EARTHQUAKE RESPONSE OF SYSTEMS WITH NONPROPORTIONAL DAMPING BY THE CONVENTIONAL RESPONSE SPECTRUM METHOD

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SUMMARY

By means of a complex modal analysis and introducing some approximations, a procedure is developed to compute, within the framework of the conventional response spectrum technique, the maximum earthquake response of systems with nonproportional damping. It is shown that this procedure may be considered as the generalization of the response spectrum method for systems without classical modes of vibration, and that it converges to this method for systems with proportional damping. Its accuracy is illustrated by comparing the approximate and exact solutions of a system.

INTRODUCTION

The analysis of systems with nonproportional damping has been customarily carried out by a conventional modal analysis in which, in order to uncouple their equations of motion, the off-diagonal elements of their generalized damping matrices are disregarded. This approach has been proved satisfactory for ordinary structures with rather uniform characteristics. There may be, however, some systems for which such a procedure may not be valid. Examples of these systems are those with great difference in the values of their various masses, stiffnesses, and damping coefficients in which, since the off-diagonal elements of their generalized damping matrices are of the order of magnitude of some of those along a main diagonal, by neglecting such off-diagonal elements one may introduce errors of considerable importance. A reliable analysis of such systems is therefore only viable through a costly and cumbersome step-by-step integration or a complex modal analysis.

In the belief, then, that the response spectrum method is a convenient method of analysis, in this paper is introduced an approximate procedure based on the theory of the aforementioned complex modal analysis, to extend the response spectrum method for the analysis of systems with nonproportional damping.

MODAL DAMPING RATIOS AND NATURAL FREQUENCIES

It is shown elsewhere [1] that the homogeneous equation of motion of a system with nonproportional damping

$$[M]\ \{\ddot{x}\} + [C]\ \{\dot{x}\} + [K]\ \{x\} = \{0\}, \qquad (1)$$

where [M], [C], and [K] represent respectively its mass, damping, and stiffness matrices, and {x} is its displacent vector, is satisfied by

$$\{x\}^{(r)} = \{w\}^{(r)}\ e^{\lambda_r t},\ r = 1,\ 2,\ \ldots,\ 2N \qquad (2)$$

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where $\{x\}^{(r)}$ represents the rth solution of Eq. 1, $\{w\}^{(r)}$ is the rth complex mode shape of the system, λ_r is the corresponding complex natural frequency, and N indicates the number of degrees of freedom of the system. Then, if Eq. 1 is premultiplied by the transpose of $\{\bar{w}\}^{(r)}$, the rth complex conjugate mode shape*, the free vibration equation of the system under consideration may be expressed as

$$\lambda_r^2 M_r^* + \lambda_r C_r^* + K_r^* = 0, \qquad r = 1, 2, \ldots, N \tag{3}$$

where

$$M_r^* = \{\bar{w}\}^{(r)^T}[M]\{w\}^{(r)} \tag{4}$$

$$C_r^* = \{\bar{w}\}^{(r)^T}[C]\{w\}^{(r)} \tag{5}$$

$$K_r^* = \{\bar{w}\}^{(r)^T}[K]\{w\}^{(r)}. \tag{6}$$

Hence, by solving for λ_r from Eq. 3 one obtains

$$\lambda_r = -\frac{1}{2}(C_r^*/M_r^*) \pm \frac{1}{2}\sqrt{(C_r^*/M_r^*)^2 - 4K_r^*/M_r^*} \tag{7}$$

and thus, since M_r^*, C_r^* and K_r^* are all real, one has that in similarity with a single-degree-of-freedom system:

(a) the condition for having an oscillatory motion in the rth mode of the system is

$$(C_r^*/M_r^*)^2 < 4K_r^*/M_r^* \tag{8}$$

(b) there exists a critical value of C_r^* with which such an oscillatory motion stops, given by

$$(C_r^*)_{cr} = 2\sqrt{K_r^* M_r^*} \tag{9}$$

(c) C_r^* may be expressed in terms of a percentage ξ_r of this critical damping value as

$$C_r^* = 2\xi_r\sqrt{K_r^* M_r^*}\ , \tag{10}$$

(d) by denoting

$$\omega_r^2 = K_r^*/M_r^* \tag{11}$$

C_r^* may be written as

$$C_r^* = 2\xi_r M_r^* \omega_r \tag{12}$$

and consequently, by substitution of these two equations into Eq. 7, λ_r may be put into the form

$$\lambda_r = -\xi_r\omega_r \pm i\omega_r' \tag{13}$$

*Throughout this paper, the complex conjugate of a complex variable will be indicated by a bar above the variable.

where

$$\omega_r' = \omega_r \sqrt{1-\xi_r^2} \quad (14)$$

Thus by means of Eqs. 2 and 13 the rth solution of Eq. 1 may be expressed as

$$\{x\}^{(r)} = \{w\}^{(r)} e^{-\xi_r \omega_r t} [\cos \omega_r' t + i \sin \omega_r' t], \quad (15)$$

from which it may be concluded that, as in the case of proportional damping, the imaginary and real parts of the rth complex frequency of a system with nonproportional damping also define, respectively, the frequency of vibration of the system in such an rth mode and the rate by which such vibrations are damped out with time. Notice, then, that the vibrational characteristics of a system with nonproportional damping may also be described by modal natural frequencies of vibration and damping ratios. Observe, however, that these frequencies of vibration and damping ratios are not necessarily those of an equivalent system with proportional damping, since the generalized parameters M_r^*, K_r^*, and C_r^* may be different.

MAXIMUM EARTHQUAKE RESPONSE

It is shown in Ref. 2 that the solution of a system with proportional damping subjected to external forces is of the form

$$\{x\} = 2 \sum_{r=1}^{N} \text{Re } [\{w\}^{(r)} Z_r] \quad (16)$$

where "Re" stands for "the real part of" and Z_r is given by

$$Z_r = \frac{\int_0^t e^{\lambda_r(t-\tau)} \{w\}^{(r)^T} \{P(\tau)\} \, d\tau}{\{w\}^{(r)^T} [2\lambda_r [M] + [C]] \{w\}^{(r)}} \quad (17)$$

in which $\{P(\tau)\}$ represents the vector of external forces applied to the system and all other symbols are as denoted before. Therefore, since for the case of an earthquake excitation the vector of external forces results as

$$\{P(t)\} = - [M]\{J\} \ddot{q}_g(t) \quad (18)$$

where $\{J\}$ is a vector of unit elements and $\ddot{q}_g(t)$ is an earthquake ground acceleration function, for earthquake forces[8] Z_r may be expressed as

$$Z_r(t) = -\gamma_r \int_0^t e^{\lambda_r(t-\tau)} \ddot{q}_g(\tau) \, d\tau \quad (19)$$

where γ_r, the rth complex participation factor of the system, is defined as

$$\gamma_r = \frac{\{w\}^{(r)^T} [M]\{J\}}{\{w\}^{(r)^T} [2\lambda_r [M] + [C]] \{w\}^{(r)}} \quad (20)$$

Thus, by substitution of Eq. 19 into Eq. 16, by writing $\{w\}^{(r)}$ and λ_r explicitly in terms of their real and imaginary parts, and if $\{w'\}^{(r)}$ denotes a complex mode shape with unit complex participation factor, i.e.,

$$\{\dot{w}'\}^{(r)} = \gamma_r\{w\}^{(r)} = \{u'\}^{(r)} + i\ \{v'\}^{(r)}, \tag{21}$$

the earthquake response of a system with nonproportional damping may be expressed as

$$\{x(t)\} = -2\sum_{r=1}^{N} [\{\bar{u}'\}^{(r)} \int_0^t e^{-\xi_r\omega_r(t-\tau)}\ddot{q}_g(\tau)\ \cos\omega_r'(t-\tau)\ d\tau - $$

$$-\{v'\}^{(r)} \int_0^t e^{-\xi_r\omega_r(t-\tau)}\ddot{q}_g(\tau)\ \sin\ \omega_r'(t-\tau)\ d\tau]. \tag{22}$$

But the second integral in this last equation may be identified as the product of ω_r' and the displacement response to the ground motion $\ddot{q}_g(t)$ of a single-degree-of-freedom system with natural frequency ω_r and damping ratio ξ_r. Similarly, for small damping ratios the first integral may be considered as the corresponding velocity response. As a consequence if $V(\omega_r,\ \xi_r,\ t)$ and $D(\omega_r,\xi_r,\ t)$ represent the aforementioned velocity and displacement responses, $\{x(t)\}$ may be expressed alternatively as

$$\{x(t)\} = 2\sum_{r=1}^{N} \{u'\}^{(r)}V(\omega_r,\xi_r,t)-\{v'\}^{(r)}\omega_r'\ D(\omega_r,\xi_r,t) \tag{23}$$

and hence the vector of maximum displacements is of the form

$$\{x_{max}\} = 2\sum_{r=1}^{N} \{u'(r)V(\omega_r,\xi_r,t_{max})-v'(r)\omega_r'D(\omega_r,\xi_r,t_{max})\} \tag{24}$$

where t_{max} signifies the time at which the maximum value of the displacement of a particular mass of the system under consideration occurs.

Notice, thus, that as in the case of proportional damping the maximum earthquake response of a system with nonproportional damping is given by the sum of the individual responses in each of its modes, and that consequently this maximum response may also be estimated from the maximum values of such individual modal responses. Notice, however, that since the maximum values of the velocity and displacement functions in Eq. 23 cannot occur at the same time (the displacement function reaches its maximum when the velocity one is zero), such a maximum response cannot be evaluated directly from a response spectrum. To determine, then, the maximum earthquake response of systems with nonproportional damping by the conventional response spectrum technique, the following approximate formulation for the aforementioned maximum modal responses is introduced.

APPROXIMATE MAXIMUM MODAL RESPONSES

It may be observed from Eq. 24 that an upper bound to the displacement response in the rth mode of a system with nonproportional damping is

$$\{x\}^{(r)} \leq 2\{|u'(r)\ V(\omega_r,\xi_r,t_{max}) + v'(r)\omega_r' D(\omega_r,\xi_r,t_{max})|\}\ , \quad (25)$$

and thus, since $V(\omega_r,\xi_r,t_{max})$ and $D(\omega_r,\xi_r,t_{max})$ are always less than or equal to their corresponding spectral values, the above inequality may also be written as

$$\{x\}^{(r)} \leq 2\{|u'(r)\ SV_r + v'(r)\ \omega_r'\ SD_r|\} \quad (26)$$

where SV_r and SD_r are respectively the velocity and displacement in the response spectrum of the ground motion $\ddot{q}_g(t)$ corresponding to a frequency ω_r and a damping ratio ξ_r.

The upper limit in Eq. 26 may be evaluated from a response spectrum and may be therefore adopted to approximate the maximum modal responses in concern. Less conservative values may be adopted, however, if the two terms in this Eq. 26 are combined, instead, on the basis of the square root of the sum of their squares. That is, $\{x\}^{(r)}$ may be approximated as

$$\{x\}^{(r)} = 2\{\sqrt{u'^2(r)\ SV_r^2 + v'^2(r)\ \omega_r'^2\ SD_r^2}\}\ . \quad (27)$$

But since this approximation does not consider the relative sign between the various modal responses of a system, and since in some instances such a sign may be an important factor in the computation of this system's maximum response (in systems with closely-spaced natural frequencies, for example), one may assume that the sign of the argument between the absolute value bars in Eq. 26 is also the sign of Eq. 27. In this manner, if it is considered that for small damping ratios $SV_r \doteq \omega_r\ SD_r$ and $\omega_r' \doteq \omega_r$, the maximum modal responses of a system with nonproportional damping may be calculated by

$$\{x\}^{(r)} = 2\{\text{sgn}(u'+v')|w'|\}^{(r)}\omega_r'\ SD_r \quad (28)$$

where $|w'|$ denotes the absolute value of an element of the complex mode shape $\{w'\}^{(r)}$.

COMBINATION OF MODAL MAXIMA

By the inspection of Eqs. 28 and 24, it is easy to see that an upper bound to the maximum response of a system with nonproportional damping may be obtained if the absolute values of its maximum modal responses are considered. Hence, in similarity with a system with proportional damping, the combination of its modes may be conservatively made by "the absolute sum of the maxima". In like manner, if among all the above mentioned maximum modal responses there is one that is significantly greater than the rest of them, it may be seen that a less conservative value of such a maximum response may be estimated by the "square root of the sum of the squares". In contrast, since the rule suggested by Rosenblueth to combine the modal responses of systems with closely-spaced natural frequencies has been derived specifically for systems with classical modes of vibration [3] this rule is not applicable to systems with nonproportional damping. Notwithstanding, it is shown in Ref. 1 that Rosenblueth's rule may be generalized for these kind of systems, and that this generalized rule is of the form

$$X_{i_{max}} = \sqrt{\sum_{r=1}^{N} X_i^2(r) + \sum_{\substack{m=1 \\ m\neq n}}^{N} \sum_{n=1}^{N} \alpha_{mn} X_i(m) X_i(n)} \tag{29}$$

where $X_{i_{max}}$ represents the maximum response of the ith mass or element of the system, $X_i(r)$ is the corresponding maximum response in its rth mode, and α_{mn}, a factor that correlates the mth and nth modes, is given by

$$\alpha_{mn} = 2\mathrm{Re}[\exp i\, [\theta_i(m)-\theta_i(n)]/(\lambda'_m + \overline{\lambda}'_n)]\ \sqrt{\xi'_m\omega_m\xi'_n\omega_n} \tag{30}$$

where $\theta_i(r)$, r=m,n, denotes the phase angle of the ith element of $\{w'\}^{(r)}$, λ'_r, r=m,n, is a corrected complex natural frequency of the form

$$\lambda'_r = -\xi'_r\omega_r + i\omega_r\sqrt{1-\xi'^2_r}\ , \tag{31}$$

and ξ'_r, r=m,n, is a corrected modal damping ratio given by

$$\xi'_r = \xi_r + 2/[\omega_r S(\xi_r)] \tag{32}$$

in which $S(\xi_r)$, a function of ξ_r, is the duration of the equivalent white noise excitation that best represents the ground motion under consideration, and may be calculated by the criteria suggested in Ref. 1.

CONVERGENCE TO THE CASE OF PROPORTIONAL DAMPING

It is well known that when the damping matrix of a system is proportional to either its mass or its stiffness matrix, or to any linear combination of them, all its mode shapes are real. For such a system, therefore, one may write

$$\{w\}^{(r)} = \{\overline{w}\}^{(r)} = \{\phi\}^{(r)}, \tag{33}$$

where $\{\phi\}^{(r)}$ represents the rth of such real mode shapes, and hence, by virtue of Eqs. 11 and 12, Eq. 20 becomes

$$\gamma_r = \{\phi\}^{(r)^T}[M]\{J\}/(i2\omega'_r M^*_r) = \alpha_r/(2i\omega'_r) \tag{34}$$

in which α_r is the conventional participation factor. In the light of Eqs. 33 and 34, Eq. 21 may then be expressed as

$$\{w'\}^{(r)} = \{u'\}^{(r)} + i\,\{v'\}^{(r)} = \alpha_r\ \{\phi\}^{(r)}/(2i\omega'_r) \tag{35}$$

from which it may be concluded that

$$\{u'\}^{(r)} = \{0\}\ ;\ \{v'\}^{(r)} = -\,\alpha_r\ \{\phi\}^{(r)}/(2\omega'_r) \tag{36}$$

and consequently by substitution of Eqs. 35 and 36 into Eq. 28 one arrives to

$$\{x\}^{(r)} = \alpha_r\{\phi\}^{(r)}\ SD_r\ . \tag{37}$$

In like manner, since for systems with proportional damping one has that $\theta_i(m)$ and $\theta_i(n)$ are equal to zero, the modal correlation factor α_{mn} in Rosenblueth's rule to combine modes turns out to be

$$\alpha_{mn} = 2(\xi'_m\omega_m+\xi'_n\omega_n)\sqrt{\xi'_m\omega_m\xi'_n\omega_n} \;/\; [(\xi'_m\omega_m + \xi'_n\omega_n)^2 + (\omega'_m - \omega'_n)^2] \tag{38}$$

which, by considering that for systems with proportional damping and with frequencies ω_m and ω_n close to each other $\sqrt{\xi'_m\,\omega_m\,\xi'_n\,\omega_n}/(\xi'_n\,\omega_m + \xi_n\,\omega_n) \doteq 1/2$, may be written approximately as

$$\alpha_{mn} = 1/[1 + (\frac{\omega_n - \omega_m}{\xi'_m\omega_m + \xi'_n\omega_n})^2] \tag{39}$$

It may be seen, thus, that the procedure suggested above may be considered as the generalization of the response spectrum method for systems with nonproportional damping, and that for the particular case of proportional damping this general procedure converges to the known expressions of this method.

COMPARATIVE RESULTS

The accuracy of the procedure herein being proposed is evaluated by comparing the exact and approximate pseudo-acceleration responses of the structure-pipeline system depicted in Fig. 1, when the base of the structure is subjected to the first ten seconds of the component S00E of El Centro (May 18, 1940) earthquake accelerogram. (The units of the mass, damping, and stiffness values indicated in Fig. 1 are T-sec^2/m, T-sec/m, and T/m, respectively.) The approximate response is computed by the proposed procedure, i.e., Eqs. 28 and 29, and the exact response is obtained by a time-history analysis.

In the determination of the response by the proposed method, the complex mode shapes and natural frequencies of the system are computed by the EISPAC eigenvalue subroutine package of the IBM 360/75 computer system at the University of Illinois [4]. For the computation of the corrected damping ratios in Rosenblueth's rule to combine modes (Eq. 32) and in accordance with the values calculated in Ref. 1, it is assumed that the equivalent earthquake durations for the first (ξ_1=0.00524) and second (ξ_2=0.01576) modes of the system are 18.5 and 16.0 sec., respectively.

The results are presented in Table 1, where the accuracy achieved with the approximate method is indicated by approximate to exact ratios.

CONCLUDING REMARKS

The derived expressions and the results of the above comparative analysis indicate that the suggested procedure is a general method of analysis that eliminates the unnecessary complications of other methods, may be applied with the same simplicity of the traditional methods for the analysis of systems with proportional damping, and furnishes an accuracy consistent with the uncertainties of the conventional response spectrum method. Thus, it is believed that this procedure provides a convenient alternative method for the analysis of systems with nonproportional damping.

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TABLE 1. APPROXIMATE AND EXACT MAXIMUM PSEUDOACCELERATIONS (g)

MASS	APPROXIMATE	EXACT	APP/EXACT
M_1	0.49806	0.50672	0.983
M_2	0.71977	0.81732	0.881
M_3	1.16004	1.39052	0.834
m_1	2.66982	2.74393	0.973
m_2	6.91567	6.66422	1.038

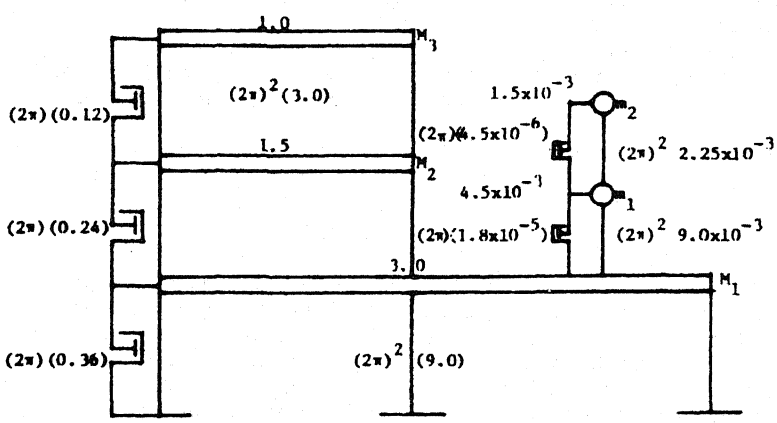

FIG. 1 SYSTEM CONSIDERED FOR COMPARATIVE ANALYSIS