UNIFIED FORMULATIONS FOR SOIL-STRUCTURE INTERACTION

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SUMMARY

The formulation phase of the soil-structure interaction problem is investigated comprehensively in a systematic manner. Starting from the three basic equations of motion of the complete system, modified response components are defined based on the pseudostatic transmission. Subsequently, unified derivations are obtained for the governing equations of the Substructure Approach. A generalized flexibility approach for the impedance problem of embedded foundations is also presented. The resulting expressions are given for different free field and support conditions.

INTRODUCTION

The dynamic interaction of soils and structures during earthquakes has been intensively investigated since 1960's. Although a number of methods and techniques have been developed during the last two decades, almost all of them can be considered to be based on either of the two basic approaches, namely, the Direct Approach (One-Step Method) or the Substructure Approach. While the formulation of the Direct Approach is straightforward(1), the Substructure Approach was developed in two directions. In the first one, the problem was analyzed in two steps as an extension of the simple half space method using directly free field surface motion(3,4). In the other development, one more step was added for the rigid and embedded structures to determine the compatible input motion at the rigid base(5). Since the Direct Approach and the two versions of the Substructure Approach were established on different concepts of formulation, considerable effort has been devoted to show their interrelationships. Although in recent years the problem was well recognized(2) and the key points in the dynamic response of embedded structures were clarified(6), there still seems to be a need for a unified approach that is capable of encompassing the various possible formulations in a systematic manner. In the first part of the present paper, three basic equations of motion are established for the complete system corresponding to certain combinations of the total and relative response components. Subsequently, modifications of the response vector in accordance with the pseudostatic transmission are described and finally substructure formulations are reached as the condensed forms of the basic and modified equations.

GENERAL FORM OF EQUATIONS OF MOTION

The equations of motion of the complete soil-structure system can be expressed in the time and frequency domains, respectively, as

$$M\ddot{r} + C\dot{r} + Kr = P \quad ; \quad \bar{K}\bar{r} = \bar{P} \tag{1}$$

in which M,C and K represent the physical property matrices; r , $\bar{r}$ and P , $\bar{P}$ are the response and effective force vectors, respectively. The representative physical property matrix in frequency domain, i.e., the dynamic

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stiffness matrix is $\bar{K} = K - \omega^2 M + i\omega C$, which depends on the frequency of excitation, ω. Referring to Fig.1a, the partitioned forms of $\bar{K}$ and $\bar{r}$ can be written as

$$\bar{K} = \begin{bmatrix} \bar{K}_{ss} & \bar{K}_{sb} & 0 \\ \bar{K}_{bs} & (\bar{K}^s_{bb} + \bar{K}^g_{bb}) & \bar{K}_{bg} \\ 0 & \bar{K}_{gb} & \bar{K}_{gg} \end{bmatrix} \qquad \bar{r} = \begin{Bmatrix} \bar{r}_s \\ \bar{r}_b \\ \bar{r}_g \end{Bmatrix} \tag{2}$$

where the subscript b denotes the common degrees of freedom (DOF) defined along the soil-structure interface (base), and subscripts and/or superscripts s and g indicate DOF defined for the structure and the soil, respectively, excluding those indicated by subscript b. For the sake of simplicity, the frequency domain notation will be used heretofore unless otherwise indicated. In many practical cases, the soil-structure interface is assumed to be perfectly rigid. Then the base response is defined as

$$\bar{r} = T_{bc}\ \bar{r}_o \tag{3}$$

in which $\bar{r}_o$ represents the base response in terms of the rigid body DOF and T_{bo} is the relevant kinematic transformation matrix.

The determination of the effective force vector and thus the ultimate formulation of soil-structure interaction, depends essentially on the definition of the response vector. In the following, three basic response vectors are defined based on certain combinations of the total and relative response components which correspond to different portions of the complete system. Resulting equations of motion provide a basis for further derivations.

BASIC FORMS OF EQUATIONS OF MOTION FOR COMPLETE SYSTEM

<u>First Basic Form :</u> This form constitutes the fundamental equations of motion in which the response of the system is represented entirely by its total motion. The response and the corresponding effective force vectors are (Fig.2a),

$$\bar{r} = \{\bar{r}^t_s \quad \bar{r}^t_b \quad \bar{r}^t_g\}^T \qquad \bar{P} = \{0 \quad 0 \quad -\bar{K}_{gf}\ \bar{v}_f\}^T \tag{4}$$

in which the superscript t denotes the total response and $\bar{v}_f$ represents the seismic input motion specified at the exterior boundaries of the complete system (Fig.1a), and $\bar{K}_{gf}$ is the coupling stiffness submatrix.

<u>Definition of Reference Soil Systems :</u> In the remaining two basic forms, some of the response components are defined relative to the response of certain "Reference Soil Systems". Three types of reference soil systems can be established as shown in Fig.1. The first and the simplest system (RSS-I), represents the physical condition which exists before excavating the soil and superposing the structure. The second one (RSS-II), represents the condition after excavation but before the superposition of the structure. Finally, the third system (RSS-III) is geometrically identical to the second one except that the rigid body motion constraints are kinematically imposed along the soil-structure interface. It should be noted that the use of the reference soil systems is based on the assumptions that the superposition of the structure (also excavation in RSS-I) will not affect the seismic input specified at the exterior boundaries, and also it will not cause any further

nonlinearity in the soil beyond the nonlinearities that would occur in the response of the reference soil system itself. Although all three systems are different in their responses due to the different boundary conditions at the surface, the equations of motion can be expressed in a general form as

$$M_R\ddot{v}_R + C_R\dot{v}_R + K_R v_R = P_R \quad ; \quad \bar{K}_R\bar{v}_R = \bar{P}_R \tag{5}$$

in which v_R represents the total response of any of the reference soil systems. Based on RSS-I, the partitioned forms can be written as

$$\bar{K}_R = \begin{bmatrix} \bar{K}_{ee} & \bar{K}_{eb} & 0 \\ \bar{K}_{be} & (\bar{K}^e_{bb}+\bar{K}^g_{bb}) & \bar{K}_{bg} \\ 0 & \bar{K}_{gb} & \bar{K}_{gg} \end{bmatrix} \quad \bar{v}_R = \begin{Bmatrix} \bar{v}_e \\ \bar{v}_b \\ \bar{v}_g \end{Bmatrix} \quad \bar{P}_R = -\begin{Bmatrix} 0 \\ 0 \\ \bar{K}_{gf} \end{Bmatrix} \bar{v}_f \tag{6}$$

in which the subscript and/or the superscript e indicates the soil portion to be excavated. It is clear that the partitioned forms of RSS-II can be obtained from those given in Eq.6 simply by deleting the terms indicated by e. In the case of RSS-III, the base motion is to be determined in terms of the rigid body degrees of freedom of the base, i.e.,

$$\bar{v}_b = T_{bo}\,\bar{v}_o \tag{7}$$

in which T_{bo} is the previously defined transformation matrix and $\bar{v}_o$ represents the vector of input motion determined for the rigid base.

Second and Third Basic Forms : These forms are based on the relative response of the soil portion of the complete system with respect to the reference soil systems described above. The response vector for the second basic form is defined as (Fig.2b),

$$\bar{r} = \{\bar{r}^t_s \quad \bar{r}^t_b \quad \bar{r}^\Delta_g\}^T \tag{8}$$

and for the third basic form as (Fig.2c),

$$\bar{r} = \{\bar{r}^t_s \quad \bar{r}^\Delta_b \quad \bar{r}^\Delta_g\}^T \tag{9}$$

in which the relative response components are

$$\bar{r}^\Delta_g = \bar{r}^t_g - \bar{v}_g \quad ; \quad \bar{r}^\Delta_b = \bar{r}^t_b - \bar{v}_b \tag{10}$$

The physical interpretation of the relative response components depends on the reference soil system used. For instance, in the case of RSS-I, they correspond to the change in the soil response due to both the excavation and the superposition of the structure. The effective force vector expressions of these basic forms are given in Table 1 for structures with flexible and rigid bases, respectively. RSS-II is not included in the table since the corresponding expressions can be obtained by deleting the terms $\bar{P}^e$ and $\bar{P}^e_o$ given for RSS-I. These terms actually represent the effect of unexcavated soil portion. At this point, a theoretical limitation of the third basic form must be remarked concerning the definition of the relative base response, r^Δ_b. Considering the case of embedment in RSS-I and RSS-II, this definition is possible in any seismic environment only if the base is flexible. If the base is assumed to be perfectly rigid, the relative base response is theoretically impossible to be defined, as indicated in Table 1; since the input base motion, $\bar{v}_b$, is not compatible with the total base response, $\bar{r}^t_o$, of the rigid base. In this case, the only applicable reference soil system is RSS-III, which requires the determination of the compatible input base motion, $\bar{v}_o$. However, in surface supported structures subjected to vertically propagating seismic waves, the free field surface motion directly satisfies the compatibility requirement, leading to a possible definition.

MODIFIED FORMS - PSEUDOSTATIC TRANSMISSION

The basic response vectors defined above, can be modified in two ways based on the pseudostatic transmission of the total input motion and the total base motion, respectively.

Pseudostatic Transmission of Total Input Motion : This can be applied to the first and third basic forms. For the first one, the response vector is modified in the time domain as (Fig.3a),

$$r = [T_{sf} \quad T_{bf} \quad T_{gf}]^T \, v_f + \{x_s \quad x_b \quad x_g\}^T \tag{11}$$

in which T_{mf} (m = s,b,g) represent the pseudostatic transfer matrices for the corresponding portions of the complete system. These matrices transfer the rigid body motion when the seismic input, v_f, is specified at the bedrock. In this case, Eq.11 leads to the traditional formulation of the One-Step Method in terms of x_m(1). In the case of the third basic form, pseudostatic transmission of the input base motion, v_b, into the complete system, requires the static analysis of the entire system(7). It is more convenient, however, to transmit the input base motion merely into the structure(8). Then the response vector of Eq.9 is modified in the time domain as (Fig.3b),

$$r = \{r_s^p \quad 0 \quad 0\}^T + \{u_s \quad r_b^\Delta \quad r_g^\Delta\}^T \tag{12}$$

in which r_s^p and u_s represent the pseudostatic and dynamic parts of the structural response, respectively. The former is obtained from the static analysis of the structure, i.e.,

$$r_s^p = T_{sb} \, v_b \qquad T_{sb} = -K_{ss}^{-1} K_{sb} \tag{13}$$

In the case of RSS-III, the pseudostatic structural response is obtained as

$$r_s^p = T_{so} \, v_o \tag{14}$$

where T_{so} can be determined directly by rigid body kinematics.

Pseudostatic Transmission of Total Base Response : As an extension of the idea of Eq.12, the total base response can also be transmitted into the structure in the pseudostatic manner (Fig.3c). Then the structural response is defined for structures with flexible and rigid bases, respectively, as

$$r_s^t = T_{sb} \, r_b^t + \delta_s \qquad r_s^t = T_{so} \, r_o^t + \delta_s \tag{15}$$

in which δ_s represents the structural response relative to the base. Note that Eq.15 results in a coordinate transformation in the system(4,8). The applications of Eq.12 and Eq.15 are given in the subsequent section with reference to substructure formulations.

CONDENSED FORMS - SUBSTRUCTURE APPROACH

The formulations based on the basic and modified response vectors provide an excellent basis to obtain the various formulations of the Substructure Approach through dynamic condensation. This operation is applied only to the soil DOF in the frequency domain. Physically, it corresponds to the release of the internal nodes of the "soil superelement" which is merely represented by DOF defined along the soil-structure interface. Resulting condensed forms are summarized in Table 2, which contains all of the fundamental substructure formulations developed so far(3,4,5) including some other possible ones. In particular, the use of RSS-I in substructure formulation has been reported only recently. In this respect, the second basic form was intruduced in Ref.6 to clarify the dynamic response of embedded foundations. RSS-I was also employed in a recently developed method(2,9). Similar to the third basic form, relative response components were used for the base and the soil, but in addition, the response of the embedded part of the structure was also defined relative to that of the unexcavated soil portion. Consequently, the structural properties of the embedded part was modified, which makes the method more complicated than the third basic form. The dynamic stiffness matrices shown in Table 2 are the condensed forms of the left-hand sides of equations of motion based on the basic and modified response vectors, in which $\bar{S}^g_{bb}$ and $\bar{S}^g_{oo}$ represent the condensed stiffness (impedance) matrices of the soil for flexible and rigid bases, respectively. In the second basic form, condensation applied to the right-hand side of equations yields the so-called driving forces which act on the base when it is held fixed in space. Note that the expressions of driving forces of the first basic form (not given in the table) are different from those of the second basic form, but they actually represent the same forces, since corresponding response vectors become identical after condensation. Thus, based on the second basic form, driving forces can be determined through the solution of the impedance problem of the flexible base combined with the free field response analysis of the simple soil system RSS-I. Equivalently, the compatible input base motion, which is required in RSS-III, can be obtained as(6),

$$\bar{v}_o = \bar{S}^{g\,-1}_{oo} T^T_{bo} (\bar{S}^g_{bb}\,\bar{v}_b + \bar{P}^e_b) \tag{16}$$

The driving forces or compatible input base motion are not needed in the third basic form unless the base is perfectly rigid. However, the impedance problem of the flexible base remains as the crucial point of the ultimate solution. In discrete systems, this problem is solved normally using RSS-II. However, it is possible to use again the simple soil system RSS-I in the solution of the problem. The basis of an approach developed for this purpose is presented in the following.

A Generalized Flexibility Approach for Impedance Problem : Considering the simplest soil system RSS-I in Fig.1, the condensed form of the dynamic stiffness matrix of Eq.6 satisfies the following familiar relationship:

$$\begin{bmatrix} \bar{K}_{ee} & \bar{K}_{eb} \\ \bar{K}_{be} & (\bar{K}^e_{bb} + \bar{S}^g_{bb}) \end{bmatrix} \begin{bmatrix} \bar{F}_{ee} & \bar{F}_{eb} \\ \bar{F}_{be} & \bar{F}_{bb} \end{bmatrix} = \begin{bmatrix} I_{ee} & 0 \\ 0 & I_{bb} \end{bmatrix} \tag{17}$$

in which the second matrix at the left-hand side represents the flexibility matrix of RSS-I in terms of DOF defined along the soil-structure interface and in unexcavated soil portion, whereas I_{bb} and I_{ee} are unit matrices of corresponding orders. From the lower diagonal element in the matrix product of Eq.17, the impedance matrix of the flexible embedded foundation is obtained as

$$\bar{S}^g_{bb} = (I_{bb} - \bar{K}_{be}\bar{F}_{eb})\bar{F}^{-1}_{bb} - \bar{K}^e_{bb} \tag{18}$$

The submatrices $\bar{F}_{bb}$ and $\bar{F}_{eb}$ are comprised of the flexibility or Green's coefficients representing the displacements in regions (b) and (e), respectively, due to application of unit loads on (b). These coefficients can be obtained from either Finite Element or continuum solutions. In the latter case, a partial discretization is necessary for points taken along and at the immediate vicinity of the soil-structure interface, since submatrix $\bar{K}_{be}$

couples only those points. As an alternative to Eq.18, further condensation of DOF associated with the unexcavated soil portion in Eq.17 results in,

$$\bar{S}^{g}_{bb} = \bar{F}^{-1}_{bb} - (\bar{K}^{e}_{bb} - \bar{K}_{be}\bar{K}^{-1}_{ee}\bar{K}_{eb}) \qquad (19)$$

for which the full discretization is required for the unexcavated soil portion, if $\bar{F}_{bb}$ is obtained from continuum solutions. In the case of RSS-II, both Eq.18 and Eq.19 reduce to the conventional flexibility approach(4), as

$$\bar{S}^{g}_{bb} = \bar{F}^{-1}_{bb} \qquad (20)$$

CONCLUSIONS

In this paper, a unified approach was presented for the formulation of the soil-structure interaction problem. It was shown that the differences in various formulations correspond to the differences in the combination of the response components as well as the different applications of the concept of pseudostatic transmission. In this sense, the table given in the paper (Table 2), can be looked upon as the summary of a Generalized Substructure Approach. From the viewpoint of substructure formulations, the two main problems are; the free field response problem of the excavated soil, and the impedance problem flexible embedded foundation. Formulation techniques were presented in the paper to handle both problems utilizing a simple, unexcavated soil system.

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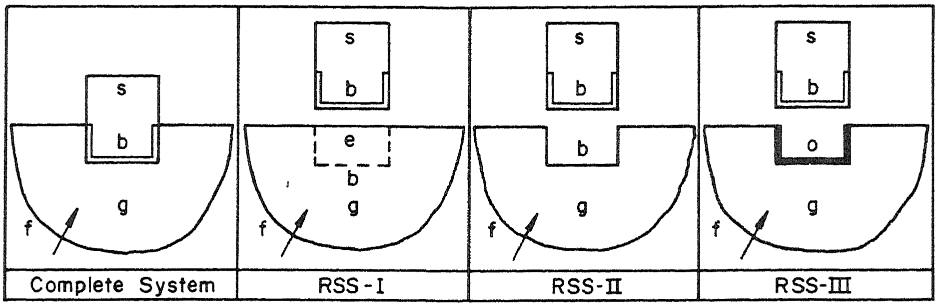

Fig. 1. Complete System and Reference Soil Systems

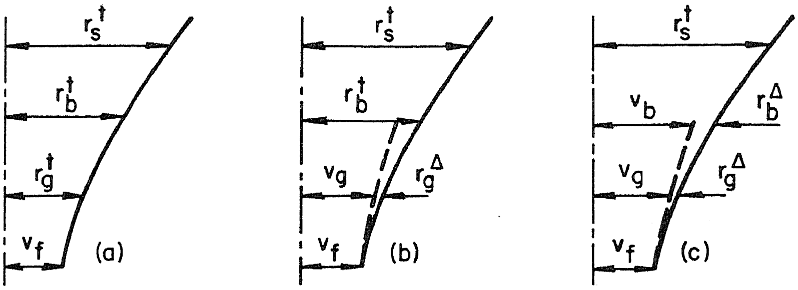

Fig.2. Schematic representation of basic response components

TABLE 1. BASIC FORMS ASSOCIATED WITH RELATIVE SOIL RESPONSE

	RESPONSE VECTOR	EFFECTIVE FORCE VECTOR RSS-I	EFFECTIVE FORCE VECTOR RSS-III
Flex. B.	$\{\bar{r}_s^t \quad \bar{r}_b^t \quad \bar{r}_g^\Delta\}^T$	$[0 \quad \bar{K}_{bb}^g \quad \bar{K}_{gb}]^T \bar{v}_b + \bar{P}^e$	$[0 \quad \bar{K}_{bb}^g \quad \bar{K}_{gb}]^T T_{bo}\bar{v}_o$ *
Flex. B.	$\{\bar{r}_s^t \quad \bar{r}_b^\Delta \quad \bar{r}_g^\Delta\}^T$	$-[\bar{K}_{sb} \quad \bar{K}_{bb}^s \quad 0]^T \bar{v}_b + \bar{P}^e$	$-[\bar{K}_{sb} \quad \bar{K}_{bb}^s \quad 0]^T T_{bo}\bar{v}_o$ *
Rigid B.	$\{\bar{r}_s^t \quad \bar{r}_o^t \quad \bar{r}_g^\Delta\}^T$	$[0 \quad T_{bo}^T \bar{K}_{bb}^g \quad \bar{K}_{gb}]^T \bar{v}_b + \bar{P}_o^e$	$[0 \quad \bar{K}_{oo}^g \quad \bar{K}_{go}]^T \bar{v}_o$
Rigid B.	$\{\bar{r}_s^t \quad \bar{r}_o^\Delta \quad \bar{r}_g^\Delta\}^T$	Undefined†	$-[\bar{K}_{so} \quad \bar{K}_{oo}^s \quad 0]^T \bar{v}_o$

Definitions and Notes:

$$\bar{P}_b^e = \bar{K}_{bb}^e \bar{v}_b + \bar{K}_{be}\bar{v}_e$$

$$\left.\begin{aligned} \bar{K}_{oo}^m &= T_{bo}^T \bar{K}_{bb}^m T_{bo} \\ \bar{K}_{mo} &= \bar{K}_{mb} T_{bo} \end{aligned}\right\} \quad m = s,g$$

$$\bar{P}^e = \{0 \quad \bar{P}_b^e \quad 0\}^T$$

$$\bar{P}_o^e = \{0 \quad T_{bo}^T P_b^e \quad 0\}^T$$

* To be used for only surface supported structures subjected to vertically propagating waves.

† Definition is possible for only surface structures and vertically incident waves. In this case, the expression given for RSS-III is to be used.

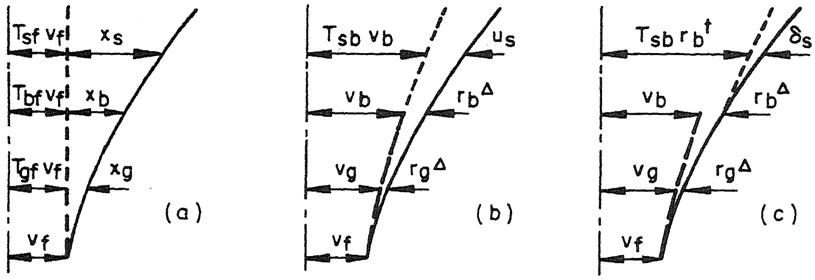

Fig. 3. Schematic Representation of pseudostatic transmission

TABLE 2. FORMULATIONS OF SUBSTRUCTURE APPROACH

	DYNAMIC STIFFNESS MATRIX	RESPONSE VECTOR	EFFECTIVE FORCE VECTOR: RSS-I	EFFECTIVE FORCE VECTOR: RSS-III
Part I: Condensed Forms of Basic Equations of Motion				
FLEX. B.	$\begin{bmatrix} \bar{K}_{ss} & \bar{K}_{sb} \\ \bar{K}_{bs} & (\bar{K}^{s}_{bb} + \bar{S}^{g}_{bb}) \end{bmatrix}$	$\{\bar{r}^{t}_{s} \quad \bar{r}^{t}_{b}\}^{T}$	$[0 \quad \bar{S}^{g}_{bb}]^{T}\,\bar{v}_{b} + \bar{p}^{e}$	$[0 \quad \bar{S}^{g}_{bb}]^{T}\,T_{bo}\bar{v}_{o}$ *
FLEX. B.		$\{\bar{r}^{t}_{s} \quad \bar{r}^{\Delta}_{b}\}^{T}$	$-[\bar{K}_{sb} \quad \bar{K}^{s}_{bb}]^{T}\,\bar{v}_{b} + \bar{p}^{e}$	$-[\bar{K}_{sb} \quad \bar{K}^{s}_{bb}]^{T}\,T_{bo}\bar{v}_{o}$ *
RIGID B.	$\begin{bmatrix} \bar{K}_{ss} & \bar{K}_{so} \\ \bar{K}_{os} & (\bar{K}^{s}_{oo} + \bar{S}^{g}_{oo}) \end{bmatrix}$	$\{\bar{r}^{t}_{s} \quad \bar{r}^{t}_{o}\}^{T}$	$[0 \quad T^{T}_{bo}\bar{S}^{g}_{bb}]^{T}\,\bar{v}_{b} + \bar{p}^{e}_{o}$	$[0 \quad \bar{S}^{g}_{oo}]^{T}\,\bar{v}_{o}$
RIGID B.		$\{\bar{r}^{t}_{s} \quad \bar{r}^{\Delta}_{o}\}^{T}$	Undefined†	$-[\bar{K}_{so} \quad \bar{K}^{s}_{oo}]^{T}\,\bar{v}_{o}$
Part II: Condensed Forms with Pseudostatic Transmission				
FLEX. BASE	See Part I	$\{\bar{u}_{s} \quad \bar{r}^{\Delta}_{b}\}^{T}$	$-[\bar{\bar{K}}_{sb} \quad \bar{\bar{K}}^{s}_{bb}]^{T}\,\bar{v}_{b} + \bar{p}^{e}$	$-[\bar{\bar{K}}_{sb} \quad \bar{\bar{K}}^{s}_{bb}]^{T}\,T_{bo}\bar{v}_{o}$ *
FLEX. BASE	$\begin{bmatrix} \bar{K}_{ss} & \tilde{\bar{K}}_{sb} \\ \tilde{\bar{K}}_{bs} & (\tilde{\bar{K}}^{s}_{bb} + \bar{S}^{g}_{bb}) \end{bmatrix}$	$\{\bar{\delta}_{s} \quad \bar{r}^{t}_{b}\}^{T}$	$[0 \quad \bar{S}^{g}_{bb}]^{T}\,\bar{v}_{b} + \bar{p}^{e}$	$[0 \quad \bar{S}^{g}_{bb}]^{T}\,T_{bo}\bar{v}_{o}$ *
FLEX. BASE		$\{\bar{\delta}_{s} \quad \bar{r}^{\Delta}_{b}\}^{T}$	$-[\tilde{\bar{K}}_{sb} \quad \tilde{\bar{K}}^{s}_{bb}]^{T}\,\bar{v}_{b} + \bar{p}^{e}$	$-[\tilde{\bar{K}}_{sb} \quad \tilde{\bar{K}}^{s}_{bb}]^{T}\,T_{bo}\bar{v}_{o}$ *
RIGID BASE	See Part I	$\{\bar{u}_{s} \quad \bar{r}^{\Delta}_{o}\}^{T}$	Undefined†	$-[\bar{\bar{K}}_{so} \quad \bar{\bar{K}}^{s}_{oo}]^{T}\,\bar{v}_{o}$
RIGID BASE	$\begin{bmatrix} \bar{K}_{ss} & \tilde{\bar{K}}_{so} \\ \tilde{\bar{K}}_{os} & (\tilde{\bar{K}}^{s}_{oo} + \bar{S}^{g}_{oo}) \end{bmatrix}$	$\{\bar{\delta}_{s} \quad \bar{r}^{t}_{o}\}^{T}$	$[0 \quad T^{T}_{bo}\bar{S}^{g}_{bb}]^{T}\,\bar{v}_{b} + \bar{p}^{e}_{o}$	$[0 \quad \bar{S}^{g}_{oo}]^{T}\,\bar{v}_{o}$
RIGID BASE		$\{\bar{\delta}_{s} \quad \bar{r}^{\Delta}_{o}\}^{T}$	Undefined†	$-[\tilde{\bar{K}}_{so} \quad \tilde{\bar{K}}^{s}_{oo}]^{T}\,\bar{v}_{o}$

Definitions and Notes:

$$\bar{S}^{g}_{bb} = \bar{K}^{g}_{bb} - \bar{K}_{bg}\bar{K}^{-1}_{gg}\bar{K}_{gb} \qquad \bar{p}^{e} = \{0 \quad \bar{p}^{e}_{b}\}^{T}$$

$$\bar{S}^{g}_{oo} = T^{T}_{bo}\bar{S}^{g}_{bb}T_{bo} \qquad \bar{p}^{e}_{o} = \{0 \quad T^{T}_{bo}\bar{p}^{e}_{b}\}^{T}$$

$$\bar{\bar{K}}_{sb} = \tilde{\bar{K}}_{sb} = -\omega^{2}(M_{sb} + M_{ss}T_{sb}) \qquad \bar{\bar{K}}_{so} = \tilde{\bar{K}}_{so} = -\omega^{2}(M_{sb}T_{bo} + M_{ss}T_{so})$$

$$\bar{\bar{K}}^{s}_{bb} = \bar{K}^{s}_{bb} + \bar{K}_{bs}T_{sb} \qquad \bar{\bar{K}}^{s}_{oo} = -\omega^{2}T^{T}_{bo}(M^{s}_{bb}T_{bo} + M_{bs}T_{so})$$

$$\tilde{\bar{K}}^{s}_{bb} = \bar{\bar{K}}_{bb} + T^{T}_{sb}\bar{\bar{K}}_{sb} \qquad \tilde{\bar{K}}^{s}_{oo} = \bar{\bar{K}}^{s}_{oo} + T^{T}_{so}\bar{\bar{K}}_{so}$$

Pseudostatic damping forces are assumed to be stiffness proportional.

* † See footnotes in Table 1.