DYNAMIC ANALYSIS OF ORTHOGONAL PLANAR FRAMES BY STORY TRANSFER MATRIX

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SUMMARY

Fundamentals of story transfer matrix method for dynamic analysis of orthogonal planar frames are presented. The story transfer matrix is derived from the transfer matrices of columns between floor levels and the slope deflection equations of the floor beams at the top of the columns. Chain multiplication of story transfer matrices results in a final matrix relationship between the state variables at the top and the bottom of the frame. Substitution of known values of state variables produces the equations needed for steady state analysis and determinant search technique. Story transfer matrix method requires a minimal amount of computer core storage and straightforward data manipulation.

1. INTRODUCTION

Building structures are commonly analyzed as planar frames in civil engineering practice, because the cost of three-dimensional solutions is not justified by the gain in accuracy except for some special cases. The majority of building frames possess orthogonal geometry and, usually, member section properties only change every two or three stories. Consequently, analysis methods based on floor-by-floor input data seem to be very efficient in utilization of small scale computers.

The transfer matrix method for static, dynamic and stability analysis is recognized as a powerful tool in structural mechanics (1). It has been successfully utilized in a large scale dynamic analysis computer program (2) for solving complex engineering problems. The story transfer matrix technique is based on the transfer matrix method. It was developed by Açıkel (3) for static analysis of orthogonal frames having inextensional members. Recently, Çıtıpıtıoğlu and Nicolas (4) have presented fundamentals of the story transfer matrix method for static analysis in a more general form which takes axial shortening of columns into account. P-Δ effect is approximated as fictitious horizontal loads which are based on story sidesway and column axial forces. Shear walls are handled by adjusting the stiffnesses of beams attached to walls and by inclusion of shear deformation effect in the analysis.

Orthogonal planar frame is modelled by lumping the masses at the joints as shown in Fig. 1a. Axial deformation of the floor beams is neglected. Consequently a story sidesway and an associated story shear are included in the state variables. In the following section of the paper, general expressions for the elements of story transfer matrix are derived. Procedures for dynamic analysis and consideration of P-Δ effects are explained in the subsequent sections.

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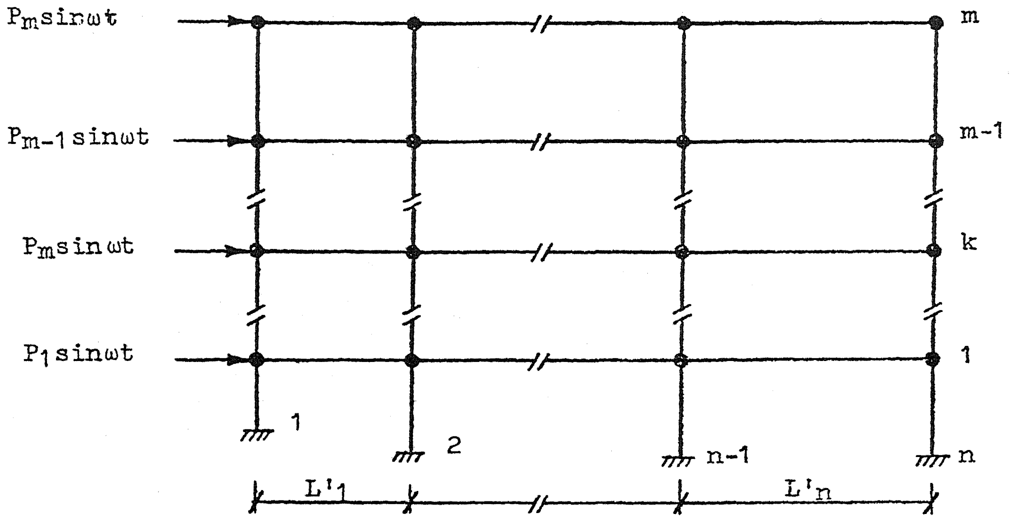
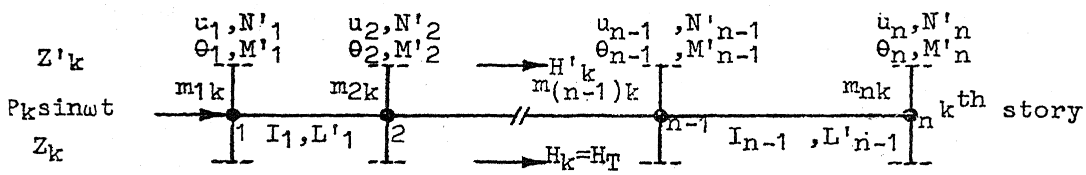
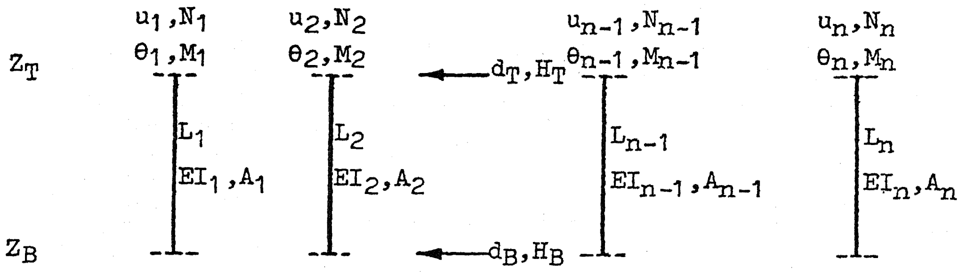


(a) Typical Orthogonal Frame Model With Lumped Masses



(c) Beam Sub-structure



(b) Column Sub-structure

Figure 1 - ORTHOGONAL PLANAR FRAME AND TYPICAL SUB-STRUCTURES

2. DERIVATION OF STORY TRANSFER MATRIX

A model for a typical orthogonal planar frame having m stories and n-1 bays is shown in Fig. 1a. Column and beam sub-structures of kth story are illustrated in Figs. 1b and 1c, respectively. The state variables which are retained in each sub-structure transfer matrix are also indicated in Figure 1. A transfer matrix for the column sub-structure (Figure 1b) is first derived by using the transfer matrices of individual columns. Subsequently, appropriate slope-deflection relations at the joints of the beam sub-structure are utilized to obtain a transfer matrix for a single story which relates state variables at the bottom of the column sub-structure to the state variables at the bottom of the columns above the beams.

2.1 Derivation of Column Sub-structure Transfer Matrix

A typical column of an orthogonal planar frame is shown in Figure 2. Positive directions of end displacements u,d, and θ and end forced N,V, and M are shown in the figure. Subscripts B and T denote the bottom and top ends, respectively.

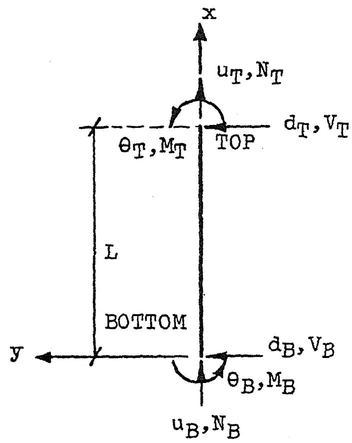


Figure 2- COLUMN END DISPLACEMENT AND FORCES

Transfer matrices for axial and flexural state variables of individual columns can be written as (see reference 1):

$$\begin{bmatrix} u \\ N \end{bmatrix}_T = \begin{bmatrix} 1 & -EA/L \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ N \end{bmatrix}_B \quad (1)$$

$$\begin{bmatrix} d \\ \theta \\ M \\ V \end{bmatrix}_T = \begin{bmatrix} 1 & L & -L^2/2EI & fL^3/6EI \\ 0 & 1 & -L/EI & L^2/2EI \\ 0 & 0 & -1 & L \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} d \\ \theta \\ M \\ V \end{bmatrix}_B \quad (2)$$

where

E = modulus of elasticity
A;I = column cross-sectional area and moment of inertia
L = length of the column
f = factor to include shear deformation effect
$f = 1 - 6EI/GA^*L^2$
GA* = effective shear rigidity
u, d, θ = end displacements
N,V,M = end forces
B,T = subscripts indicating bottom and top ends.

From the first equation of the matrix relation in Eq. 2, column bottom shear V_B is expressed as

$$V_B = \frac{6EI}{fL^3}(d_T - d_B) - \frac{6EI}{fL^2}\Theta_B + \frac{3}{fL}M_B. \qquad (3)$$

Story shear equation can now be written as

$$H_B = \Sigma V_B{}^i = \Delta\Sigma\left(\frac{6EI}{fL^3}\right)^i - \Sigma\left(\frac{6EI}{fL^2}\right)^i \Theta_B{}^i + \Sigma\left(\frac{3}{fL}\right)^i M_B{}^i \qquad (4)$$

where i = superscript indicating ith column
Σ = summation i=1, n for all columns of the story
B = subscript indicating column bottom
Δ = relative sidesway $(d_T - d_B)$.

Solving Δ from Eq. 4 and substituting into Eqs. 3 and 2 results in column substructure transfer matrix (CSTM) as

$$\begin{bmatrix}\Theta\\ M\\ d\\ H\\ u\\ N\end{bmatrix}_T = \begin{bmatrix}A_{11} & A_{12} & 0 & A_{14} & 0 & 0\\ A_{21} & A_{22} & 0 & A_{24} & 0 & 0\\ A_{31} & A_{32} & A_{33} & A_{34} & 0 & 0\\ 0 & 0 & 0 & A_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & A_{55} & A_{56}\\ 0 & 0 & 0 & 0 & 0 & A_{66}\end{bmatrix}\begin{bmatrix}\Theta\\ M\\ d\\ H\\ u\\ N\end{bmatrix}_B \qquad (5)$$

where

Θ, M, u, N = Column vectors of state variables (n x 1)
d,H = story sidesway and story shear
A_{11} = nxn submatrix; $A_{11}(I,J) = C\,(3/fL)^I(6EI/fL^2)^J + \delta_{IJ}(1-3/f)^I$
A_{12} = nxn submatrix; $A_{12}(I,J) = -\,C(3/fL)^I\,(3/fL)^J + \delta_{IJ}(1.5f-1)^I(L/EI)^I$
A_{14} = nx1 submatrix; $A_{14}\,(I,1) = C\,(3/fL)^I$
A_{21} = nxn submatrix; $A_{21}\,(I,J) = C(6EI/fL^2)^I(6EI/fL^2)^J - \delta_{IJ}(6EI/fL)^I$
A_{22} = nxn submatrix; $A_{22}(I,J) = -\,C\,(6EI/fL^2)^I(3/fL)^J + \delta_{IJ}[(3-f)/f]^I$

A_{24} = nx1 Submatrix; $A_{24}(I,1)=C\,(6EI/fL^2)^I$
A_{31} = 1xn Submatrix; $A_{31}(1,J)=C\,(6EI/fL^2)^J$
A_{32} = 1xn Submatrix; $A_{32}(1,J) = -\,C\,(3/fL)^J$
A_{33} = 1x1 Submatrix; $A_{33}(1,1) = 1$
A_{34} = 1x1 Submatrix; $A_{66}\,(1,1) = C$
$C = 1.0/\sum_{i=1}^{n}(6EI/fL^3)^i$, δ_{IJ} = Kronecker delta

A_{44} = 1x1 Submatrix, $A_{44}\,(1,1) = -\,1$
A_{55} = nxn unit submatrix
A_{56} = nxn diagonal submatrix; $A_{56}\,(I,I) = -\,(EA/L)^I$
A_{66} = nxn negative unit submatrix

n = number of columns in a story
T,B = subscripts indicating top and bottom column ends.

The symbolic form of Eq.5 is written as:

$$Z_T = AZ_B$$

where

Z_T = column vector of column top state variables for a story (ℓx1)
A = column sub-structure transfer matrix (CSTM) (ℓxℓ)
Z_B = column vector of column bottom state variables (ℓx1)
ℓ = 4n + 2.

It should be noted that formulas for the elements of the CSTM can be simplified greatly if all of the columns have the same length (3). In general, however, different column lengths are encountered at the first floor due to foundation conditions.

2.2 Introduction of Slope-deflection Relations

Expressions for the elements of story transfer matrix (STM) can be derived by using floor joint slope-deflection equations and the elements of CSTM (Eq.5) A typical floor joint i with neighboring spans (i-1) and (i) is shown in Fig. 3.

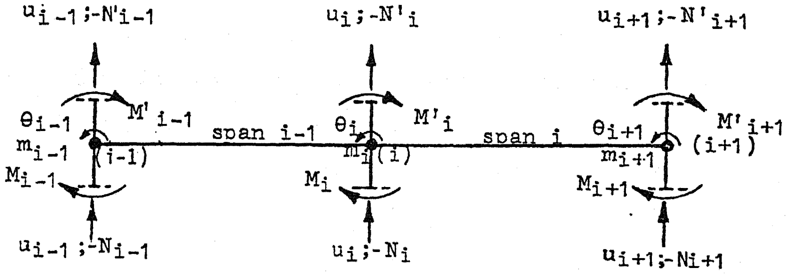


Figure 3- TYPICAL FLOOR JOINT (i)

The state variables at column ends above and below the joints are indicated in the figure. Note that the roation, vertical and horizontal displacements are the same for all member ends attached to a joint i due to compatibility requirements.

The axial forces N_i' and moments M_i' above the floor level at joint i can be determined from joint slope deflection equations as

$$M_i' = - M_i + CK_{i-1}^{\Theta M}\Theta_{i-1} + (K_{i-1}^{\Theta M} + K^{\Theta N})\Theta_i + CK_i^{\Theta M}\Theta_{i+1}$$
$$+ CK_{i-1}^{UM}u_{i-1} + (K_{i-1}^{UM} + K_i^{UM})\ u_i + CK_i^{UM}u_{i+1} \qquad (6)$$

$$N_i' = -N_i + CK_{i-1}^{\Theta N}\Theta_{i-1} + (K_{i-1}^{\Theta N} + K_i^{\Theta N})\ \Theta_i + CK_i^{\Theta N}\ \Theta_{i+1}$$
$$+ CK_{i-1}^{UN}u_{i-1} + (K_{i-1}^{UN} + K_i^{UN})\ u_i + CK_i^{UN}\ u_{i+1} - \omega^2 m_i u_i \qquad (7)$$

where

$CK_{i-1}^{\Theta M}$, $CK_i^{\Theta M}$ = moment at joint i due to rotations at far ends of members i-1 and i, respectively

$K_{i-1}^{\Theta M}$, $K_i^{\Theta M}$ = moment at joint i due to Θ_i rotation of members i-1 and i, respectively.

CK_{i-1}^{UM}, CK_i^{UM} = moment at joint i due to vertical displacement at far ends of members i-1 and i, respectively.

K_{i-1}^{UM}, K_i^{UM} = moment at joint i due to u_i displacements of members i-1 and i, respectively.

m_i = lumped mess at joint i

ω = frequency.

All other stiffness coefficients having superscript N indicate shear force rather than bending moment. Signs of stiffness terms are assumed to be adjusted for proper directions.

Shear equilibrium at floor level k (Fig. 1a and b) results in

$$H_k' = -H_k - d_k \, \omega^2 \Sigma m_i - P_k \tag{8}$$

where

d_k = story sidesway at floor k

Σm_i = total mass at floor k

P_k = applied harmonic horizontal force amplitude.

It is now possible to assemble the story transfer matrix directly by considering Eqs. 6,7,8 and the elements of column sub-structure transfer matrix. A general form of story transfer matrix equations is given by Eq. 9.

$$\begin{bmatrix} \theta \\ M \\ d \\ H \\ u \\ N \end{bmatrix}_{T'} = \begin{bmatrix} B_{11} & B_{12} & 0 & B_{14} & 0 & 0 \\ B_{21} & B_{22} & 0 & B_{24} & B_{25} & B_{26} \\ B_{31} & B_{32} & B_{33} & B_{34} & 0 & 0 \\ B_{41} & B_{42} & B_{43} & B_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{55} & B_{56} \\ B_{61} & B_{62} & 0 & B_{64} & B_{65} & B_{66} \end{bmatrix} \begin{bmatrix} \theta \\ M \\ d \\ H \\ u \\ N \end{bmatrix}_{B} + \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{bmatrix} \tag{9}$$

where T' = subscript indicating the state variables above the floor level
B = subscript indicating the state variables at the column bottom ends

Since $\Theta_{T'} = \Theta_T$ and $u_{T'} = u_T$, the following submatrix relations exist.

$B_{11} = A_{11}$ $B_{55} = A_{55}$ $R_1 = 0$ $R_2 = 0$ $R_6 = 0$

$B_{12} = A_{12}$ $B_{56} = A_{56}$ $R_3 = 0$ $R_5 = 0$

$B_{14} = A_{14}$

From Eqs. 5,6 and 7, expressions for the elements of other submatrices are obtained as:

$$B_{21}(I,J) = -A_{21}(I,J) + [K^{\Theta M}(I-1) + K^{\Theta M}(I)] \, A_{11}(I,J) + CK^{\Theta M}(I-1) \, A_{11}(I-1,J) + CK^{\Theta M}(I) \, A_{11}(I+1,J)$$

$$B_{22}(I,J) = -A_{22}(I,J) + [K^{\Theta M}(I-1) + K^{\Theta M}(I)]\, A_{12}(I,J) + CK^{\Theta M}(I-1)\, A_{12}(I-1,J) + CK^{\Theta M}(I)\, A_{12}(I+1,J)$$

$$B_{24}(I,1) = -A_{24}(I,1)$$

$$B_{25}(I,J) = [K^{UM}(I-1) + K^{UM}(I)]\, A_{55}(I,J) + CK^{UM}(I-1)\, A_{55}(I-1,J) + CK^{UM}(I)\, A_{55}(I+1,J)$$

$$B_{26}(I,J) = [K^{UM}(I-1) + K^{UM}(I)]\, A_{56}(I,J) + CK^{UM}(I-1)\, A_{56}(I-1,J) + CK^{UM}(I)\, A_{56}(I+1,J)$$

$$B_{31} = A_{31},\ B_{32} = A_{32},\ B_{33} = A_{33},\ B_{34} = A_{34}$$

$$B_{41} = -\omega^2 \Sigma m_i A_{31},\ B_{42} = -\omega^2 \Sigma m_i A_{32},\ B_{43} = -\omega^2 \Sigma m_i A_{33}$$

$$B_{44}(1,1) = -[A_{34}(1,1) + A_{66}(1,1)]\, \omega^2 \Sigma m_i$$

$$R_4(1) = -P_k$$

$$B_{61}(I,J) = [K^{\Theta N}(I-1) + K^{\Theta N}(I)]\, A_{11}(I,J) + CK^{\Theta N}(I-1)\, A_{11}(I-1,J) + CK^{\Theta N}(I)\, A_{11}(I+1,J)$$

$$B_{62}(I,J) = [K^{\Theta N}(I-1) + K^{\Theta N}(I)]\, A_{12}(I,J) + CK^{\Theta N}(I-1)\, A_{12}(I-1,J) + CK^{\Theta N}(I)\, A_{12}(I+1,J)$$

$$B_{64}(I,1) = [K^{\Theta N}(I-1) + K^{\Theta N}(I)]A_{14}(I,1) + CK^{\Theta N}(I-1)\, A_{14}(I-1,1) + CK^{\Theta N}(I)\, A_{14}(I+1,1)$$

$$B_{65}(I,J) = [K^{\Theta N}(I-1) + K^{UN}(I)]\, A_{55}(I,J) + CK^{UN}(I-1)\, A_{55}(I-1,J) + CK^{UN}(I)\, A_{55}(I+1,J) - \omega^2 m(I)\, A_{55}(I,J)$$

$$B_{66}(I,J) = -[A_{66}(I,J) + \omega^2 m(I)\, A_{56}(I,J)] + [K^{UN}(I-1) \_ K^{UN}(I)]\, A_{56}(I,J) + CK^{UN}(I-1)\, A_{56}(I-1,J) + CK^{UN}(I)\, A_{56}(I+1,J)$$

In the above expressions invalid subscripts such as $I-1 = 0$ and $I+1 > n$ should not be considered in the computations.

The story transfer matrix relation (Eq.9) for the kth story can be written in compact symbolic form as:

$$Z^k_{T'} = B^k Z^k_B + R^k \tag{10}$$

where

$Z^k_{T'}$ = State vector containing column end rotations, moments, axial displacements, axial forces story shear and sidesway above the floor level (ℓx1)

B^k = Story transfer matrix (ℓxℓ)

Z^k_B = State vector at the bottom of story columns (ℓx1)

R^k = Vector of constants in steady state analysis (ℓx1)

ℓ = 4n + 2

3. DYNAMIC ANALYSIS

Utilization of story transfer matrix in dynamic analysis follows the same principles as those explained in reference 1. Steady state solution requires the solution of linear equations. Free vibration analysis can be performed by using determinant search technique (1). Computer core storage requirement is independent of the number of stories in all computations for dynamic analysis.

3.1 Consideration of P - Δ Effect

P - Δ effect is a geometric non-linearity. Consequently, sensitivity of fundamental frequency to P - Δ effect can be approximated in a linear analysis. The following procedure is proposed for this purpose.

1. Determine the fundamental frequency and the associated mode shape by using determinant search technique.
2. Set the top floor sway of the mode shape to some percent of allowable building code sway value.
3. Include P - Δ effect in Eq. 8 by considering the fictitious horizontal loads due to relative story sidesways based on the mode shape as explained in reference 5.
4. Recalculate the fundamental frequency by considering the frequency dependent horizontal loads.

A range for the fundamental frequency is determined by this technique which allows the designer to judge the sensitivity of the structure to P - Δ effect. Also, a more realistic value for the fundamental frequency can be selected for determination of equivalent static loads by using one of the seismic design codes.

4. SUMMARY AND CONCLUSIONS

A general formulation of the story transfer matrix method which includes the shear deformations, column shortening and non-prismatic members is presented for dynamic analysis of orthogonal planar frames. P - Δ effect is considered in determination of the fundamental frequency in an approximate way.

STM method is an extension of the well-known transfer matrix method and utilizes the sub-structure concept. It is suitable for implementation on small computer systems having limited core storage.

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