mathbf{Q}$$

in which $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the inertia, damping and stifness matrices; $\mathbf{q}$ and $\mathbf{Q}$ are the vectors of generalized coordinates and forces; and time derivation is represented by a dot.

Let $\mathbf{q}^b$ denot the coordinates modelling the points of support of the structure (base) and $\mathbf{q}^f$ the other (free) coordinates. Thus, after reordering and partitioning to explicit support displacement, the following matrices are identified:

$$\mathbf{M}\equiv\begin{bmatrix}\mathbf{M}^{ff} & \mathbf{M}^{fd}\\ \mathbf{M}^{df} & \mathbf{M}^{dd}\end{bmatrix}\qquad \mathbf{C}\equiv\begin{bmatrix}\mathbf{C}^{ff} & \mathbf{C}^{fd}\\ \mathbf{C}^{df} & \mathbf{C}^{dd}\end{bmatrix}$$

$$\mathbf{K}\equiv\begin{bmatrix}\mathbf{K}^{ff} & \mathbf{K}^{fd}\\ \mathbf{K}^{df} & \mathbf{K}^{dd}\end{bmatrix}\qquad \mathbf{q}\equiv\begin{bmatrix}\mathbf{q}^{f}\\ \mathbf{q}^{d}\end{bmatrix}\qquad \mathbf{Q}\equiv\begin{bmatrix}\mathbf{Q}^{f}\\ \mathbf{Q}^{d}\end{bmatrix}$$

It is assumed that there are no forces acting on the structure $(\mathbf{Q}^f=0)$ and the vector $\mathbf{q}^b$ is a linear function of a vector of earthquake motions $\mathbf{a}$; hence:

$$2)\quad \mathbf{q}^d=\mathbf{T}\,\mathbf{a}$$

where $\mathbf{T}$ is a matrix that establishes the correspondence between the idealization of the structure and the idealization of the earthquake motions. After defining the matrices $\mathbf{M}^{da}$, $\mathbf{M}^{fa}$, $\mathbf{C}^{da}$, $\mathbf{C}^{fa}$, $\mathbf{K}^{da}$ and $\mathbf{K}^{fa}$ by the equations:

(I) - Research Officer, Applied Dynamics Division, National Laboratory for Civil Engineering (LNEC), Lisbon, Portugal.

$$M^{da}=M^{dd}\,T$$
$$M^{fa}=M^{fd}\,T$$
....

Eq. 1 can be written fractioned in the two following equations:

$$3)\quad M^{ff}\,\ddot{q}^f+C^{ff}\,\dot{q}^f+K^{ff}\,q^f=-M^{fa}\,\ddot{a}-C^{fa}\,\dot{a}-K^{fa}\,a$$

$$4)\quad Q^d=M^{df}\,\ddot{q}^f+C^{df}\,\dot{q}^f+K^{df}\,q^f+M^{da}\,\ddot{a}+C^{da}\,\dot{a}+K^{da}\,a$$

Eq. 3 is the equations of motion for the structure subjected to the base constraints, and Eq. 4 is only an algebraic equation for computing the base reactions.

Modal Analysis with Damping

The consideration of the modes of vibration can significantly simplify the solution of dynamic problems. In the absence of disturbances the motion of a structure is governed by the equation :

$$5)\quad M^{ff}\,\ddot{q}^f+C^{ff}\,\dot{q}^f+K^{ff}\,q^f=0$$

This systems of second-order differential equations can be reduced to a set of N^f second-order differential equations in a single variable, one equa tion for each mode of vibration, N^f being the number of free coordinates. However, this is only possible if the mode shapes are all solenoidal or ir - rotacional, in terms of viscous elastodynamics (Leitman and Fisher, 1973), or if the matrix $(M^{ff^{-1}}\,C^{ff})$ comutes with the matrix $(M^{ff^{-1}}\,K^{ff})$, in terms of matrix analysis (Caughey and O'Kelly, 1963). As is obvious, these conditions seldom occurr, but it as been demonstrated (Foss, 1958) that if Eq. 5 is rewritten as

$$6)\quad \begin{bmatrix} O & M^{ff} \\ M^{ff} & C^{ff} \end{bmatrix} \begin{bmatrix} \ddot{q}^f \\ \dot{q}^f \end{bmatrix} + \begin{bmatrix} M^{ff} & O \\ O & K^{ff} \end{bmatrix} \begin{bmatrix} \dot{q}^f \\ q^f \end{bmatrix} = 0$$

it is possible to find a set of $2N^f$ complex valued damped modes of vibration which reduce Eq.3 to a set of $2N^f$ first-order differential equations. These damped modes may be computed by methods presented in standard references, e.g. in Ref. 4 and 5.

Let w_m be the complex valued mode shape and p_m the complex valued natu ral frequency of the m - th damped mode, and W the matrix defined by $W\equiv[w_1, w_2, \ldots w_n]$ where N may be less than $2N^f$, as the significant response is generally contained in a relatively small number of the lowest modes (Clough and Mojtahedi, 1976), but shall be an even number because the damped modes appear in complex conjugate pairs.

The vectores w_m are normalized by the condition

$$7)\quad w_m^T\,(2p_m\,M^{ff}+C^{ff})\,w_m=1$$

where T denotes the transpose of the vector. The use of the orthogonality conditions:

$$8)\quad \begin{bmatrix} p_m\,w_m \\ w_m \end{bmatrix}^T \begin{bmatrix} O & M^{ff} \\ M^{ff} & C^{ff} \end{bmatrix} \begin{bmatrix} p_n\,w_n \\ w_n \end{bmatrix} = \begin{cases} 0 & m\neq n \\ 1 & m=n \end{cases}$$

$$9)\quad \begin{bmatrix} p_m\,w_m \\ w_m \end{bmatrix}^T \begin{bmatrix} M^{ff} & O \\ O & K^{ff} \end{bmatrix} \begin{bmatrix} p_n\,w_n \\ w_n \end{bmatrix} = \begin{cases} 0 & m\neq n \\ -q_m & m=n \end{cases}$$

when applied to Eq. 3 rewritten in a form similar to Eq. 6, allows to write the equivalent N first-order differential equations:

$$10)\quad \dot{y}_m - p_m\, y_m = \mathbf{m}_m^{ya}\, \ddot{\mathbf{a}} + \mathbf{c}_m^{ya}\, \dot{\mathbf{a}} + \mathbf{k}_m^{ya}\, \mathbf{a}$$

where the y_m are the modal coordinates defined by:

$$11)\quad \mathbf{q}^f = \mathbf{W}\,\mathbf{y} \qquad \mathbf{y} = [y_1, y_2, \ldots y_n]^T$$

and the vectors $\mathbf{m}_m^{ya}$, $\mathbf{c}_m^{ya}$ and $\mathbf{k}_m^{ya}$ (I) are defined by

$$\mathbf{m}_m^{ya} \equiv -\mathbf{w}_m^T\, \mathbf{M}^{fa}$$

$$\mathbf{c}_m^{ya} \equiv -\mathbf{w}_m^T\, \mathbf{c}^{fa}$$

$$\mathbf{k}_m^{ya} \equiv -\mathbf{w}_m^T\, \mathbf{K}^{fa}$$

Eq. 10 is the fundamental equation for the analysis of structures with non--proportional damping.

GROUND MOTION IDEALIZATION

The model of the earthquake motion to be considered is a non-stationary process obtained by the superposition of time-segments of a number of elementary stationary Gaussian vector processes stochasticly independent. Let $\mathbf{a}_r(t)$, $T'_r < t < T''_r$ be a sample function of the r-th elementary process, restricted to the time interval $\{T'_r, T''_r\}$, and make $\mathbf{a}_r(t) \equiv \mathbf{0}$ for values of t outside $\{T'_r, T''_r\}$. The characteristics and frequency content of each elementary process is determined by its spectral densities matrix $\mathbf{S}_r$. The non-stationarity is due to the differences in the time intervals at which the elementary processes act. The main problem in establishing a model for the earthquake motion lies in the quantification of the matrices $\mathbf{S}_r$. This problem is comprehensively dealt with in Ref. 7. The fundamentals for its solution are presented, in an abridged form, in Ref. 8. For the present purposes it sufices to state that the matrices $\mathbf{S}_r$ may be quantified either from the peak values of the ground motion or from response spectra ; and that this formulation is sufficiently general to include translational and rotational components of the ground motion, and differences in ground motion along the base of the structure due to P, S, Love and Raleigh waves, or any unassigned cause (Duarte, 1978).

RESPONSE COMPUTATION

Maximum Values

When dealing with a probabilistic ensemble of earthquake motions and associated structural responses, there is a need to establish rules relating the probabilistic distributions, that are the true results, to conventional "deterministic" results for use in design. For the present purposes, any quantity will be represented conventionally by its mean value. The probability distribution of the maximum responses will be assumed to be a Gumbel distribution $P(X) = \exp(-\exp(-a(X-u)))$ Its mean value is given by

$$12)\quad \bar{X} = u + \gamma/a$$

(I) - In the aprendix 1, m_{im}^{ya} (c_{im}^{ya}, k_{im}^{ya}) denotes element i of vector $\mathbf{m}_m^{ya}$ ($\mathbf{c}_m^{ya}$, $\mathbf{k}_m^{ya}$)

where $\gamma \cong 0.5777222$ is Euler's constant. It is not possible to estimate the value of u and a directly from the stochastic properties of the response. Thus, a and u are quantified for a Gumbel distribution equivalent to a Vanmarcke first-passage distribution P_v. This distribution may be computed from the three first time-dependent spectral moments, λ_1, λ_2 and λ_3 (Corotis et al., 1972) and is given by

$$13) \quad P_v(X>l) = A \exp\left(-\int_0^{\infty} \alpha(t)\, dt\right)$$

where α is the limiting decay rate of the first crossing probability and A is the probability of the process starting below threshold :

$$14) \quad \alpha = 2\, \phi_l \frac{(1-\exp(-lq^{1.2}(2\pi/\lambda_0)^{1/2}))}{1-\exp(-l^2/2\lambda_0)}$$

$$15) \quad A = 1-\exp(-l^2/2\lambda_0)$$

The time-dependent frequency of crossings of the level l, ϕ_l , and the time-dependent spectral density function shape parameter, q, can also be expressed as functions of the three first time-dependent spectral moments:

$$16) \quad \phi_l = (2\pi)^{-1}(\lambda_2/\lambda_0)^{1/2} \exp(-l^2/2\lambda_0)$$

$$17) \quad q = (1-\lambda_1^2/\lambda_0\lambda_2)^{1/2}$$

The equivalence between the Gumbel and the Vanmarcke distributions is established by making coincident the 0.05 and 0.95 probability levels, l_1 and l_2. These levels must be calculated iteratively, but as the Vanmarcke distribution is very near the Gumbel distribution, the convergence is very fast, and usually only two or three iteractions are necessary. The iteration process is as follows:

a) Make $l_1 = 0.25\sqrt{\lambda_0}$ and $l_2 = 3\sqrt{\lambda_0}$

b) Compute:

$$a = ln\,(ln\, P_v\,(X>l_1)\, /\, ln\, P_v\,(X>l_2))\, /\, (l_2-l_1)$$

$$u = l_1 + ln\,(-ln\, P_v\,(X>l_1))\, /\, a$$

c) Compute new values for l_1 and l_2 from the Gumbel distribution with the new values of a and u.

d) Repeat steps b) and c) until a sufficient approximation is reached.

After a and u are determined, the maximum value is found from Eq. 12.

Generalized Spectral Moments

It will be considered that response l_i can be any linear combination of the modal coordinates and its time derivatives, and of the base displacements, velocities and accelerations. Hence:

$$18) \quad \mathbf{l} = \mathbf{M}^y \ddot{\mathbf{y}} + \mathbf{C}^y \dot{\mathbf{y}} + \mathbf{K}^y \mathbf{y} + \mathbf{M}^a \ddot{\mathbf{a}} + \mathbf{C}^a \dot{\mathbf{a}} + \mathbf{K}^a \mathbf{a}$$

where the matrices $\mathbf{M}^y$, ..., $\mathbf{K}^a$ are established from static structural analysis methods, and $\mathbf{l}$ is a vector incorporating all the responses l_i.

It is a well-known result of the theory of random vibrations that for stationary processes the spectral density matrix of the responses $\mathbf{S}^l$ may be calculated from

$$19)\quad \mathbf{S}^l = \mathbf{H}^{T*}\,\mathbf{S}\,\mathbf{H}$$

where $\mathbf{H}$ is the transfer function matrix, $\mathbf{S}$ is the spectral density matrix of the excitation and * denotes conjugate. The derivation of the transfer function matrix is straightforward from Eq. 18 and Eq. 10, and its details may be found in Ref. 7. Representing by X_{ij} the (i,j) element of matrix $\mathbf{X}$, X standing for any letter, Eq. 19 is develloped as:

$$20)\quad S^l_{ij} = \sum_k \sum_l \Big(\sum_e C^{pp}_{eijkl}\,\omega^e + \sum_m R_m(\omega) \sum_e C^{dp}_{eijklm}\,\omega^e$$

$$+ \sum_m R^*_n(\omega) \sum_e C^{pd}_{eijklm}\,\omega^e + \sum_m \sum_n R_m(\omega)\, R^*_n(\omega) \sum_e C^{dd}_{eijklmn}\,\omega^e \Big) S_{kl}$$

Coefficients C^{pp}_{eijkl}, $C^{dd}_{eijklmn}$, C^{dp}_{eijklm} and C^{pd}_{eijkln} are defined in apendix 1, and they respectively affect the pseudostatic parcel of the response, the dynamic parcel of the response, and the correlation between these two parcels. The functions $R_m(\omega)$ are defined as

$$21)\quad R_m(\omega) = (i\omega - p_m)^{-1}$$

and, in a certain sense, are the transfer functions of a one-degree-of-freedom oscillators with non-proportional damping with the same complex valued frequencies as the damped modes of vibration of the structure.

As in the classical modes of vibration case (Vanmarcke, 1972) the product $R_m(\omega)\, R^*_n(\omega)$ may be decomposed as:

$$22)\quad R_m(\omega)\, R^*_n(\omega) = (C_{omn} + C_{1mn}\,\omega^{-1})\,|R_m(\omega)|^2 + (C^1_{onm} + C^1_{1nm}\,\omega^{-1})\,|R_n(\omega)|^2$$

Coefficients C_{omn}, C_{1mn}, C'_{onm} and C'_{1nm} depend only on the complex valued frequencies $p_m = \alpha_m + i\,\beta_m$ and $p_n = \alpha_n + i\,\beta_n$ and are defined in apendix 2. Full details of this decomposition may be found in Ref. 7.

The notion of spectral moments, usually restricted to power spectral density functions, is broadned to include spectral density matrices. Acoordingly, the c – th spectral moment of the excitation is defined as:

$$23)\quad \lambda_{ckl} \equiv \int_0^\infty \omega^c\, S_{kl}\, d\omega$$

The generalized spectral moments are of three kinds, and are defined by:

$$24)\quad \lambda^{I}_{cklm} \equiv \int_0^\infty \omega^c\, R_m(\omega)\, S_{kl}\, d\omega$$

$$25)\quad \lambda^{II}_{cklm} \equiv \int_0^\infty \omega^c\, R^*_m(\omega)\, S_{kl}\, d\omega$$

$$26)\quad \lambda^{III}_{cklm} \equiv \int_0^\infty \omega^c\, |R_m(\omega)|^2\, S_{kl}\, d\omega$$

To compute maximum values only the spectral moments defined for the diagonal elements of $\mathbf{S}^l$ are necessary. These moments may be computed from Eq.20. Using the results expressed by Eq. 22 and the definitions of spectral moments, the spectral moments of the response may be written as:

$$27)\quad \lambda_{ci}=\int_0^{\infty} \omega^c S_{ii}^l \, d\omega =$$
$$=\sum_k \sum_l \Big(\sum_e C_{eijkl}^{pp} \lambda_{(c+e)kl}+\sum_m \sum_e C_{eijklm}^{dp} \lambda_{(c+e)klm}^{\mathrm{I}}$$
$$+\sum_n \sum_e C_{eijklm}^{pd} \lambda_{(c+e)klm}^{\mathrm{II}}+\sum_m \sum_n \sum_e C_{eijklm}^{dd} \Big(\sum_p C_{pmn} \lambda_{(c+e-p)klm}^{\mathrm{III}}+$$
$$+\sum_p C'_{pnm} \lambda_{(c+e-p)kln}^{\mathrm{III}}\Big)\Big)$$

Thus, in the stationary case, the spectral moments of the response of a structure with non-proportional damping can be expressed exactly in terms of the spectral moments and generalized spectral moments of the excitation.

Non-stationarity Idealization

Because it is assumed that only maximum values are of interest, the non-stationarity of the structural response may be idealized by a very simple model, and described only in terms of the spectral moments of the excitation and the generalized spectral moments.

For each complex valued frequency $p_m=\alpha_m+i\,\beta_m$ is associated a real circular frequency $\omega_m=(\alpha_m^2+\beta_m^2)^{1/2}$ and a real percentual damping $\zeta_m=\alpha_m\,/\,(\alpha_m^2+\beta_m^2)^{1/2}$ Consider now the response to the r-th elementary stationary Gaussian vector process, acting during the time interval $\{T_r', T_r''\}$. The time-evolution of the dynamic parcel of the response may be roughly evaluated, although with sufficient accuracy for the present purposes, multiplying the sample functions of the stationary response of each modal coordinate y_m by a time function ψ_{mr} defined as:

$$28)\quad \begin{cases} \psi_{mr}=0 & t<T_r' \\ \psi_{mr}=(1-\exp(-2\zeta_m\omega_m(t-T_r')))^{1/2} & T_r'\leq t\leq T_r'' \\ \psi_{mr}=(\exp(-2\zeta_m\omega_m(t-T_r'')))^{1/2}-\exp(-2\zeta_m\omega_m(t-T_r')))^{1/2} & T_r''<t \end{cases}$$

This function is defined bearing in mind the transient behaviour of structures when suddenly exposed to, or suddenly shielded from a stationary random excitation. The errors envolved in this approach arise basically from the neglect of the more important high-frequency content of the starting-up modal response, as compared to its stationary content, and from the variability that remains in the modal response after the excitation has ceased. Happily, these errors have a tendency to canceal each other, and numerical simulation has shown they are not important for usual structures and usual models of ground motion.

The time-behaviour of the pseudostatic parcel is taken in account with the help of a function ψ_{0r} obviously defined as

$$29)\quad \begin{cases} \psi_{or}=0 & t<T_r' \text{ or } T_r''<t \\ \psi_{or}=1 & T_r'\leq t\leq T_r'' \end{cases}$$

Let $\lambda_{cklr}, \ldots, \lambda^{III}_{cklmr}$ be the spectral moments computed from the spectral densities matrix $\mathbf{S}_r$. Then, the time-dependent spectral moments of the response may be computed from a generalized version of Eq. 27:

$$30)\quad \lambda_{ci}(t)=\sum_k \sum_l \Big(\sum_e C^{pp}_{eijkl} \sum_r \psi^2_{or}\, \lambda_{(c+e)klr} +$$

$$+\sum_m \sum_e C^{dp}_{eijklm} \sum_r \psi_{mr}\, \psi_{or}\, \lambda^{I}_{(c+e)klmr} + \sum_n \sum_e C^{pd}_{eijkln} \sum_r \psi_{or}\, \psi_{mr}\, \lambda^{II}_{(c+e)klnr} +$$

$$+\sum_m \sum_n \sum_e C^{dd}_{eijklmn} \Big(\sum_p C_{pmn} \sum_r \psi^2_{mr}\, \lambda^{III}_{(c+e-p)klmr} + \sum_p C'_{pnm} \sum_r \psi^2_{nr}\, \lambda^{III}_{(c+e-p)klnr}\Big)\Big)$$

With these time-dependent spectral moments, maximums values of the response can be computed.

FINAL REMARKS

The principal usefulness of non-proportional damping models is to make straightforward the idealization of systems for which the damping cannot su̲posed to be uniformly distributed. Such is the case, for instance, in soil-structure interaction situations, in reinforced concrete-steel composite structures and in spring supported flexible equipment. The principal difficulty in a non-proportional damping analysis is to compute the damped modes of vibration as, to the author's knowledge, there is no method equivalent, in numerical efficiency, to the space iteration or determinant search method for non-damped modes of vibration. However, the use of non-damped modes as the assumed shapes for a standard Ritz analysis has proved to be a sensible way to reduce the dimension of the problem to workable proportions.

REFERENCES

1 - Leitman, M.J. and Fisher, G.M.C., 1973 - The Linear Theory of Viscoelasticity, in Encyclopedia of Physics, Vol. VI a/13, pp. 1-123, Springer-Verlag, Berlim

2 - Caughey, T.K. and O'Kelly, M.E.J., 1963 - General Theory of Damped Linear Dynamic Systems, Dynamics Laboratory Report, California Institute of Technology.

3 - Foss, K.A., 1958 - Coordinates which Uncouple the Equations of Motion of Damped Linear Systems, Jour. Appl. Mech., Vol. 25, No. 3, pp. 361-364.

4 - Hurty, W.C. and Rubinstein, M.F., 1964 - Dynamics of Structures, Prentice-Hall Inc..

5 - Meirovitch, L., 1967 - Analytical Methods in Vibration, The MacMillan Company, New-York.

6 - Clough, R.W. and Mojtahedi, S., 1976 - Earthquake Response Analysis Considering Non-Proportional Damping, Jour. Earthquake Engrg. Structural Dynamics, Vol. 4, No. 5, pp. 489-496.

7 - Duarte, R.T., 1978 - Uma Teoria Linear do Comportamento Dinâmico das Estruturas Sujeitas a Acçoes Sismicas, LNEC, Lisbon.

8 - Duarte, R.T., 1979 - A Stochastic Model of Non-Stationary Ground Motion, Paper presented to the 6.th Europ. Conf. in Earthquake Engrg., revised version, LNEC.

9 - Corotis, R.B., Vanmarcke, E.H., and Cornell, C.A., 1972 - First Passage of Nonstationary Random Processes, Jour. Engrg. Mechanics Div., ASCE, Vol. 98, No. EM2, p. 401-414.

10 - Vanmarcke, E.H., 1972 - Properties of Spectral Moments with Applications to Random Vibration, Jour. Engrg. Mechanics Div., Vol. 98, No. EM2, pp. 425-446.

APENDIX 1

$$a_{inj}=-M^a_{in}\, m^{ya}_{nj}$$
$$b_{inj}=C^a_{in}\, m^{ya}_{nj}+M^a_{in}\, c^{ya}_{nj}$$
$$c_{inj}=K^a_{in}\, m^{ya}_{nj}+C^a_{in}\, c^{ya}_{nj}+M^a_{in}\, k^{ya}_{nj}$$
$$d_{inj}=-K^a_{in}\, c^{ya}_{nj}-C^a_{in}\, k^{ya}_{nj}$$
$$e_{inj}=-K^a_{in}\, k^{ya}_{nj}$$

$$C^{pd}_{0ijkln}=K^a_{ik}\, e_{jnl} \qquad C^{pd}_{1ijkln}=i(C^a_{ik}\, e_{jnl}-K^a_{ik}\, d_{jnl})$$
$$C^{pd}_{2ijkln}=-M^a_{ik}\, e_{jnl}+C^a_{ik}\, d_{jnl}+K^a_{ik}\, c_{jnl}$$
$$C^{pd}_{3ijkln}=i(M^a_{ik}\, d_{jnl}+C^a_{ik}\, c_{jnl}-K^a_{ik}\, b_{jnl})$$
$$C^{pd}_{4ijkln}=-M^a_{ik}\, c_{jnl}+C^a_{ik}\, b_{jnl}+K^a_{ik}\, a_{jnl}$$
$$C^{pd}_{5ijkln}=i(M^a_{ik}\, b_{jnl}+C^a_{ik}\, a_{jnl}) \qquad C^{pd}_{6ijkln}=-M^a_{ik}\, a_{jnl}$$

$$C^{pp}_{0ijkl}=K^a_{ik}\, K^a_{jl}$$
$$C^{pp}_{1ijkl}=i(C^a_{ik}\, K^a_{jl}-K^a_{ik}\, C^a_{jl})$$
$$C^{pp}_{2ijkl}=-(M^a_{ik}\, K^a_{jl}-C^a_{ik}\, C^a_{jl}+K^a_{ik}\, M^a_{jl})$$
$$C^{pp}_{3ijkl}=-i(-M^a_{ik}\, C^a_{il}+C^a_{ik}\, M^a_{jl})$$
$$C^{pp}_{4ijkl}=M^a_{ik}\, M^a_{jl}$$

$$C^{dp}_{0ijklm}=K^a_{jl}\, e_{imk} \qquad C^{dp}_{1ijklm}=i(C^a_{jl}\, e_{imk}+K^a_{jl}\, d_{imk})$$
$$C^{dp}_{2ijklm}=-M^a_{jl}\, e_{imk}-C^a_{jl}\, d_{imk}+K^a_{jl}\, c_{imk}$$
$$C^{dp}_{3ijklm}=i(-M^a_{jl}\, d_{imk}-C^a_{jl}\, c_{imk}+K^a_{jl}\, b_{imk})$$
$$C^{dp}_{4ijklm}=-M^a_{jl}\, c_{imk}-C^a_{jl}\, b_{imk}+K^a_{jl}\, a_{imk}$$
$$C^{dp}_{5ijklm}=i(-M^a_{jl}\, b_{imk}-C^a_{jl}\, a_{imk}) \qquad C^{dp}_{6ijklm}=-M^a_{jl}\, a_{imk}$$

$$C^{dd}_{0ijklmn}=e_{imk}\, e_{jnl}$$
$$C^{dd}_{1ijklmn}=i(d_{imk}\, e_{jnl}-e_{imk}\, d_{jnl})$$
$$C^{dd}_{2ijklmn}=c_{imk}\, e_{jnl}-d_{imk}\, d_{jnl}+e_{imk}\, c_{jnl}$$
$$C^{dd}_{3ijklmn}=i(b_{imk}\, e_{jnl}-c_{imk}\, d_{jnl}+d_{imk}\, c_{jnl}-e_{imk}\, b_{jnl})$$
$$C^{dd}_{4ijklmn}=a_{imk}\, e_{jnl}-b_{imk}\, d_{jnl}+c_{imk}\, c_{jnl}-d_{imk}\, b_{jnl}+e_{imk}\, a_{jnl}$$
$$C^{dd}_{5ijklmn}=i(-a_{imk}\, d_{jnl}+b_{imk}\, c_{jnl}-c_{imk}\, b_{jnl}+d_{imk}\, a_{jnl})$$
$$C^{dd}_{6ijklmn}=a_{imk}\, c_{jnl}+b_{imk}\, b_{jnl}+c_{imk}\, a_{jnl}$$
$$C^{dd}_{7ijklmn}=i(-a_{imk}\, b_{jnl}+b_{imk}\, a_{jnl})$$
$$C^{dd}_{8ijklmn}=a_{imk}\, a_{jnl}$$

APENDIX 2

$$r=\frac{\alpha_m^2+\beta_m^2}{\alpha_n^2+\beta_n^2} \qquad B'_{nm}=\frac{2((\alpha_m-\alpha_n)^2+(\beta_m-\beta_n)^2)}{(2r-r^2-1)(\alpha_m^2+\beta_m^2)+2(\beta_m-r\beta_n)} \qquad B_{mn}=-r\,B'_{nm} \qquad A_{mn}=(2+B_{mn}+B'_{nm})/2$$
$$A'_{nm}=(2-B_{mn}-B'_{nm})/2$$

$$D'_{nm}=\frac{4(\alpha_m\,\beta_n-\alpha_n\,\beta_m)(\beta_n-\beta_m)-2(\alpha_n-\alpha_m)(\alpha_n^2+\beta_n^2)(r-1)}{4(\beta_m^2+r\,\beta_n^2-(1+r)\,\beta_m\,\beta_n)-(1-r^2)(\alpha_n^2+\beta_n^2)}$$

$$D_{mn}=-r\,D'_{nm} \qquad C'_{nm}=\frac{\alpha_m-\alpha_n+(1-r)\,D'_{nm}}{2(\beta_m-\beta_n)} \qquad C_{mn}=-C'_{nm}$$

$$C_{0mn}=(A_{mn}+i\,C_{mn})/2 \qquad C'_{0mn}=(A'_{nm}+i\,C'_{nm})/2 \qquad C_{1mn}=(B_{mn}+i\,D_{mn})/2 \qquad C'_{1nm}=(B'_{nm}+i\,D'_{nm})/2$$