RESEARCH ON THE DYNAMIC CHARACTERISTICS OF THE FOUNDATION BY THE USE OF NUMERICAL INTEGRAL EXPRESSIONS FOR THE WAVE EQUATIONS

Isao Toriumi[i], Akira Kowada[ii], Toshiharu Hisatoku[iii], Teruo Segawa[iv]

SUMMARY

Analytical expressions and numerical examples are presented for a method for dynamic response analysis of the foundation as the structure-soil systems. In this method the soil region is represented as a homogeneous, isotropic, linearly elastic half space in plane strain. And the dynamic problem of structur-soil systems is treated as the wave propagation problem expressed by the wave equations in the form of numerical integral expressions in terms of potential functions ϕ and ψ. The procedure is based on solutions of the wave equations in the soil region and solutions of the mixed boundary value problem which is expressed by stresses at the surface of the soil and displacements at the structure-soil interface.

INTRODUCTION

Basically there are two methods of evaluating the dynamic characteristics of the foundation. One is associated with half-space analyses[1,2] of elastic or viscoelastic layered system, and the other is through the use of finite element method[3]. In the former method, the treatment of the problem generally results in the solution of the mixed boundary value problem which should be expressed by stresses and displacements. But in most of the previous studies, boundary conditions were expressed only by stresses assuming the stress distributions beneath the foundation, and also the surface waves were not considered for the embedded foundation. In the latter, it seems difficult to define the boundary conditions which represent the effects of wave energy dissipations at the outer side of the soil region.

The features of the method which is proposed in this paper are the numerical intearal expressions for the wave equations, and the treatment of boundary conditions in which the mixed boudary conditions can be directry expressed by simple terms of potential functions ϕ, ψ, even for the embedded foundation. Furthermore, since the soil region is fully represented by the medium of wave propagation without defining the outer side boundary, this method can express the effects of wave energy dissipations more clearly than the finite element method.

BASIC EQUATIONS FOR ELASTIC MEDIA

With reference to an orthogonal cartesian coordinate system (x, z) shown in Fig. 1, the equations relating σxx, σzz, σxz, the stress components, to Ux, Uz, the displacement components, applicable to a homogeneous, isotropic, linearly elastic body in the state of plane strain are

$$\sigma xx = (\lambda + 2\mu)\frac{\partial Ux}{\partial x} + \lambda\frac{\partial Uz}{\partial z}$$

$$\sigma zz = (\lambda + 2\mu)\frac{\partial Uz}{\partial z} + \lambda\frac{\partial Ux}{\partial x} \qquad (1)$$

i Prof. of Civ. Eng., Fukui University, Japan
ii Kansai Electric Power Co., Inc., Japan
iii Dr. Civ. Eng., Takenaka Komuten Co., Ltd., Japan
iv Takenaka Komuten Co., Ltd., Japan

$$\sigma xz = \mu \left(\frac{\partial Uz}{\partial x} + \frac{\partial Ux}{\partial z}\right)$$

where λ and μ are the Lamé constants.

The Navier equations of equmprium are

$$(\lambda + \mu) \frac{\partial}{\partial x} \left(\frac{\partial Ux}{\partial x} + \frac{\partial Uz}{\partial z}\right) + \mu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) Ux = \rho \frac{\partial^2 Ux}{\partial t^2}$$
$$(\lambda + \mu) \frac{\partial}{\partial z} \left(\frac{\partial Ux}{\partial x} + \frac{\partial Uz}{\partial z}\right) + \mu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) Uz = \rho \frac{\partial^2 Uz}{\partial t^2} \qquad (2)$$

where ρ is the density of thc medium.

The displacements can be expressed in terms of potential functions ϕ and ψ:

$$Ux = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}$$
$$Uz = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \qquad (3)$$

Eq. 3 will be satisfied if the functions ϕ and ψ are solution of the wave equations

$$\frac{\partial^2 \phi}{\partial t^2} = \alpha^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \phi$$
$$\frac{\partial^2 \psi}{\partial t^2} = \beta^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \psi \qquad (4)$$

where

$$\alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}, \quad \beta = \sqrt{\frac{\mu}{\rho}} \qquad (5)$$

α is the velocity of dilatational waves, and β is the velocity of distortional waves.

The stress components can, from Eqs. 1 and 3, be expressed in terms of potential functions:

$$\sigma xx = \lambda \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right) + 2\mu \left(\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial z}\right)$$
$$\sigma zz = \lambda \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right) + 2\mu \left(\frac{\partial \phi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x \partial z}\right) \qquad (6)$$
$$\sigma xz = \mu \left(2 \frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2}\right)$$

NUMERICAL INTEGRAL EXPRESSIONS FOR WAVE EQUATIONS

The wave equations, Eq. 4 can be rewritten in the form of numerical integral expressions by Takahashi method.[4]

$$\phi(x,z,t+\tau)=\frac{2}{2s_1\cdot 2s_1}\int_{-s_1}^{s_1}\int_{-s_1}^{s_1}\phi(x+\xi,z+\zeta,t)\,d\xi d\zeta$$
$$-\phi(x,z,t-\tau)+O(s_1^4)+O(\tau^4) \tag{7}$$
$$\psi(x,z,t+\tau)=\frac{2}{2s_2\cdot 2s_2}\int_{-s_2}^{s_2}\int_{-s_2}^{s_2}\psi(x+\xi,z+\zeta,t)\,d\xi d\zeta-\psi(x,z,t-\tau)$$
$$+O(s_2^4)+O(\tau^4)$$

where τ is time increment and s_1, s_2 are integral intervals. Eq. 7 will be satisfied if s_1, s_2, α, β and τ are related as

$$s_1=\sqrt{3}\,\alpha\tau \quad , \quad s_2=\sqrt{3}\,\beta\tau \tag{8}$$

Since the right-hand side of Eq. 7 contains a double integral term, it is inconvenient to calculate numerically.
The double integral can be replaced by single integral and differential terms, with the aid of Taylor expansion, as follows:

$$\phi(x,z,t+\tau)=\frac{1}{2s_1}\left\{\int_{-s_1}^{s_1}\phi(x+\xi,z,t)\,d\xi+\int_{-s_1}^{s_1}\phi(x,z+\zeta,t)\,d\zeta\right\}$$
$$+\frac{s_1^2}{6}\left(\frac{\partial^2\phi(x,z,t)}{\partial x^2}+\frac{\partial^2\phi(x,z,t)}{\partial z^2}\right)-\phi(x,z,t-\tau)$$
$$+O(s_1^4)+O(\tau^4) \tag{9}$$
$$\psi(x,z,t+\tau)=\frac{1}{2s_2}\left\{\int_{-s_2}^{s_2}\psi(x+\xi,z,t)\,d\xi+\int_{-s_2}^{s_2}\psi(x,z+\zeta,t)\,d\zeta\right\}$$
$$+\frac{s_2^2}{6}\left(\frac{\partial^2\psi(x,z,t)}{\partial x^2}+\frac{\partial^2\psi(x,\bar{z},t)}{\partial z^2}\right)-\psi(x,z,t-\tau)$$
$$+O(s_2^4)+O(\tau^4)$$

Applying Simpson's 1/3 rule and numerical differentiation by **Collatz** to the first term and the second term respectively in right-hand side of Eq. 9, the following equations are obtained as below:

$$\phi(x,z,t+\tau)=\frac{1}{3}\{\phi(x,z+s_1,t)+\phi(x-s_1,z,t)+2\phi(x,z,t)$$
$$+\phi(x+s_1,z,t)+\phi(x,z-s_1,t)\}-\phi(x,z,t-\tau)$$
$$+O(s_1^4)+O(\tau^4) \tag{10}$$
$$\psi(x,z,t+\tau)=\frac{1}{3}\{\psi(x,z+s_2,t)+\psi(x-s_2,z,t)+2\psi(x,z,t)$$
$$+\psi(x+s_2,z,t)+\psi(x,z-s_2,t)\}-\psi(x,z,t-\tau)$$
$$+O(s_2^4)+O(\tau^4)$$

As shown in Eq. 8, the integral intervals s_1 and s_2 are different because in general α is not equal to β. Thus it becomes complicated to evaluate stresses and displacements. To avoid this complication, Taylor expansion is used for the numerical reduction of the wave equations. $\phi(x,z,t+\tau)$ and $\phi(x,z,t-\tau)$ can be expressed by using Taylor expansion:

$$\phi(x,z,t+\tau) = \phi(x,z,t) + \tau\frac{\partial\phi(x,z,t)}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2\phi(x,z,t)}{\partial t^2} + \frac{\tau^3}{6}\frac{\partial^3\phi(x,z,t)}{\partial t^3} + O(\tau^4) \quad (11)$$

$$\phi(x,z,t-\tau) = \phi(x,z,t) - \tau\frac{\partial\phi(x,z,t)}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2\phi(x,z,t)}{\partial t^2} - \frac{\tau^3}{6}\frac{\partial^3\phi(x,z,t)}{\partial t^3} + O(\tau^4) \quad (12)$$

Combining Eqs. 11 and 12, $\phi(x,z,t+\tau)$ is expressed as follows:

$$\phi(x,z,t+\tau) = 2\,\phi(x,z,t) + \tau^2\frac{\partial^2\phi(x,z,t)}{\partial t^2} - \phi(x,z,t-\tau) + O(\tau^4) \quad (13)$$

Substituting the wave equation, Eq. 4 into Eq. 13 and using Collatz differentiation, $\phi(x,z,t+\tau)$ can be described in the form similar to Eq. 10:

$$\phi(x,z,t+\tau) = \frac{1}{P_1}\{\phi(x,z-s_1,t)+\phi(x-s_1,z,t)-4\phi(x,z,t) + \phi(x+s_1,z,t) + \phi(x,z+s_1,t)\} + 2\phi(x,z,t) - \phi(x,z,t-\tau) + O(s_1^4) + O(\tau^4) \quad (14)$$

where $P_1 = (s_1/\alpha\tau)^2$. Replacing ϕ, s_1 and P_1 by ψ, s_2 and P_2, $\psi(x,z,t+\tau)$ is obtained, provided that $P_2 = (s_2/\beta\tau)^2$.

Since s_1 equals to s_2 when P_2 equals to $(\alpha/\beta)^2 P_1$, $\phi(x,z,t+\tau)$ and $\psi(x,z,t+\tau)$ are expressed with the same integral interval s:

$$\phi(x,z,t+\tau) = \frac{1}{P}\{\phi(x,z-s,t) + \phi(x-s,z,t) - 4\phi(x,z,t) + \phi(x+s,z,t) + \phi(x,z+s,t)\} + 2\,\phi(x,z,t) - \phi(x,z,t-\tau) + O(s^4) + O(\tau^4)$$

$$\psi(x,z,t+\tau) = \left(\frac{\beta}{\alpha}\right)^2\frac{1}{P}\{\psi(x,z-s,t)+\psi(x-s,z,t)-4\psi(x,z,t) + \psi(x+s,z,t)+\psi(x,z+s,t)\} + 2\psi(x,z,t) - \psi(x,z,t-\tau) + O(s^4) + O(\tau^4) \quad (15)$$

where

$$s = s_1 = s_2\ ,\ P = P_1 = \left(\frac{\beta}{\alpha}\right)^2 P_2 \quad (16)$$

NUMERICAL DIFFERENTIAL EXPRESSIONS FOR BOUNDARY CONDITIONS

The boundary conditions for the dynamic problems of the structure-soil systems are expressed both by displacements at the structure-soil interface and by stresses at the surface of the soil.

Displacement boundary. There are two portions in the structure-soil interface. One is along the base line of the structure, and the other is along the side line of the structure. Each portions are denoted as Db and Ds respectively in Fig. 1. The numerical differential expressions of the prescribed displacements $\bar{U}_x(x,z,t)$ and $\bar{U}_z,(x,z,t)$ in Db or Ds are written from Eq. 3 as follows:

$$\bar{U}x(x,z,t)=\frac{1}{2s}\{\phi(x+s,z,t)-\phi(x-s,z,t)-\psi(x,z+s,t)+\psi(x,z-s,t)\}+O(s^2)$$
$$\bar{U}z(x,z,t)=\frac{1}{2s}\{\psi(x+s,z,t)-\psi(x-s,z,t)+\phi(x,z+s,t)-\phi(x,z-s,t)\}+O(s^2) \tag{17}$$

In the interface Db, the virtual potentials $\phi(x,z-s,t)$, $\psi(x,z-s,t)$ which are outside the soil region are calculated by the prescribed displacements and the potentials in the soil region determined from Eqs. 13 and 14. In the interface Ds, the virtual potential $\phi(x-s,z,t)$, $\psi(x-s,z,t)$ are calculated in the same manner.

Stress boundary. The boudary condition at the surface of the soil region is described as follows:

$$\sigma zz(x,o,t) = 0, \quad \sigma xz(x,o,t) = 0 \tag{18}$$

The numerical differential expressions of the prescribed stress $\sigma zz\ (x,o,t)$ and $\sigma xz\ (x,o,t)$ are written from Eq. 6 as below:

$$\sigma zz(x,o,t)=\frac{\lambda}{s^2}\{\phi(x,-s,t)+\phi(x-s,o,t)-4\phi(x,o,t)+\phi(x+s,o,t)+\phi(x,s,t)\}+\frac{\mu}{2s^2}\{4\phi(x-s,o,t)-8\phi(x,o,t)+4\phi(x+s,o,t)-\psi(x-s,s,t)+\psi(x+s,s,t)+\psi(x-s,-s,t)-\psi(x+s,-s,t)\}+O(s^2)$$
$$\sigma xz(x,o,t)=\frac{\mu}{2s^2}\{\phi(x-s,-s,t)-\phi(x+s,-s,t)-\phi(x-s,s,t)+\phi(x+s,s,t)+2(-\psi(x,-s,t)+\psi(x-s,o,t)+\psi(x+s,o,t)-\psi(x,s,t))\}+O(s^2) \tag{19}$$

Since Eq. 19 contains six virtual potentials $\phi(x-s,-s,t)$, $\psi(x-s,-s,t)$, etc., it is impossible to obtain all of them directly. The virtual potentials $\phi(x,-s,t)$, $\psi(x,-s,t)$ are determined approximately using other virtual potentials at time $(t-\tau)$ instead of time (t).

The calculation of the dynamic response of the structure-soil systems is performed as follows: (i) Determine the virtual potentials at the structure-soil interface from Eq. 17. (ii) Determine the potential functions in the soil region from Eqs. 15 and 16. (iii) Determine the virtual potentials at the surface of the soil from Eq. 19. (iv) Determine the virtual potentials at the structure-soil interface from Eq. 17. (v) Obtain displacements and stresses in the soil region from Eqs. 3 and 6. (vi) Repeat the operation from (ii) through (v).

EXAMPLE PROBLEM

The structure-soil system for an embedded foundation and parameters used in the exaimple problem are presented in Fig. 1. Forced displacement on the foundation is rotation at the center of the foundation base line. Two types of forced displacement are used. One is static type (case 1) and the other is harmonic vibration type (case 2). Forced rotational angle Θ of the foundation base line is prescribed as below:

$$\Theta = 0.5 \text{ rad.} \qquad \text{(case 1)}$$
$$\Theta = 0.5 \sin\left(\frac{\pi}{10\tau} t\right) \text{ rad.} \qquad \text{(case 2)} \qquad (20)$$

Calculated displacement distributions of the structure-soil system are presented in Fig. 2 (case 1) and Fig. 3 (case 2).
The time histories of the normal stress distributions at the structure-soil interface in the case 2 are shown in Figs. 4 and 5. The time histories in Figs. 4 and 5 are of σ_{zz} at the foundation base line and of σ_{xx} at the foundation side line, respectively. In both figures displacements of the foundation are indicated by the dashed lines. Since the time length calculated in the case 2 is short, vibration of the foundation is not yet stationary. It is indicated, however, by Fig. 4 that the stress distributions at the base line become linear as the time passes. And the phase lag between displacements and stresses is observed apparently.

CONCLUSION

Analytical expressions and numerical examples have been presented for a method for dynamic response analysis of the foundation as the structure-soil systems. The dynamic responses of the foundation were calculated by solving the wave equations and the mixed boundary value problem which were simply expressed in terms of the potential functions ϕ and ψ. This paper proposes a method for the evaluation of the dynamic characteristics of the foundation, and the numerical examples used here show only an application of this method. Further studies on the dynamic characteristics of the foundation should be conducted by using various parameters and longer time length for the dynamic response calculation.

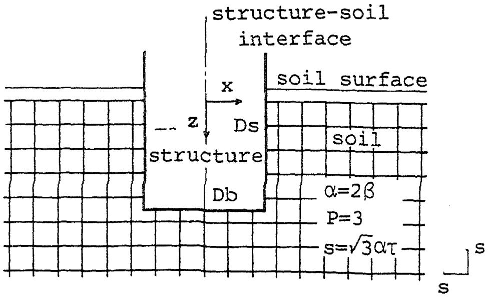

Fig. 1 Structure-soil system

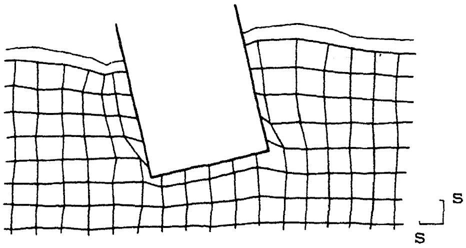

Fig. 2 Displacement distributions (case 1) t = 25τ

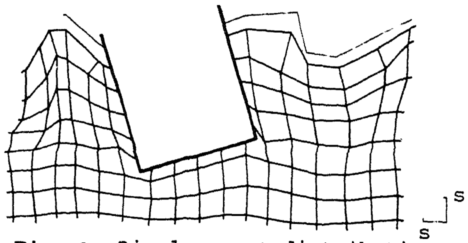

Fig. 3 Displacement distributions (case 2) t = 29τ

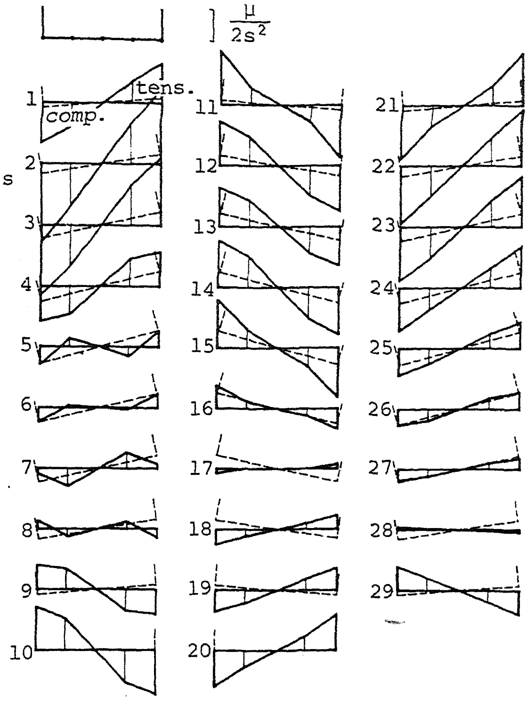

Fig. 4 Time history of σ_{zz} at base line

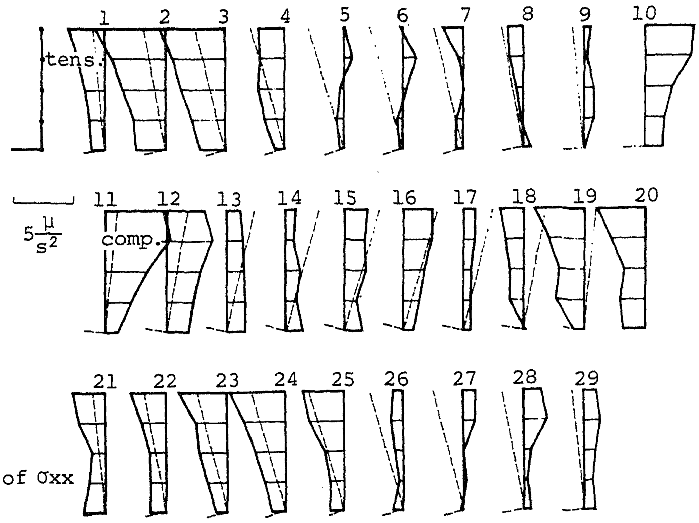

Fig. 5 Time history of σ_{xx} at side line

REFERENCES

1. Toriumi, I., Sato, Y. and Yamaguchi, R.: Vibration in Foundation, Structure and its Vicinity on the Elastic Ground, Proceedings of the Second World Conference on Eathquake Engineering. 1960, pp. 1413-1429.
2. Thomson, W.T., Kobori, T.: Dynamical Compliance of Rectangular Foundations on an Elastic Half-Space, Journal of Applied Mechanics, Trans. ASME, Vol. 30, Series E, No.4, 1963, pp. 579-584.
3. Seed, H.B., Lysmer, J., and Hwang, R.: Soil-Structure Interaction Analyses for Seismic Response, Journal of the Geotechnical Engineering Division, ASCE, Vol. 101, No.GT5, 1975, pp. 439-457.
4. Hidaka, K.: Numerical Integration, Iwanami-Shoten Co., Ltd., 1943, pp. 274-283, (in Japanese).