LIMIT ANALYSIS OF SEISMIC SLOPE STABILITY

S. W. Chan[1], W. F. Chen[2] and S. L. Koh[3]

SUMMARY

Earlier work on the limit analysis of slope stability under seismic loads (Chen et al., 1978) is reformulated to include the case of general body force distribution allowing variation of both gravitational loads as well as seismic loads throughout the height of the slope. It is shown that the upper bound method of limit analysis of perfect plasticity using the logspiral failure mechanism yields results comparable to published results for more restricted cases.

INTRODUCTION

Limit analysis techniques have been successfully applied to the study of the stability of slopes. Among the first to use this method, Drucker and Prager (1952) determined the lower and upper bounds of the collapse load for a given slope subjected solely to gravitational effects. The lower bound is determined by assuming a stress field which satisfies equilibrium conditions but does not violate the yield criterion. The upper bound of the collapse load is determined with the application of the flow rule in which the rate of work of external forces is not less than the rate of internal energy dissipation. Improved upper bound values were obtained by Chen, Giger and Fang (1969) with the assumption of a rotational logarithmic spiral mechanism for failure. A detailed discussion of this approach is given by Chen (1975).

In these early studies, only the static case of slope stability was treated. The slope material was considered homogeneous with constant specific weight. The collapse load was taken to be entirely due to the weight of the slope material. More recently, Chen, Chang and Yao (1978) extended the application of the upper bound method of limit analysis of slope stability to include horizontal pseudo-static forces simulating seismic loads. Assuming that the failure mechanism is that of a rotational logarithmic spiral, they obtained values for the critical height where the seismic coefficient is taken to be constant throughout the slope or constant along a finite number of layers of the slope (at most four).

In the present study, we are interested in the more general case where the slope material is nonhomogeneous with the seismic coefficient varying through the height of the slope. The upper bound method of limit analysis is also used in this study. In the next section, we show that the critical failure mechanism for this more general formulation is still the logarithmic spiral. The analysis is then presented for the case where both the vertical and horizontal components of the body force, simulating varying specific weight and seismic loads, may be expressed as power functions of the elevation of the material point. Numerical data obtained from this new formulation are then compared with previously published results.

1 Graduate Research Assistant, Purdue University, W. Lafayette, IN, U.S.A.
2 Professor of Civil Engineering, Purdue University, same address.
3 Professor of Mechanical Engineering, Purdue University, same address.

FAILURE MECHANISM

As with previous studies, we consider the soil to satisfy the Coulomb yield criterion and its associated flow rule. (Cf. Chen, 1975.) The shearing resistance of the material is characterized by two material constants: cohesion c, and internal friction angle ϕ. A basic assumption of the analysis considers the body forces to vary only with the elevation of the material point. Thus, these forces are treated as constant throughout an infinitesimal soil layer at a given height. These forces may then be summed over each infinitesimally small layer and replaced by a statically equivalent concentrated force acting at the center of gravity of that layer. In this study other forms of nonhomogeneity are ignored. Pore water pressure and anisotropy are similarly neglected.

It has been shown, e.g. by Chen (1975), that for the static case, the critical failure mechanism for the slope stability is the logspiral:

$$r = r_o \, e^{(\theta - \theta_o)\tan\phi} \tag{1}$$

where the geometrical parameters r and θ are as shown in Fig. 1. For the general problem under consideration here, we reanalyze the problem incorporating our assumption of nonhomogeneous body forces.

The soil is treated as perfectly plastic, limiting the failure to a rigid body sliding of the upper portion of the slope along a slip surface. The critical failure mechanism is one which would require the minimum external load to induce failure. We assume that the resultant body force acting on each layer element may be expressed in terms of horizontal and vertical components, dF_x and dF_y, with force component profiles given by:

$$F_x(h) = \gamma\, K_x(h) = \gamma\, [a_o + a_1 h + a_2 h^2 + \ldots + a_m h^m] \tag{2}$$

$$F_y(h) = \gamma\, K_y(h) = \gamma\, [b_o + b_1 h + b_2 h^2 + \ldots + b_n h^n] \tag{3}$$

Here h is the elevation of the element above the toe T as shown in Fig. 2, γ is the specific weight, and K_x and K_y are the loading coefficients. The applied load on the slope may be calculated by integrating the elemental body force components through the height of the slope:

$$W = \int_H [(dF_x)^2 + (dF_y)^2]^{1/2} = \int_H \gamma\, \xi\, (K_x^2 + K_y^2)^{1/2}\, dh \tag{4}$$

where $\xi = \xi(h)$ is the breadth of the element dh.

Because of the break in the slope at the "knee", point D, different expressions for the breadth ξ will be required, and two sets of integration will have to be performed. For simplicity, we consider an equivalent slope, with the upper secondary slope leveled off as shown by the dotted line in Fig. 2. The elevation of this horizontal surface is now H^* such that Region I will compensate for the removed Region II. The integration indicated in Eq. 4 may then be expressed as

$$W = \int_{\theta_o^*}^{\theta_h} \gamma\, \xi(\theta, r, \beta)(K_x^2 + K_y^2)^{1/2} f(\theta, r, \zeta)\, d\theta \tag{5}$$

with K_x and K_y expressed in terms of r and θ, and ξ expressed as:

$$\xi = r \cos\theta - (r_h \sin\theta_h - r \sin\theta)\cot\beta - r_h \cos\theta_h = \xi(\theta, r, \beta) \qquad (6)$$

The equilibrium equations have to be satisfied under the restriction that the slope material satisfies Coulomb's criterion. This results in a set of three homogeneous equations which serve as constraint conditions in the minimization of W. The Lagrange multiplier technique is then used to determine the critical failure configuration under the minimum W. It may be shown that this is given by

$$r = r_o^* \, e^{(\theta - \theta_o^*) \tan\phi} \qquad (7)$$

Without difficulty one may then show that this is identical to Eq. 1 for the given combination of a primary slope with slope angle β and a secondary slope of angle α. The logspiral configuration is then shown to be the critical failure mechanism.

UPPER BOUND SOLUTION

An upper bound on the critical height of the slope is obtained by the application of the upper bound theorem which asserts that the slope will collapse under the prescribed loading conditions for any compatible failure mechanism when the rate of work done by the loading exceeds the rate of internal energy dissipation. Thus, the critical height is obtained for the logspiral failure mechanism determined above by equating the rates of external and internal energies.

In the following analysis, we consider only the case of the rotational logspiral passing through or above the toe. In Fig. 3, the block DBED above the slip surface BE will rotate as a rigid body about the center of rotation 0, as yet unknown, once the condition of the upper bound theorem is satisfied. The rate of external work $\dot{W}_E$ done by the body forces on the rotating block DBED is best calculated in parts:

$$\dot{W}_E = \dot{W}_1 - \dot{W}_2 - \dot{W}_3 \qquad (8)$$

where $\dot{W}_1$, $\dot{W}_2$, $\dot{W}_3$ are the rates of external work due to the body forces acting on blocks ABEFA, ABDCA and CDEFC, respectively.

For convenience, a change in the variables from elevation h to y, the vertical distance from the center of rotation 0, or to (r,θ) is effected as shown in Fig. 4. Correspondingly, the loading coefficients K_x and K_y, defined in Eqs. 2 and 3 and graphically illustrated in Fig. 4, are expressed in equivalent forms:

$$K_x = \sum_{j=0}^{m} \nu_j \, y^j = \sum_{j=0}^{m} \nu_j \, (r \sin\theta)^j \qquad (9)$$

$$K_y = \sum_{j=0}^{n} \mu_j \, y^j = \sum_{j=0}^{n} \mu_j \, (r \sin\theta)^j \qquad (10)$$

After some lengthy but straight-forward calculations, we obtain for the total rate of external work $\dot{W}_E$ the expression:

$$\dot{W}_E = \gamma\Omega \, r_o^3 \, (f_{1x} + \frac{1}{2} f_{1y} - f_{2x} - \frac{1}{2} f_{2y} - f_{3x} - \frac{1}{2} f_{3y}) \qquad (11)$$

where Ω is the angular velocity of the pertinent region DBED, and the functions have the forms:

$$f_{1x} = r_o^{-3} \sum_{j=0}^{m} \nu_j \rho^{j+3} \{ I[(j+3)\tan\phi,(j+1)]_{\theta_o}^{\theta_h} - I[(j+3)\tan\phi,(j+3)]_{\theta_o}^{\theta_h} + \tan\phi \; J[(j+3)\tan\phi,(j+2)]_{\theta_o}^{\theta_h} \} \qquad (12)$$

$$f_{1y} = r_o^{-3} \sum_{j=0}^{n} \mu_j \rho^{j+3} \{ J[(j+3)\tan\phi,j]_{\theta_o}^{\theta_h} - J[(j+3)\tan\phi,(j+2)]_{\theta_o}^{\theta_h} + \tan\phi \; I[(j+3)\tan\phi,(j+1)]_{\theta_o}^{\theta_h} - \tan\phi \; I[(j+3)\tan\phi,(j+3)]_{\theta_o}^{\theta_h} \} \qquad (13)$$

$$f_{2x} = r_o^{-3} \sum_{j=0}^{m} \nu_j \left[\frac{\xi_2 \, y^{j+2}}{j+2} - \frac{y^{j+3}}{(j+3)\tan\alpha} \right]_{y_B}^{y_D} \qquad (14)$$

$$f_{2y} = r_o^{-3} \sum_{j=0}^{n} \mu_j \left[\frac{\xi_2^2 \, y^{j+1}}{j+1} - \frac{2\xi_2 \, y^{j+2}}{(j+2)\tan\alpha} + \frac{y^{j+3}}{(j+3)\tan^2\alpha} \right]_{y_B}^{y_D} \qquad (15)$$

$$f_{3x} = r_o^{-3} \sum_{j=0}^{m} \nu_j \left[\frac{\xi_3 \, y^{j+2}}{j+2} - \frac{y^{j+3}}{(j+3)\tan\beta} \right]_{y_D}^{y_E} \qquad (16)$$

$$f_{3y} = r_o^{-3} \sum_{j=0}^{n} \mu_j \left[\frac{\xi_3^2 \, y^{j+1}}{j+1} - \frac{2\xi_3 \, y^{j+2}}{(j+2)\tan\beta} + \frac{y^{j+3}}{(j+3)\tan^2\beta} \right]_{y_D}^{y_E} \qquad (17)$$

In Eqs. 12-17, we have used the notation:

$$\rho = r_o \, e^{-\theta_o \tan\phi} \; ; \quad I[A,B] = \int e^{A\theta} \sin^B\theta \, d\theta \; ; \; J[A,B] = \int e^{A\theta} \sin^B\theta \cos\theta d\theta$$

$$\xi_2 = r_o \cos \theta_o + \frac{r_o \sin\theta_o}{\tan\alpha} \quad ; \quad \xi_3 = r_h \cos \theta_h + \frac{r_h \sin \theta_h}{\tan\beta}$$

$$y_B = r_o \sin \theta_o \quad ; \quad y_D = r_o \sin \theta_o + L \sin \alpha \quad ; \quad y_E = r_h \sin \theta_h$$

$$L = \frac{r_o}{\sin(\beta-\alpha)} \left[\sin(\theta_o + \beta) - e^{(\theta_h - \theta_o)\tan\phi} \sin(\theta_h + \beta) \right]$$

Internal dissipation of energy occurs along the slip surface BE. From earlier analyses, Refs. 1-3, the total rate of energy dissipation is

$$\dot{W}_I = \frac{c \, r_o^2 \, \Omega}{2 \tan\phi} \left[e^{2(\theta_h - \theta_o) \tan\phi} - 1 \right] \qquad (18)$$

In applying the upper bound theorem of limit analysis, we equate the total rate of external work, Eq. 11, to the total rate of internal energy dissipation, Eq. 18. After some simplification, we obtain an expression for the elevation of the knee, C, above the lowest point of the slip surface, E:

$$H_C = \frac{c}{\gamma} F(r_o, \theta_o, \theta_h) \qquad (19)$$

where

$$F(r_o, \theta_o, \theta_h) = \frac{\sin\beta \left[e^{(\theta_h-\theta_o)\tan\phi} \sin(\theta_h+\alpha) - \sin(\theta_o+\alpha) \right] \left[e^{2(\theta_h-\theta_o)\tan\phi} - 1 \right]}{2 \tan\phi \sin(\beta-\alpha) \left[f_{1x} - f_{2x} - f_{3x} + \frac{1}{2}(f_{1y} - f_{2y} - f_{3y}) \right]} \tag{20}$$

The critical height H^*_C of the slope, i.e. the maximum height required to prevent slip along the critical logspiral failure surface, may be expressed in terms of a dimensionless number N^*_s, the "stability factor":

$$H^*_C = \frac{c}{\gamma} N^*_s \tag{21}$$

where $N^*_s = \min F(r_o, \theta_o, \theta_h) = F(r^*_o, \theta^*_o, \theta^*_h)$ such that r^*_o, θ^*_o, θ^*_h satisfy the conditions:

$$\frac{\partial F}{\partial r_o} = 0; \quad \frac{\partial F}{\partial \theta_o} = 0; \quad \frac{\partial F}{\partial \theta_h} = 0$$

DISCUSSION OF RESULTS AND CONCLUSIONS

Explicit results can be obtained from the above analysis for specific situations, i.e. when the body force components expressed in Eqs. 2 and 3 are prescribed. The seismic load is expressed through the loading coefficient K_x, which may be treated as a variable through the height of the slope. Vertical loads, including the gravitational forces, may be expressed through the loading coefficient K_y. For the classical case discussed by Chen(1975), we consider K_x identically zero and $K_y(h) = 1$. A numerical comparison between our calculations and the results presented by Chen (1975), Table 9.1, show excellent correspondence.

The cases considered by Chen, Chang and Yao (1978) are special cases of the analysis discussed in the present paper. A comparison between the numerical results obtained by Chen et al. for the case of constant seismic coefficient, for $K = 0.325$, and those obtained using our formulation also show good correlation, except for small values of ϕ. It is, of course, recognized that $\phi = 0^o$ is a limiting case requiring special consideration.

The work presented here considers a more complicated loading condition that would permit the inclusion of seismic analysis of slopes for which the seismic coefficient takes on any distribution varying along the height of the slope. Computer codes may be developed to analyze specific cases of varying material constants, varying seismic loads and different slope geometries. Extensions to benchmark cases of earthquake-induced slope failure may then be considered and design codes subsequently developed.

ACKNOWLEDGEMENTS

The authors of this paper are grateful to the National Science Foundation for the support of their work through a grant, No. PRF-7809326. The help and encouragement extended by colleagues at the Foundation, in particular Drs. S. C. Liu and W. Hakala, are especially acknowledged.

REFERENCES

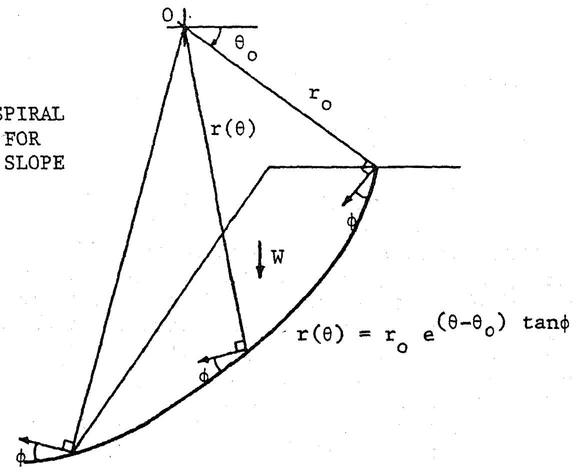
[1] Chen, W. F., 1975, Limit Analysis and Soil Plasticity, Elsevier, Amsterdam.

[2] Chen, W. F., Giger, M. W. and Fang, H. Y., 1969, Soils and Foundations, Japanese Society of Soil Mechanics and Foundation Engineering, 9, 23-32.

[3] Chen, W. F., Chang, C. J. and Yao, J.T.P., Proceedings, 15th Annual Meeting of the Society of Engineering Science, 533-538, 1978.

[4] Drucker, D. C. and Prager, W., 1952, Q. Appl. Math., 10, 157-165.

Fig. 1 - LOGARITHMIC SPIRAL FAILURE MECHANISM FOR LIMIT ANALYSIS OF SLOPE STABILITY



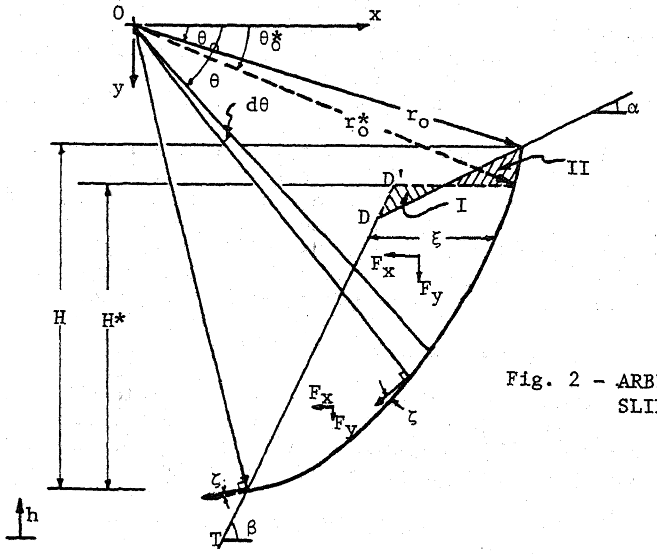


Fig. 2 - ARBITRARY POTENTIAL SLIP SURFACE

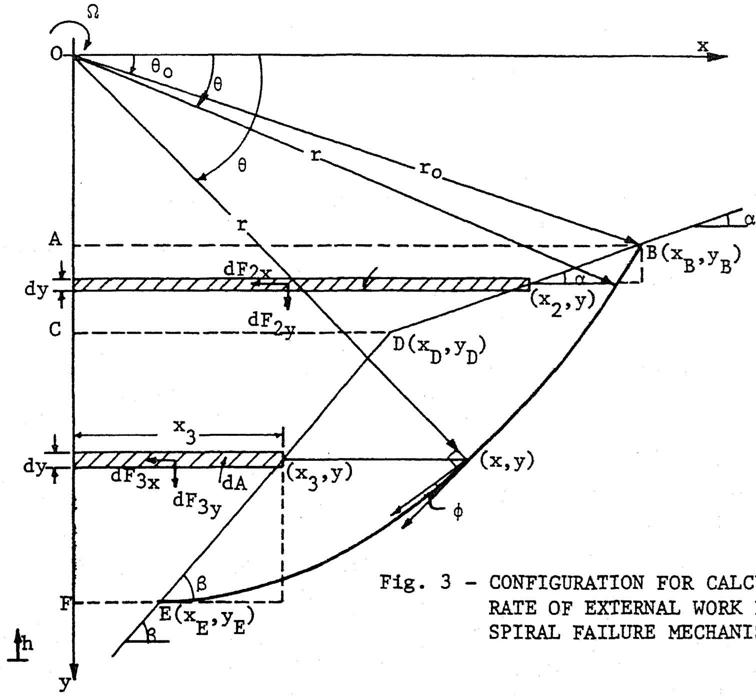
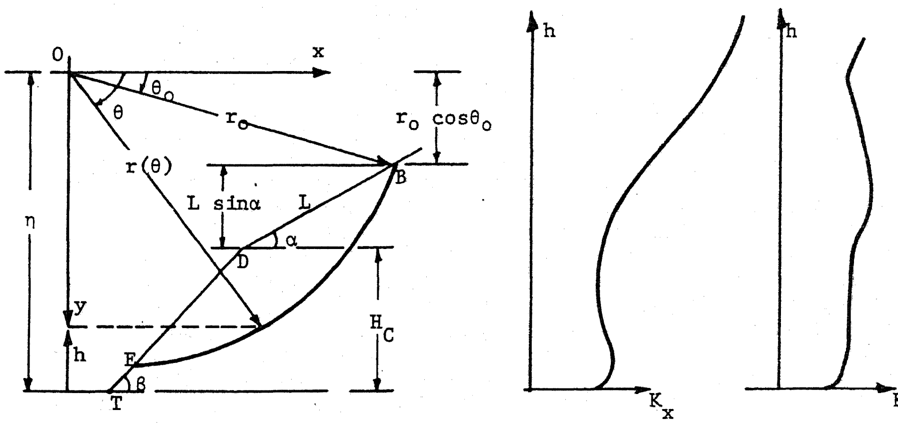


Fig. 3 - CONFIGURATION FOR CALCULATION OF RATE OF EXTERNAL WORK FOR LOG-SPIRAL FAILURE MECHANISM



Profiles of Horizontal and Vertical Body Force Components

Fig. 4 - SLOPE GEOMETRY AND LOADING PROFILES