THE STOCHASTIC FINITE-ELEMENT METHOD

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SUMMARY

A generic stochastic finite-element modeling method with the capability of analyzing and designing structures in a probabilistic framework has been developed. Stochastic differential equations are combined with the finite-element method in such a way that structural loads are idealized as stochastic processes and incorporated into multidimensional finite-element dynamic models with uncertainty in their parameters. An estimate of the probability of failure, necessary for making decisions between alternative designs, can also be computed. Known and established procedures in second-moment reliability analysis are used, taking the joint-probability function of the parameters that define the safety and serviceability of the structure as gaussian, with a transformation of the random parameters and variables to a gaussian space.

INTRODUCTION

Structural engineers must face two fundamental difficulties: building adequate mathematical models of structures and representing external disturbances. For those designing important structures or developing codes, the severe consequences of structural failure make a probabilistic approach inescapable. The application of stochastic processes to the representation of loads and disturbances and the use of stochastic finite elements to model structures could be convenient tools for multidimensional analysis of structures. By employing stochastic differential equations [1] in combination with the finite-element method [2], a solution of problems associated with time in dynamic models of structures could be accomplished by a step-by-step integration procedure that could be implemented in a general-purpose finite-element program. To account for uncertainty in structural parameters, two methods are proposed: one for considering the *a priori* uncertain parameters [3] and another, which uses measurement of the behavior of the real structure, for identifying the *a posteriori* uncertain parameters.

Stochastic Differential Equations

Stochastic dynamic systems whose representations are continuous mathematical expressions and whose state spaces are finite n-dimensional vectors can be modeled by finite-dimensional Markov processes that are the output of stochastic differential or difference equations. This mathematical tool, based on Itô's calculus from a bayesian point of view, gives the optimum estimate of a system's state for a particular *a priori* mathematical model and *a posteriori* data. Thus, measurements and uncertainty in the parameters of the model can be incorporated into a dynamic system.

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Let the continuous stochastic dynamic system be described by the (generally nonlinear) stochastic differential equation:

$$\dot{x}_t = f(x_t,t) + G(x_t,t)w_t, \text{ when } t \geq t_0 \tag{1}$$

where f is a generally nonlinear real n-vector function; x_t is an n-vector; G is an nxr matrix; and $\{w_t, t>t_0\}$ is a white-noise r-vector with E $w_t w_\tau' = Q\delta_{t\tau}$, where E and δ are the expected-value and Dirac operators, respectively, and $(\cdot)'$ is the transpose of $(\cdot)$. This system can be measured continuously with the m-vector, z, such that:

$$z = h(x_t,t) + v_t, \text{ when } t \geq t_0 \tag{2}$$

where h is an m-vector real function and $\{v_t, t>t_0\}$ is an m-vector white-noise process with E $v_t v_\tau' = R\delta_{t\tau}$, when $R > 0$. We suppose that w_t, v_t, and x_t are independent. If the white-noise processes w_t and v_t are gaussian Eqs. 1 and 2 constitute a vector stochastic differential equation as per Itô [1].

A computer solution to this problem necessitates the discretization in time of Eqs. 1 and 2. The equivalent stochastic difference equations are:

$$x_{k+1} = \psi(x_k,k+1,k) + \Gamma(x_k,k)w_{k+1}, \text{ when } k = 0,1,\ldots \tag{3}$$

where k is a parameter associated with time, t_k; x_k is a state-space n-vector; ψ is an n-dimensional real function; Γ is an nxr matrix; and $\{w_k, k=1,2,\ldots\}$ is an r-vector sequence of white noise such that E $w_k w_l' = Q\delta_{xl}$.

Observations (measurements related to the state of the system) will be expressed by the m-vector, z, such that:

$$z_k = \eta(x_k,k) + v_k, \text{ when } k = 1,2,\ldots \tag{4}$$

where η is an m-vector real function and $\{v_k, k=1,2,\ldots\}$ is an m-vector white-noise sequence with E $v_k v_l' = R\delta_{kl}$, where $R > 0$. As before, w_k, v_k, and x_k are assumed to be independent.

The linear formulation of this equation can be solved by Kalman filtering or other appropriate algorithms [4]. To solve the generally nonlinear problem, either linearization around the mean, as in the extended Kalman filtering [5], or the Monte Carlo technique with reduction of variance [6] can be used.

The state-space vector can be augmented with any desired parameters, p, of the system:

$$x_{p_t} = \begin{Bmatrix} x_t \\ p \end{Bmatrix} = \begin{Bmatrix} \text{time-varying state} \\ \text{system parameters} \end{Bmatrix} \tag{5}$$

In this augmented state-space vector, the parameters, p, can be estimated (in the presence of measured observations) or considered uncertain and discarded at each integration step.

Finite-Element Method

The finite-element method has developed rapidly with the increasing use of digital computers. It is a valuable tool for solving boundary problems. The method was first used intuitively by engineers for structural analysis [7]. Mathematicians, however, soon recognized the underlying principle to be a form of the Rayleigh-Ritz-Galerkin method [8]. This understanding of the finite-element method extends its application to all physical sciences.

Stochastic Finite-Element Method

Partial differential equations, representing a system at a certain time, t, can be discretized by the finite-element method and solved to yield the initial state of the system at instant $t = t_0$. Integration in time can be accomplished through the use of known solution procedures for the stochastic differential equations. Combination of these methods can be one of the most powerful analytic tools available.

Let the linear dynamic equation of a structure modeled with finite elements be given by [2]:

$$M\ddot{s} + C\dot{s} + Ks = \zeta \tag{6}$$

where M, C, and K are mass, damping, and stiffness matrices of dimension $\frac{n}{2} \times \frac{n}{2}$, respectively, of the structure; s is the generalized displacement vector; and ζ is an external-force vector. In general, M, C, K, and ζ are functions of time.

We present Eq. 6 in state-space format by making $x \equiv \{s\ \dot{s}\}'$ such that:

$$\begin{Bmatrix} \dot{s} \\ \ddot{s} \end{Bmatrix} = \begin{bmatrix} \underline{0} & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{Bmatrix} s \\ \dot{s} \end{Bmatrix} + \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & M^{-1} \end{bmatrix} \begin{Bmatrix} \underline{0} \\ \zeta \end{Bmatrix} \tag{7}$$

or, in a simplified form:

$$\dot{x} = Ax + By \tag{8}$$

To discretize in time, we have to find the state transition matrix $\Phi(k + 1,k)$, defined as:

$$\Phi(k + 1,k) = e^{A(t_{k+1}-t_k)} = \sum_{j=0}^{\infty} \frac{1}{j!} A^j (t_{k+1} - t_k)^j \tag{9}$$

The discretized system is then:

$$x_{k+1} = \Phi(k + 1,k)x_k + \Lambda(k + 1,k)y_k \tag{10}$$

where:

$$\Lambda(k + 1,k) = \int_{t_k}^{t_{k+1}} e^{A\xi} B d\xi \tag{11}$$

If we add to Eq. 10 a term accounting for white noise, we have:

$$x_{k+1} = \Phi(k+1,k)x_k + \Lambda(k+1,k)y_k + \Gamma(k)w_{k+1} \tag{12}$$

which is a linear stochastic difference equation as defined above. Here w_k is a vector white-noise sequence and $\Gamma(k)$ a scaling matrix.

A block diagram representation of the stochastic finite-element method can be seen in Fig. 1. The model allows the introduction of control, parameter identification, and uncertain parameters within the same basic mathematical formulation.

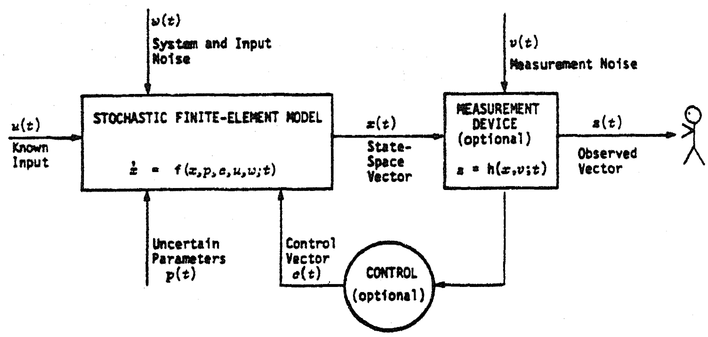

FIGURE 1 GENERAL STOCHASTIC FINITE-ELEMENT MODEL

STOCHASTIC MODELS OF LOADS

To develop stochastic processes to represent external disturbances, a thorough knowledge of the statistical properties of the physical events that are associated with the disturbances is needed. For a particular type of phenomenon, measured results are erratic and dissimilar to one another, thus justifying the use of a probabilistic model and statistical inference techniques to identify its parameters. Fig. 2 shows how a probabilistic model and a physical phenomenon can be linked through measurement and statistical inference. Available data related to the phenomenon under study are collected and processed with the tools of statistical inference to obtain information suitable for use in the stochastic models.

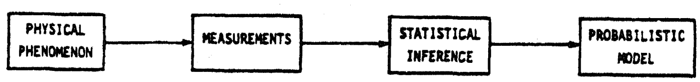

FIGURE 2 RELATIONSHIP BETWEEN PHYSICAL PHENOMENON AND PROBABILISTIC MODEL

Earthquake Simulation

The generation of stochastic processes to simulate loads will be illustrated with examples of seismic disturbances. It is usually assumed [9] that seismic waves arrive at the underlying bedrock in accordance with a nonhomogeneous Poisson process, which can be represented as a white-noise process multiplied by a deterministic envelope function [10]. Up to their second moments, the two processes are equivalent.

Stochastic differential equations are especially well suited to simulating seismic loads. The white noise, w, once filtered through a series of second-order filters, can be subjected to the deterministic envelope function in order to obtain the final earthquake acceleration process. This process is easily introduced into the formulation of stochastic differential equations.

Discretization of White-Noise Disturbance

A stochastic difference equation that can be used to represent the stochastic finite-element method is given by Eq. 12. In the equation, the last term, $\Gamma(k)w_{k+1}$, must be discretized from the given stochastic process that simulates the outside disturbance. If we take $\Gamma \equiv I$, we can, without losing generality, write Eq. 12 as:

$$x_{k+1} = \Phi(k+1,k)\, x_k + \Lambda(k+1,k)\, y_k + w_{k+1} \tag{13}$$

where:

$$w_{k+1} = \int_{t_k}^{t_{k+1}} \Phi(k+1,\tau) G(\tau) w_\tau \, d\tau \tag{14}$$

and it can be proved [5] that w_k is a zero-mean white-noise sequence with

$$E w_{k+1}\, w'_{k+1} = \int_{t_k}^{t_{k+1}} \Phi(k+1,\tau) G(\tau) Q(\tau) G'(\tau) \Phi'(k+1,\tau)\, d\tau \tag{15}$$

and

$$E w_{k+1}\, w'_{k+1} = Q_{k+1} = \Gamma_k Q_{k+1} \Gamma'_k, \text{ as } \Gamma_k = I \tag{16}$$

Eq. 15 can be integrated numerically, or even explicitly [11] for simple stochastic process representations.

STRUCTURAL RELIABILITY

Second-moment reliability formats [12,13] provide a means of obtaining estimates of reliability based on sample moments of the design parameters. The joint-probability distribution of these parameters must be known, or at least assumed; to apply the second-moment reliability methods, a transformation of the probability function of the parameters to a gaussian space is necessary. To assume that all the parameters follow a gaussian distribution will lead to estimates of reliability that could compare poorly with the ones obtained by an approach that considers the sample distribution of the parameters. Methods of transformation to gaussian spaces are used; thereafter, the transformed variables are once more transformed, to canonic forms, to achieve independence; from these, the reliability (or probability of failure) of the structure can be evaluated.

Transformation to Gaussian Space

The vector $p = \{p_1\ p_2\ \ldots\ p_n\}$ of the design parameters has a particular multivariate distribution; in general, the parameters p_i are dependent. It is assumed that the distribution of p is unknown but that sample moments (including sample covariances) can be obtained. If the first four moments of a random variable p_i are known, it is possible to use the S_L, S_B, or S_U transformations [14]. The choice of the particular transformation is defined by the values of the third and fourth moments. Table 1 presents the three transformations S_L, S_B, and S_U and the limits of application of each one in terms of skewness coefficient γ_1 and the kurtosis γ_2.

TABLE 1

TRANSFORMATIONS S_U, S_B, AND S_L

Random variable X with known first four sample moments:

Mean	$\overline{X}$	= first moment
Variance	σ^2	= second central moment
Skewness	γ_1	= (third central moment)σ^{-3}
Kurtosis	γ_2	= (fourth central moment)σ^{-4}

Transformation to a Z random variable with standard normal distribution $Z \sim N(\overline{Z} = 0, \sigma_Z = 1)$:

$$Z = a + bf(X)$$

Transformation Type	f(X)	Limits of Application
S_U (unbounded)	$\sinh^{-1} \frac{X - c}{d}$	$-\infty \le X \le +\infty$; $b,d > 0$; $3 + 1.87\gamma_1 < \gamma_2 < \infty$
S_B (bounded)	$\ln \frac{X - c}{c + d - X}$	$c < X < c + d$; $b,d > 0$; $1 + \gamma_1 \le \gamma_2 < 3 + 1.87\gamma_1$
S_L (lognormal)	$\ln(X - c)$	$c \le X \le +\infty$; $b > 0$; $\gamma_2 = 3 + 1.87\gamma_1$

Another approach is to assign a particular distribution form based on the theoretical behavior of the parameter and, with the known values of, say, the first and second moments, to establish an approximate relationship between the assigned and the normal distributions. The extreme distributions have been used to represent the strength of materials, lifetime of components of systems, and other phenomena that are thought to follow asymptotic laws. The three extreme distributions can be related through logarithmic transformations. Any continuous unimodal distribution $T(x)$ can be transformed to an exponential type of distribution through the relationship $[-\ln T(x)]$; the extreme Type I distribution, normalized with zero mean and unit standard deviation, will be used as an illustration of the method without loss of generality. A least-squares scheme was utilized to fit a logarithmic transformation of the normal distribution. There is close agreement of the least-squares-fitted normal model and the extreme distribution, especially in the upper "tail" values, as shown in Table 2; the fitted lognormal distribution gives values that could also be accurate enough for some applications.

TABLE 2

APPROXIMATIONS OF THE EXTREME TYPE I WITH NORMAL AND LOGNORMAL DISTRIBUTIONS

X	Extreme Type I	Normal, $\Phi[u(x)]^{+}$	Fitted Lognormal
5.0	0.999079	0.999034	0.998762
6.0	0.999745	0.999714	0.999591
7.0	0.999929	0.999912	0.99986
8.0	0.999980	0.999972	0.999949
10.0	0.999998	0.999997	0.9999926
12.0	0.99999988	0.99999954	0.99999876

$^{+}u(x) = 2.8364 \ln (x + 2.7886) - 2.72$

To establish the joint multivariate distribution, it is reasonable to suppose that the variables transformed to gaussian space follow a multivariate normal distribution. The covariances must then be determined from the sample data previously transformed to gaussian space using the relationship employed for each parameter. Through further orthogonal transformations, the multivariate normal distribution of the parameters, p_N, can be presented in the canonical form, p_C; then the covariance matrix of p_C will be diagonal and normalized.

Reliability in Gaussian Space

Following the technique described above, the original design parameters, p, can be transformed to independent random variables p_C, each having a standard normal distribution. The reliability, R, is then given by:

$$R = \frac{1}{(2\pi)^{n/2}} \int_{\Omega_C} \exp(-\frac{1}{2} p_C p_C') \, dp_C \qquad (17)$$

where Ω_C defines the transformed failure surface in the C space. Two approximate solutions exist for this expression: for linear failure surface, Ω_C^L; $R = \Phi(\beta)$, and for spherical failure surface, Ω_C^S; $R = \chi_n^2(\beta^2)$, where β is the distance of closest approach of the failure surface, Ω_C, to the origin. The last expression provides also a lower reliability bound for any form of failure surface. Φ is the standardized normal cumulative distribution function, and χ_n^2 is the Chi-square cumulative distribution function of n degrees of freedom.

NUMERICAL EXAMPLE

We will illustrate the stochastic finite-element method by applying it to a shear-frame structure with two degrees of freedom. Assume that the base of the structure is excited by modulated white noise, $\ddot{x}_0$, given by:

$$\ddot{x}_0 = m_t w_t \tag{18}$$

where w_t = gaussian white noise, and m_t = a deterministic modulation function, such that:

$$m_t = 3.06 \left(e^{-0.25t} - e^{-0.63t}\right) H_t \tag{19}$$

with H_t = Heaviside function. The structural values at levels 1 and 2 are: mass, $m_1 = m_2 = 40{,}000$; stiffness, $k_1 = 4k_2 = 640{,}000$; and damping, $c_1 = 2c_2 = 6{,}400$. Variation coefficients of 0.2 and 0.3 are used for k and c, respectively. The results of solving the stochastic system are shown in Fig. 3, where the value of $\sigma_i = \sqrt{Ex_i^2}$ (i for levels 1 and 2) is compared for deterministic and uncertain parameters k and c.

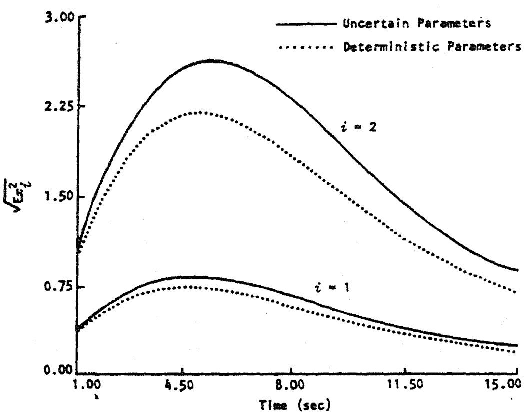

FIGURE 3 RESPONSE OF STRUCTURAL MODEL TO MODULATED WHITE-NOISE EXCITATION

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