A STIFFNESS MATRIX APPROACH FOR LAYERED SOILS

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SUMMARY

The Haskell-Thompson transfer matrix method for layered media is used to derive layer stiffness matrices which may be applied in much the same way as stiffness matrices in conventional structural analysis. The exact expressions are given for the matrices, and an approximation for thin layers is presented and illustrated with common applications in geotechnical engineering.

INTRODUCTION

The determination of the response of a soil deposit to dynamic loads, caused either by a seismic excitation or by prescribed forces at some location in the soil mass, falls mathematically into the area of wave propagation theory. The formalism to study the propagation of waves in layered media was presented by Thomson (2) and Haskell (1) more than 25 years ago, and it is based on the use of transfer matrices in the frequency—wave number domain. The solution technique for arbitrary loadings necessitates resolving the loads in terms of their temporal and spatial Fourier transforms, so that they are assumed to be harmonic in time and space. This corresponds formally to the use of the method of separation of variables to find solutions to the wave equation.

In the case of a unit, harmonic point load in the three-dimensional space (line load in the two-dimensional space) the associated displacement field in the soil defines the so-called Green functions. These functions are necessary in certain formulations of geotechnical problems, and they embody all the fundamental dynamic properties of the soil system. Since the two-dimensional Fourier transform of a point load $\delta(x,y)$ (in Cartesian coordinates) or the Hankel transform of a point load $\delta(r)$ (in cylindrical coordinates) have constant spectral amplitude in the wave number domain, it is possible to find the Green functions by inverse Fourier/Hankel transformations of the harmonic displacements. Closed-form solutions are then found for simple cases by contour integration, while numerical solutions are needed for arbitrarily layered soils. The details of the procedures are well known, and need not be repeated here.

The first step in the computation for dynamic loads is then to find the harmonic displacements at the layer interfaces due to these unit harmonic loads. In the transfer matrix approach, the (harmonic) displacements and internal stresses at a given interface define the state vector, which is related through the transfer matrix to the state vectors at neighboring interfaces.

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State Vectors. Consider a layered soil system as shown in Fig. 1. The interfaces are dictated by discontinuities in material properties in the vertical direction, or by the presence of external loads at a given elevation. We define then the state vectors

$$\mathbf{Z} = \left\{ \bar{u}_x,\ \bar{u}_y,\ i\bar{u}_z,\ \bar{\tau}_{xz},\ \bar{\tau}_{yz},\ i\bar{\sigma}_z \right\}^T = \begin{Bmatrix} \bar{\mathbf{U}} \\ \bar{\mathbf{S}} \end{Bmatrix} \tag{1}$$

for Cartesian coordinates, or

$$\mathbf{Z} = \left\{ \bar{u}_\rho,\ \bar{u}_\theta,\ \bar{u}_z,\ \bar{\tau}_{\rho z},\ \bar{\tau}_{\theta z},\ \bar{\sigma}_z \right\}^T = \begin{Bmatrix} \bar{\mathbf{U}} \\ \bar{\mathbf{S}} \end{Bmatrix} \tag{2}$$

for cylindrical coordinates. In these expressions, $\bar{u}$, $\bar{\tau}$, $\bar{\sigma}$ are the displacement, shearing stress and normal stress components at a given elevation in the direction identified by the subindex; and T stands for the transposed vector. The factor $i = \sqrt{-1}$ has been introduced for $\bar{u}_z$, $\bar{\sigma}_z$ in the Cartesian coordinates case for reasons of convenience. The superscript bar, on the other hand, is a reminder that the displacement vector $\bar{\mathbf{U}}$ and stress vector $\bar{\mathbf{S}}$ are functions of z only, that is, it is assumed that the variation of displacements and stresses in the horizontal plane is harmonic.

For Cartesian coordinates, the actual displacements at a point are obtained by multiplying $\bar{\mathbf{U}}$, $\bar{\mathbf{S}}$, by the factor $\exp i(\omega t - kx - \ell y)$, i.e.,

$$\begin{Bmatrix} \mathbf{U} \\ \mathbf{S} \end{Bmatrix} = \begin{Bmatrix} \bar{\mathbf{U}} \\ \bar{\mathbf{S}} \end{Bmatrix} \exp i(\omega t - kx - \ell y) \tag{3}$$

in which ω = frequency of excitation, and k, ℓ are the wavenumbers. If we restrict our attention to a plane strain condition (i.e., plane waves), it follows that $\ell = 0$ and the factor becomes simply $\exp i(\omega t - kx)$.

For cylindrical coordinates, on the other hand, the variation of displacements in the azimuthal direction is obtained by multiplying $\bar{u}_\rho$, $\bar{u}_z$, $\bar{\tau}_{\rho z}$, $\bar{\sigma}_z$ and $\bar{u}_\theta$, $\bar{\tau}_{\theta z}$ by $\cos \mu\theta$ and $-\sin \mu\theta$, (or by $\sin \mu\theta$ and $\cos \mu\theta$), respectively, with $\mu = 0, 1, 2, \ldots$ being an integer. The variation in the radial direction is obtained multiplying $\bar{\mathbf{U}}$, $\bar{\mathbf{S}}$ by the matrix $\mathbf{C}$ (which is common to all layers):

$$\begin{Bmatrix} \mathbf{U} \\ \mathbf{S} \end{Bmatrix}_\mu = \begin{Bmatrix} \mathbf{C}\bar{\mathbf{U}} \\ \mathbf{C}\bar{\mathbf{S}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{C} & \\ & \mathbf{C} \end{Bmatrix} \begin{Bmatrix} \bar{\mathbf{U}} \\ \bar{\mathbf{S}} \end{Bmatrix} \tag{4}$$

$$\mathbf{C} = -\begin{Bmatrix} \frac{1}{k}\frac{d}{d\rho} C_\mu & \frac{\mu}{k\rho} C_\mu & \\ \frac{\mu}{k\rho} C_\mu & \frac{1}{k}\frac{d}{d\rho} C_\mu & \\ & & -C_\mu \end{Bmatrix} \tag{5}$$

in which $C_\mu = C_\mu(k\rho)$ are cylindrical functions of μ^{th} order and first, second or third kind (Bessel, Neumann or Hankel functions, respectively). The argument k is the wave number. This corresponds to the well-known decomposition of the displacements in a Fourier series in the azimuthal direction, and cylindrical functions in the radial direction. The variation with time is given again by the factor exp $i\omega t$.

In the transfer matrix method, the state vector at the $(j+1)^{th}$ interface is related to that at the preceding one by the expression

$$\mathbf{Z}_{j+1} = \mathbf{T}_j \mathbf{Z}_j \tag{6}$$

where $\mathbf{T}_j$ is the transfer matrix of the j^{th} layer. It is a function of the frequency of excitation ω, the wavenumbers k, ℓ, the soil properties, and the thickness of the layer. Again in the particular case of plane waves (plane strain), the second wavenumber (ℓ) is zero. The transfer matrix has then a structure such that motions in a vertical plane (SV-P waves) uncouple from motions in a horizontal plane (SH waves). It is a remakable fact that the transfer matrix for cylindrical coordinates is identical to that of the plane strain case, and is independent of the Fourier index μ. This implies, among other things, that the solution for point loads can be derived, in principle, from the solution for the three line load cases of the plane strain case. (This is referred to as the inversion of the descent of dimensions).

Stiffness Matrix Approach. Referring to Fig. 1, we isolate a specific layer and preserve equilibrium by application of external loads $\bar{\mathbf{P}}_1 = \bar{\mathbf{S}}_1$ at the upper interface, and $\bar{\mathbf{P}}_2 = -\bar{\mathbf{S}}_2$ at the lower interface. From Eq. 6 we have

$$\begin{Bmatrix} \bar{\mathbf{U}}_2 \\ -\bar{\mathbf{P}}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{Bmatrix} \begin{Bmatrix} \bar{\mathbf{U}}_1 \\ \bar{\mathbf{P}}_1 \end{Bmatrix} \tag{7}$$

where $\mathbf{T}_{ij}$ are submatrices of the transfer matrix $\mathbf{T}_j$. After some straightforward matrix algebra, we obtain

$$\begin{Bmatrix} \bar{\mathbf{P}}_1 \\ \bar{\mathbf{P}}_2 \end{Bmatrix} = \begin{Bmatrix} -\mathbf{T}_{12}^{-1}\mathbf{T}_{11} & \mathbf{T}_{12}^{-1} \\ \mathbf{T}_{22}\mathbf{T}_{12}^{-1}\mathbf{T}_{11} - \mathbf{T}_{21} & -\mathbf{T}_{22}\mathbf{T}_{12}^{-1} \end{Bmatrix} \begin{Bmatrix} \bar{\mathbf{U}}_1 \\ \bar{\mathbf{U}}_2 \end{Bmatrix} \tag{8}$$

or briefly

$$\bar{\mathbf{P}} = \mathbf{K}\,\bar{\mathbf{U}} \tag{9}$$

where $\mathbf{K}$ = stiffness matrix of the layer; $\bar{\mathbf{P}}$ = "load" vector; and $\bar{\mathbf{U}}$ = displacement vector. A rather tedious analysis will demonstrate that it is symmetric. The matrices of the individual layers are used in the same way as in conventional structural analysis, assembling the global stiffness matrix by overlapping the contribution of the layer matrices at each "node" (interface) of the system. The loading vector corresponds in this case to prescribed external stresses at the interfaces.

Stiffness Matrices (Plane-Waves and Cylindrical Waves)

a) Exact Solution. Due to space limitations, the details of the formulation will be omitted and only the final results will be given. The elements of the stiffness matrix were obtained solving the wave equation in Cartesian and cylindrical coördinates (Fig. 1). The elements of the layer stiffness matrices are given in tables 1, 2. The following notation is used:

ω = frequency of excitation
k = wave number
h = layer thickness
G = shear modulus

$\alpha = C_s/C_p$ = shear wave velocity/dilatational wave velocity

$r = \sqrt{1 - (\omega/kC_p)^2}$

$$s = \sqrt{1 - (\omega/kC_s)^2}$$

To save space, the results are given in partitioned form: First the matrices for SV-P waves (representing rows/columns 1,3,4,6) (Table 1) and then the matrices for SH waves (representing rows/columns 2,5) (Table 2).

b. Discrete Solution. If the layer thicknesses are small as compared to the wavelengths of interest, it is possible to linearize the transcendental functions which govern the displacements in the vertical direction. This procedure was presented by Waas and Lysmer in a different context, and is restricted to layered strāta over rigid rock only. The principal advantage of the method is the substitution of algebraic expressions in place of the more involved transcendental functions. Hence, the eigenvalue problems for the natural modes of wave propagation are algebraic, and may be solved by standard techniques. The layer stiffness matrices in the discrete case may be obtained as

$$\mathbf{K} = \mathbf{A}k^2 + \mathbf{B}k + \mathbf{G} - \omega^2\mathbf{M} ,$$

where k = wave number, ω = frequency of excitation; and **A**, **B**, **G**, **M** are the matrices given in table 3.

Example of Application. Consider the layered soil over rigid rock shown in Fig. 1. The equilibrium problem is then

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \mathbf{K}_{23} \\ & \mathbf{K}_{32} & \mathbf{K}_{33} \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{U}}_1 \\ \bar{\mathbf{U}}_2 \\ \bar{\mathbf{U}}_3 \end{Bmatrix} = \begin{Bmatrix} \bar{\mathbf{P}}_1 \\ \bar{\mathbf{P}}_2 \\ \bar{\mathbf{P}}_3 \end{Bmatrix}$$

where the $\mathbf{K}_{ij}$ are obtained, overlapping corresponding elements. The stiffness matrix is tridiagonal. Bedrock is held fixed ($\bar{\mathbf{U}}_4 = \mathbf{0}$).

For given loads (stresses) $\bar{\mathbf{P}}_j$ at the interfaces, we can solve for the displacements $\bar{\mathbf{U}}_j$ by Gaussian elimination. As an example of the above, the Green functions for a horizontal line load at the free surface would be obtained solving for $\bar{\mathbf{U}}_j$ with a loading $\bar{\mathbf{P}}_1 = (1,0,0)^T$, $\bar{\mathbf{P}}_2 = \bar{\mathbf{P}}_3 = \mathbf{0}$, as a function of k, and computing the inverse Fourier Transform (for instance, with the FFT algorithm).

Natural modes of wave propagation on the other hand are obtained from the eigenvalue problem resulting from setting the right-hand side to zero (a quadratic eigenvalue problem in k for the discrete case).

Finally, a wave amplification problem is obtained when prescribing an amplitude of motion $\bar{\mathbf{U}}_4$ at (rigid) bedrock (which is equivalent to prescribing a loading $\bar{\mathbf{P}}_3 = -\mathbf{K}_{34}\,\bar{\mathbf{U}}_4$ at the third interface). The associated wave-number in this case is related to the direction of propagation of the incident wave through bedrock. For instance, for vertical incidence, k=0.

In all these problems, advantage should be taken of the uncoupling of the equations implied by the structure of the layer matrices. For elastic bedrock, the stiffnesses of the halfspace (tables 1, 2) should be used, either using the exact solution throughout, or using the exact solution only for the halfspace (a hybrid formulation).

Advantages over Transfer Matrix Method. The following advantages over the transfer matrix method can be observed:

a) stiffness matrices are symmetric (less storage required);
b) number of operations is reduced (Gaussian elimination of symmetric matrix requires much fewer operations than product of non-symmetric transfer matrices;
c) treatment of interfaces at which loadings are applied is simpler. This implies an easier coding, particularly if solving simultaneously for various loadings;
d) greater familiarity of the engineers with stiffness matrices than with transfer matrices.

ACKNOWLEDGEMENTS

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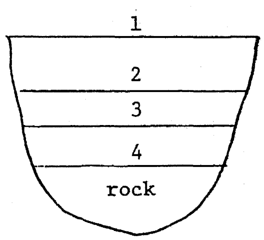

Fig. 1

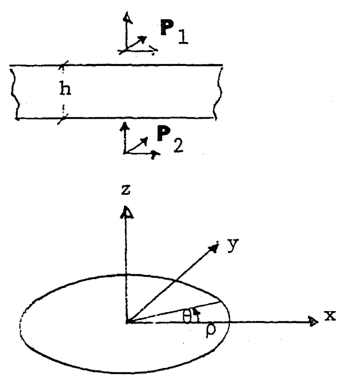

TABLE 1 - SV and P Waves

a) Stiffness matrix for non-zero frequency, non-zero wave number:

$$\omega \neq 0 \qquad k \neq 0$$

$$C^r = \cosh krh \qquad S^r = \sinh krh \qquad \mathbf{K} = 2kG \begin{Bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{Bmatrix}$$

$$C^s = \cosh ksh \qquad S^s = \sinh ksh$$

$$D = 2(1 - C^r C^s) + (\frac{1}{rs} + rs)\, S^r S^s$$

$$\mathbf{K}_{11} = \frac{1-s^2}{2D} \begin{Bmatrix} \frac{1}{s}(C^r S^s - rs\, C^s S^r) & -(1 - C^r C^s + rs\, S^r S^s) \\ -(1 - C^r C^s + rs\, S^r S^s) & \frac{1}{r}(C^s S^r - rs\, C^r S^s) \end{Bmatrix} - \frac{1+s^2}{2} \begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix}$$

$$\mathbf{K}_{12} = \frac{1-s^2}{2D} \begin{Bmatrix} \frac{1}{s}(rs\, S^r - S^s) & -(C^r - C^s) \\ C^r - C^s & \frac{1}{r}(rs\, S^s - S^r) \end{Bmatrix}$$

$\mathbf{K}_{22}$ = same as $\mathbf{K}_{11}$, with off-diagonal signs changed; $\mathbf{K}_{21} = \mathbf{K}_{12}^T$

b) Stiffness matrix for zero frequency, non-zero wave number:

$$\omega = 0 \qquad k \neq 0 \qquad \kappa = kh$$

$$C = \cosh kh \qquad S = \sinh kh \qquad \mathbf{K} = 2kG \begin{Bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{Bmatrix}$$

$$D = (1 + \alpha^2)^2 S^2 - \kappa^2 (1 - \alpha^2)^2$$

$$\mathbf{K}_{11} = \frac{1}{D} \begin{Bmatrix} (1+\alpha^2)SC - \kappa(1-\alpha^2) & (1+\alpha^2)S^2 \\ (1+\alpha^2)S^2 & (1+\alpha^2)SC + \kappa(1-\alpha^2) \end{Bmatrix} - \begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix}$$

$$\mathbf{K}_{12} = \frac{1}{D} \begin{Bmatrix} \kappa(1-\alpha^2)C - (1+\alpha^2)S & -\kappa(1-\alpha^2)S \\ \kappa(1-\alpha^2)S & -(\kappa(1-\alpha^2)C + (1+\alpha^2)S) \end{Bmatrix}$$

$\mathbf{K}_{22}$ = same as $\mathbf{K}_{11}$, with off-diagonal signs changed; $\mathbf{K}_{21} = \mathbf{K}_{12}^T$

c) Stiffness matrix for non-zero frequency, and zero wave number:

$$\omega \neq 0 \qquad k = 0 \qquad \eta = \omega h / C_s \qquad \alpha\eta = \omega h / C_p$$

$$\mathbf{K} = \rho\, C_s\, \omega \begin{Bmatrix} \cot\eta & & -\frac{1}{\sin\eta} & \\ & \frac{1}{\alpha}\cot\alpha\eta & & -\frac{1}{\alpha}\frac{1}{\sin\alpha\eta} \\ -\frac{1}{\sin\eta} & & \cot\eta & \\ & -\frac{1}{\alpha}\frac{1}{\sin\alpha\eta} & & \frac{1}{\alpha}\cot\alpha\eta \end{Bmatrix}$$

d) <u>Stiffness matrix for zero frequency, and zero wavenumber</u> ($\omega = 0$; $k = 0$)

$$\mathbf{K} = \frac{G}{h}\begin{Bmatrix} 1 & & -1 & \\ & \frac{1}{\alpha^2} & & -\frac{1}{\alpha^2} \\ -1 & & 1 & \\ & -\frac{1}{\alpha^2} & & \frac{1}{\alpha^2} \end{Bmatrix}$$

e) <u>SV+P waves for a halfspace</u> ($h = \infty$)

$$\mathbf{K} = 2kG\left[\frac{1-s^2}{2(1-rs)}\begin{Bmatrix} r & 1 \\ 1 & s \end{Bmatrix} - \begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix}\right]; \quad \mathbf{K} = \frac{2kG}{1+\alpha^2}\begin{Bmatrix} 1 & -\alpha^2 \\ -\alpha^2 & 1 \end{Bmatrix}$$

$(k \neq 0 \quad \omega \neq 0)$ $\qquad$ $(k \neq 0 \quad \omega = 0)$

$$\mathbf{K} = i\omega\theta C_s \begin{Bmatrix} 1 & \\ & \frac{1}{\alpha} \end{Bmatrix} (k = 0,\ \omega \neq 0): \quad \mathbf{K} = \mathbf{0} \text{ (the null matrix)} \; (k = 0,\ \omega = 0)$$

<u>TABLE 2 - SH Waves</u>

$$\mathbf{K} = \frac{ksG}{\sinh ksh}\begin{Bmatrix} \cosh ksh & -1 \\ -1 & \cosh ksh \end{Bmatrix} \quad (k \neq 0;\ \text{for } \omega = 0,\ \text{substitute } s = 1)$$

$$\mathbf{K} = \frac{\rho C_s\,\omega}{\sin\eta}\begin{Bmatrix} \cos\eta & -1 \\ -1 & \cos\eta \end{Bmatrix}, \quad \eta = \frac{\omega h}{C_s} \neq 0; \quad k = 0; \quad \mathbf{K} = \mathbf{0} \text{ (the null matrix)} \; (k = 0,\ \omega = 0).$$

<u>SH waves in a halfspace</u>:

$$K = \begin{cases} ksG & \text{for } k \neq 0,\ \omega \neq 0 \\ kG & \text{for } k \neq 0,\ \omega = 0 \\ i\omega\rho C_s & \text{for } k = 0,\ \omega \neq 0 \\ 0 & \text{for } k = 0,\ \omega = 0 \end{cases}$$

$$\mathbf{A} = \frac{h}{6}\begin{Bmatrix} 2(\lambda+2G) & \cdot & \cdot & \lambda+2G & \cdot & \cdot \\ \cdot & 2G & \cdot & \cdot & G & \cdot \\ \cdot & \cdot & 2G & \cdot & \cdot & G \\ \lambda+2G & \cdot & \cdot & 2(\lambda+2G) & \cdot & \cdot \\ \cdot & G & \cdot & \cdot & 2G & \cdot \\ \cdot & \cdot & G & \cdot & \cdot & 2G \end{Bmatrix} \qquad \mathbf{B} = \frac{1}{2}\begin{Bmatrix} \cdot & \cdot & \lambda-G & \cdot & \cdot & -(\lambda+G) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda-G & \cdot & \cdot & \lambda+G & \cdot & \cdot \\ \cdot & \cdot & \lambda+G & \cdot & \cdot & -(\lambda-G) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -(\lambda+G) & \cdot & \cdot & -(\lambda-G) & \cdot & \cdot \end{Bmatrix}$$

$$\mathbf{G} = \frac{1}{h}\begin{Bmatrix} G & \cdot & \cdot & -G & \cdot & \cdot \\ \cdot & G & \cdot & \cdot & -G & \cdot \\ \cdot & \cdot & \lambda+2G & \cdot & \cdot & -(\lambda+2G) \\ -G & \cdot & \cdot & G & \cdot & \cdot \\ \cdot & -G & \cdot & \cdot & G & \cdot \\ \cdot & \cdot & -(\lambda+2G) & \cdot & \cdot & \lambda+2G \end{Bmatrix} \qquad \mathbf{M} = \frac{\rho h}{6}\begin{Bmatrix} 2 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 2 \end{Bmatrix}$$

λ = Lame constant, G = shear modulus — for soil with damping, use complex values.
ρ = mass density
h = layer thickness

$$\mathbf{K} = \mathbf{A}k^2 + \mathbf{B}k + \mathbf{G} - \omega^2\mathbf{M}$$

TABLE 3