THE USE OF INTERPOLATION FUNCTIONS IN RESPONSE CALCULATIONS

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SUMMARY

This paper presents a method for the calculation of the response of linear elastic systems. The basis of the method is to express the acceleration diagram by a suitable interpolation function and then to perform an exact integration of the Duhamel Integral. Complex algebra is very convenient for the necessary formulation. The imaginary part of a complex quantity is proportional to the required displacement response. Since it is essentially an exact method, no problems of accuracy and convergence arise in the calculations. The only type of error is due to truncation, if a very large number of response ordinates are to be calculated.

FORMULATION

The displacement response of a damped single-degree-of-freedom system during earthquakes can be expressed by

$$y(t) = -\frac{1}{p_d}\ \mathrm{Im}\{z(t)\} \qquad \text{where} \qquad z(t)= e^{\lambda t}\int_0^t a(\tau)e^{-\lambda\tau}d\tau\,,\ \lambda=-\xi p+ip_d$$

and ξ is the fraction of critical damping, p and p_d are undamped and damped natural frequencies, respectively, and $a(t)$ is the acceleration-time history of the earthquake.

In general, the acceleration-time history of the earthquake is digitized at equal time intervals h. Since $t_j=jh$, then the $z(t)$ function is

$$z(t)=z(jh)$$

and this can be designated simply as z_j.

The acceleration-time history $a(t)$ of the earthquake can be resresented by continuous functions $g_j(t)$ in the interval between t_{j-1} and t_j. Then z_j can be calculated as

$$\begin{aligned} z_j &= e^{\lambda t_j}\int_0^{t_j} a(\tau)e^{-\lambda\tau}d\tau \\ &= e^{\lambda t_j}\Big[\int_0^{t_1} g_1(\tau)e^{-\lambda\tau}d\tau+\int_{t_1}^{t_2} g_2(\tau)e^{-\lambda\tau}d\tau+\ \dots\ +\int_{t_{j-1}}^{t_j} g_j(\tau)e^{-\lambda\tau}d\tau\Big] \end{aligned}$$

where $g_j(t)$ is any continuous interpolation function of time which can reasonably represent the change of acceleration. Let the integral of its product with the exponential function be

$$\int g_j(\tau)e^{-\lambda\tau}d\tau = G_j(\tau)e^{-\lambda\tau}$$

Then

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$$\int_{t_{j-1}}^{t_j} g_j(\tau)e^{-\lambda\tau}d\tau = \left|G_j(\tau)e^{-\lambda\tau}\right|_{t_{j-1}}^{t_j} = G_j(t_j)e^{-j\lambda h} - G_j(t_{j-1})e^{-(j-1)\lambda h}$$

$$= e^{-j\lambda h}\underbrace{\left[G_j(t_j) - e^{\lambda h}G_j(t_{j-1})\right]}_{f_j} = e^{-j\lambda h}f_j$$

Therefore

$$z_j = e^{j\lambda h}\left[e^{-\lambda h}f_1 + e^{-2\lambda h}f_2 + \dots + e^{-(j-1)\lambda h}f_{j-1} + e^{-j\lambda h}f_j\right]$$

$$= e^{\lambda h}\underbrace{e^{(j-1)\lambda h}\left[e^{-\lambda h}f_1 + e^{-2\lambda h}f_2 + \dots + e^{-(j-1)\lambda h}f_{j-1}\right]}_{z_{j-1}} + f_j$$

$$z_j = e^{\lambda h}z_{j-1} + f_j$$

This is a recurrence formula for the determination of z_j with the initial value of z_o. If the system is at rest before the earthquake, $z_o=0$.

INTERPOLATION FUNCTIONS

Three different interpolation functions, namely the linear, quadratic, and cubic, are discussed in this paper. The necessary formulae are presented in Table 1.

COMPUTATION OF INTERMEDIATE RESPONSE VALUES

If the response of a single-degree-of-freedom system is calculated only at discrete points of acceleration ordinates, peak ordinates which may occur in intermediate points are missed. In this case, the interval h is divided into m equal subintervals where $h'=h/m$. The z_k value at any k-th discrete point is calculated by using the formulae given in Table 2. The coefficients s_ℓ'.are computed from Table 1, by substituting h' for h.

COMPUTATION OF RESPONSE SPECTRA

In the computation of response spectra, only the maximum value of the response, rather than the response values at each step, is needed. For this reason, a procedure which can find out the maximum value without necessitating the calculation of response values at each step is more efficient. The sign of the velocity is the necessary criterion to establish such a procedure.

It can be shown that the expression of velocity is

$$v(t) = -\mathrm{Re}\{z(t)\} + \frac{\xi p}{p_d}\mathrm{Im}\{z(t)\}$$

If the response is assumed to be a sinusoidal motion at the natural frequency of the oscillator, within a time interval shorter than the half of the *natural period of the oscillator, the response may have at most one peak.* Change of sign in the velocity shows the existence of such a peak. The time of this peak coincides exactly with the zero crossing of the velocity value.

Table 1

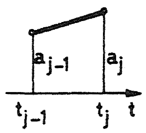
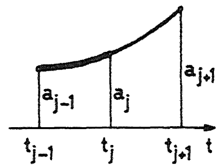
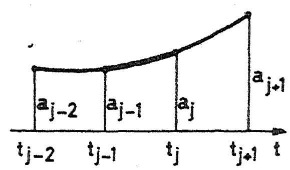
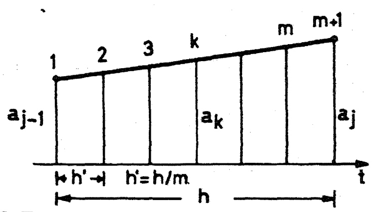
Linear	Quadratic	Cubic
		
$f_j = s_1 a_{j-1} + s_2 a_j$	$f_j = s_1 a_{j-1} + s_2 a_j + s_3 a_{j+1}$	$f_j = s_o a_{j-2} + s_1 a_{j-1} + s_2 a_j + s_3 a_{j+1}$
$s_1 = k_o - \frac{k_1}{h}$ $s_2 = \frac{k_1}{h}$	$s_1 = k_o - (3k_1 - \frac{k_2}{h}) \frac{1}{2h}$ $s_2 = (2k_1 - \frac{k_2}{h}) \frac{1}{h}$ $s_3 = (-k_1 + \frac{k_2}{h}) \frac{1}{2h}$	$s_o = (-k_1 + \frac{3k_2}{2h} - \frac{k_3}{2h^2}) \frac{1}{3h}$ $s_1 = k_o - (k_1 + \frac{2k_2}{h} - \frac{k_3}{h^2}) \frac{1}{2h}$ $s_2 = [k_1 + (k_2 - \frac{k_3}{h}) \frac{1}{2h}] \frac{1}{h}$ $s_3 = (-k_1 + \frac{k_3}{h^2}) \frac{1}{6h}$
$k_o = (e^{\lambda h} - 1) \frac{1}{\lambda}$ $k_1 = (k_o - h) \frac{1}{\lambda}$	$k_2 = (2k_1 - h^2) \frac{1}{\lambda}$	$k_3 = (3k_2 - h^3) \frac{1}{\lambda}$

Table 2



$$z_k = e^{\lambda h'} z_{k-1} + \sum_{\ell=\ell_a}^{\ell_b} p_{\ell,k} a_{j+\ell-2}$$

$$p_{\ell,k} = \sum_{i=\ell_a}^{\ell_b} s_i' r_{\ell,k+i-2} \quad (\ell = \ell_a, \dots \ell_b)$$

$(k=2,3,\dots m)$

Linear $\ell_a = 1$ $\ell_b = 2$	Quadratic $\ell_a = 1$ $\ell_b = 3$	Cubic $\ell_a = 0$ $\ell_b = 3$
$r_{1,k} = \frac{m-k+1}{m}$ $r_{2,k} = \frac{k-1}{m}$	$r_{1,k} = 1 + \frac{(k-1)(k-3m-1)}{2m^2}$ $r_{2,k} = \frac{(k-1)(2m-k+1)}{m^2}$ $r_{3,k} = \frac{(k-1)(k-m-1)}{2m^2}$	$r_{o,k} = \frac{k-1}{3m} [\frac{1}{2m^2} (k-1)(3m-k+1) - 1]$ $r_{1,k} = 1 - \frac{k-1}{2m} [1 + \frac{(k-1)(2m-k+1)}{m^2}]$ $r_{2,k} = \frac{k-1}{2m} [2 + \frac{(k-1)(m-k+1)}{m^2}]$ $r_{3,k} = \frac{k-1}{6m} [(\frac{k-1}{m})^2 - 1]$
$(k = 1,2,\dots,m)$	$(k=1,2,\dots,m+1)$	$(k=0,1,\dots,m+1)$

Suppose that for division of m equal subintervals, the necessary coefficients are already calculated, then any z_k can be obtained from z_{j-1}. If m is chosen in powers of two, the bisection method is the most efficient procedure for the determination of the peak to the desired accuracy. The details are given in Ref.1.

Necessary formulae are given in Table 3. In the example, the peak near k=8 is obtained only after four computations.

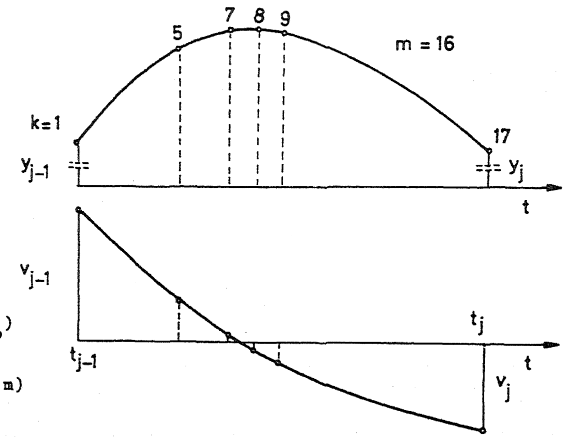
Table 3

$$z_k = \alpha^{k-1} z_{j-1} + \sum_{\ell=\ell_a}^{\ell_b} q_{\ell,k} a_{j+\ell-2} \qquad (k=2,3,..m)$$

where

$$\alpha = e^{\lambda h'}$$

$$\left.\begin{aligned} q_{\ell,2} &= p_{\ell,2} \\ q_{\ell,k} &= \alpha q_{\ell,k-1} + p_{\ell,k} \end{aligned}\right\} \begin{aligned} &(\ell=\ell_a,..\ell_b) \\ &(k=3,4,..m) \end{aligned}$$



CONCLUSIONS

1- Since the integral performed on the assumed interpolation function is exact, the result obtained is accurate. The only kind of error introduced in the computations is the truncation error at each step.

2- After the determination of the complex coefficients, the whole integration can be performed by using only real algebra.

3- For the simplest interpolation function, the straight line, the number of numerical operations required in each step is the same as the linear acceleration method. In case of higher-order interpolation functions, more accurate results are obtained by performing a few more numerical operations.

4- The procedure is applicable to the division of the time interval into shorter time intervals, without loss of accuracy.

5- For the purpose of calculating the response spectra, it is also shown that, in combination with the bisection method, the proposed procedure is very efficient.

6- In Ref.1 it is shown that the procedure is also applicable to multi-degree-of-freedom systems.

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REFERENCES

1- Ipek M., 1979, "The Use of Interpolating Functions in Response Calculations", (Part I and Part II), Research Report, Department of Architecture, Faculty of Engineering, University of Tokyo.