STOCHASTIC FUNCTIONS IN EARTHQUAKE OCCURRENCE MODELS: A CRITICAL SURVEY

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SUMMARY

When one deals with time distribution of events such as earthquake occurrences, some attentions must be paid to the way one fixes the origin and the end of the time interval, to the procedure of counting events and to conditions implicitly set up and often not in the open. All these circumstances lead to the definitions of a number of stochastic functions with hazard implications. Authors suggest a critical reading of these functions and their structure in the case of uncorrelated and correlated time distributions of events.

BACKGROUND LITERATURE

Statistical functions have been used in the past in order to describe the time dependence of earthquake sequences or catalogues. Gaisky (1966) studied the time distribution of large and deep earthquakes in Pamir-Hindu-Kush and observed a tendency for the recurrence times to be somewhat regular. Vere-Jones (1970) produced the most extensive and complete presentation of stocastic models for earthquake occurrence. His article represents the strongest trial to bridge the gap between seismology and statistics: he introduced, among the many subjects, appropriate definitions of hazard functions, the useful adoption of second-order statistical analysis, the fundamental concept of cluster-type occurrence and so on. Rice (1974) discussed the estimation of the hazard function, the intensity function and the variance-time curve. He concentrated firstly to give rigorous definitions of these statistical functions: as it appears in the rest of the survey, definitions are often critical and not always without ambiguity.

According to Rice, the hazard function, a.k.a. the age specific occurrence rate, is defined

$$h(t)dt = P\{\text{an event in } (t,t+dt) \mid \text{last event at } t=0\} .$$

The difference from a probability density, f(t) say, consists of the imposition that no event occurs between 0^+ and t.

According to Rice, whereas the hazard function is a measure of the aftereffect of an event on the rate of occurrence of the immediately succeeding event, the intensity function measures the effect of an event on all succeeding events (the first, second, third, etc.). The intensity function is defined as

$$m(t)dt = P\{\text{an event in } (t,t+dt) \mid \text{event at } t=0\} .$$

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It must be noted that m(t)dt changes conspicuously according to the choice of the event at t=0. In fact, it can be read as

$$m(t)dt = P\{\text{an event in } (t,t+dt) \mid \text{an event at } t=0\}$$

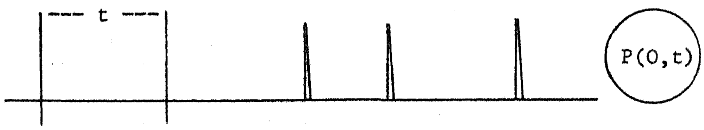
and also as

$$m(t)dt = P\{\text{an event in } (t,t+dt) \mid \text{each event taken as time-mark } t=0\}$$

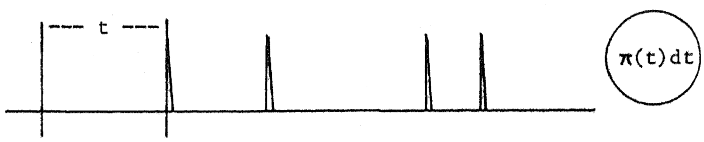
FIVE HAZARD-RELATED PROBABILITIES

The hazard and intensity functions are not the only ones which can be related to the probability of an earthquake occurring somewhere along a time axis. If one specifies origin and end of the time interval and where events are expected (e.g. at the beginning, within, at the end, beyond the chosen time interval), at least five probability functions can be defined. Here they are:

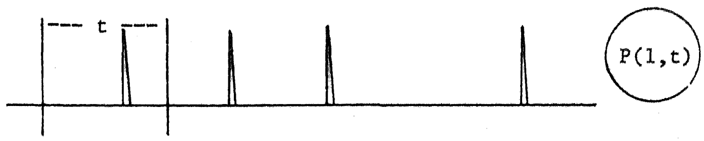
(a) the probability that in a time interval (0,t) no event occurs, i.e. the event is likely to occur after t (instants 0 and t are arbitrarily chosen and then uncorrelated)



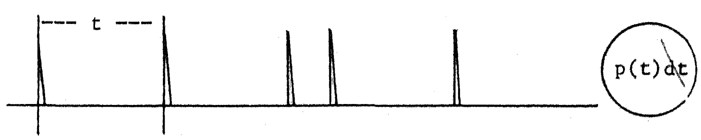
(b) the probability that, after an arbitrarily chosen time origin, the next event occurs between t and t+dt



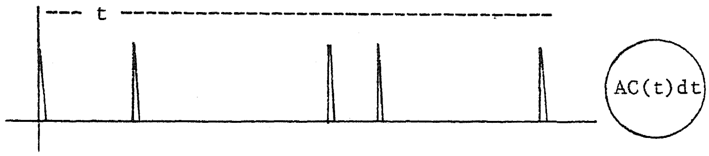
(c) the probability that in the time interval (0,t) one event occurs (instants 0 and t are uncorrelated)



(d) the probability that, after an event at time t=0, the next event occurs between t and t+dt



(e) the probability that, after an event at time t=0, another event occurs between t and t+dt



The hazard function coincides with the probability p(t)dt. The autocorrelation function AC(t)dt determines the conditional probability distribution of the time interval intercurring among <u>each</u> event of a sequence and <u>all</u> the later events: this function releases the maximum information contained in the series under analysis and, in particular, features the hazard function as subensemble. Under these explicit circumstances, the intensity function m(t)dt coincides with the autocorrelation function.

OPERATIONAL PROCEDURE

The five probabilities listed in the previous section are not a mathematical fantasy. They suggest not only a philosophy but also practical approaches for the handling of earthquake sequences and catalogues. In order to clarify their operational role, it is better to split them into two groups.

<u>Probabilities</u> - P(0,t) and P(1,t) belong to this group. The implied procedure consists of the following steps: (1) subdivide the overall length of the data record into a large number of time intervals of equal duration $t=t_1$; (2) count the number of intervals with content equal to 0 event and 1 event; (3) divide the previous fugures by the total number of analyzed intervals and obtain experimental values for $P(0,t_1)$ and $P(1,t_1)$; (4) repeat identical procedures for $t=t_2$, $t=t_3$ and so on; (5) finally plot P(0,t) vs t and P(1,t) vs t for as many points as the adopted value of the time variable. At this point a comparison must be made with some theoretical values of P(0,t) and P(1,t) belonging to the most popular discrete-variable probability distributions. Details on this last operation are given further on.

<u>Conditional probabilities</u> - $\pi(t)dt$ p(t)dt and AC(t)dt belong to this group. The implied procedure for $\pi(t)dt$ consists of the following steps: (1) select a random time generator which sets up a large number of time origins (O pulses) over a duration equal to that of the original data set; (2) overlap the previous sequence to that of seismic data (E pulses); (3) measure the time distance between each O pulse and the first contiguous E pulse; (4) form an istogram carrying the fraction of data consisting of time distances OE equal to 0--t_1 , t_1--t_2, t_2--t_3 and so on. The implied procedure for p(t)dt consists of the following steps: (1) take each event of the seismic sequence; (2) measure the time distance between the selected event and the next event; (3) form an istogram carrying the fraction of data consisting of time distances EE equal to 0--t_1 , t_1--t_2 , t_2--t_3 and so on. The implied procedure for Ac(t)dt consists of the following steps: (1) take each event of the seismic sequence; (2) measure the time distance between the selected event and ALL the following events; (3) form an istogram carrying the fraction of data corresponding to time distances equal to 0--t_1 , t_1--t_2 , t_2--t_3 etc.

At this point a comparison must be made with some theoretical values of $\pi(t)dt$, $p(t)dt$ and $Ac(t)dt$ belonging to the well known probability distributions given in literature. Details are specified in the coming sections.

THE GENERATING FUNCTION CONNECTION

Needless to say, the five probabilities introduced are, in general, quite different: nevertheless they are strictly related from a mathematical viewpoint. Their mutual relationships can be elegantly expressed by using the probability generating function (pgf) algorithm.

The pgf is defined as

$$G(z,t) = \sum_{N=0}^{\infty} P(N,t)\ z^N$$

where $P(N,t)$ is the probability of occurrence of N events in a time interval 0--t with instants 0 and t chosen in an arbitrary way. It can be shown that (see Appendix for details)

$$P(0,t) = G(0,t)$$

$$\pi(t) = -\frac{dG(0,t)}{dt}$$

$$P(1,t) = \left.\frac{dG(z,t)}{dz}\right|_{z=0}$$

$$p(t) = \frac{1}{\lambda}\frac{d^2G(0,t)}{dt^2}$$

$$AC(t) = \frac{1}{2}\frac{d^2}{dt^2}\left[\frac{d^2}{dz^2}\ G(z,t)\right]_{z=1}$$

where λ is the rate of occurrence of the events or the mean number of events per unit time.

Once the expression of the pgf is made explicit by the choice of a certain probability distribution, it can be inserted into the previously introduced definitions of the five hazard-related probabilities and a test of hypothesis is performed against the same probabilities computed from seismic data.

In the following sections, two tests are proposed with essentially demonstrative purposes.

THE POISSON DISTRIBUTION TEST OF HYPOTHESIS

If the case of a Poisson distribution with mean λt is considered, the corresponding pgf is $\exp[\lambda t(z-1)]$ so that the following useful tableau can be compiled. It is ready to be compared with the corresponding operational probabilities computed from earthquakes sequences or catalogues.

mean = λt	variance/mean = 1
variance = λt	$P(0,t) = \exp(-\lambda t)$

$\pi(t) = \lambda \exp(-\lambda t)$	$p(t) = \lambda \exp(-\lambda t)$
$P(1,t) = \lambda t \exp(-\lambda t)$	$AC(t) = \lambda^2$

The following remarks seem appropriate:

- recalling the definition of $\pi(t)$ and $p(t)$, one can note that they differ in the initial event: an arbitrarily chosen time origin for $\pi(t)$, an event of the sequence for $p(t)$. In the case of a Poisson, i.e. uncorrelated, distributed sequence, the two initial events are actually the very same, so that $\pi(t)$ and $p(t)$ coincide ;
- since $AC(t)$ is proportional to the conditional probability that two events are t-spaced in time, it is no surprise that in the case of Poisson distributed events $AC(t)$ is constant. In a Poisson distributed sequence the occurrence of an event is not influenced by earlier events and does not affect later events. That is to say that $AC(t)$ does not depend on time ;
- $AC(t)dt$ is the probability distribution of the time interval intercurring among each event of the sequence and all the following ones; $p(t)dt$ is the probability distribution of the time interval intercurring between each event of the sequence and the next one only. All the information related to time intervals between non-contiguous events is omitted in $p(t)dt$. This probability is then decreasing in time because the larger the time interval the higher the chance of t-spaced events being omitted ;
- $P(1,t)$ is larger (smaller) than $P(0,t)$ in correspondence to λt being larger (smaller) than unity .

THE NEGATIVE BINOMIAL DISTRIBUTION TEST OF HYPOTHESIS

If the case of a Negative Binomial distribution with mean λt and correlation index $Y(t)$ is considered, the corresponding pgf is $\exp\{-[\lambda t/Y(t)] \log [1+Y(t)(1-z)]\}$ so that an anologous tableau can be compiled.

mean $= \lambda t$	variance/mean $= 1+Y(t)$
variance $= \lambda t[1+Y(t)]$	$P(0,t) = \exp[-\lambda t\, H(t)]$
$\pi(t) = -\frac{d}{dt}\exp[-\lambda t\, H(t)]$	$p(t) = \frac{1}{\lambda}\frac{d^2}{dt^2} \exp[-\lambda t H(t)]$
$P(1,t) = [\lambda t/1+Y(t)]\ P(0,t)$	$AC(t) = \lambda^2 + \frac{1}{2}\frac{d^2}{dt^2}[\lambda t Y(t)]$

where $H(t) = [1/Y(t)] \log[1+Y(t)]$

The following remarks are appropriate:

- the role of the correlation index $Y(t)$ is denoted by the following important property

$$\lim_{Y(t)\to 0} \text{(Negative Binomial)} = \text{(Poisson)} \quad ;$$

- recalling the definition of $\pi(t)$ and $p(t)$, one can be convinced that in in this case the two probabilities have no reason to be alike since randomly generated time origins and seismic events are totally uncorrelated ;

- autocorrelation function AC(t) is composed of two addenda: the first one, constant with time, is identical to that present in the Poisson distribution case. The second one is made time-dependent by the presence of the terms Y(t) which expresses the property - usually named correlation or contagious effect - that the presence of an event in the sequence makes the probability increase of another event close by in time ;
- P(1,t) is larger (smaller) than P(0,t) in correspondence to λt being larger (smaller) than 1+Y(t) .

COMMENTS AND CONCLUSIONS

What is the distributions of earthquakes in time ? Earthquakes do not occur periodically but randomly, i.e. they are statistically distributed in time. Is the occurrence of an earthquake influenced by the occurrence of an earlier event ? Is the occurrence of an earthquake affecting the occurrence of later events ? If the two described circumstances do not happen, then earthquake occurrences are defined as uncorrelated in time; if the contrary holds earthquake occurrences are defined as correlated in time. The presence of correlation is somewhat visualized by the fact that seismic events might be clustered in time, i.e. long periods occur with no event at all, then a pair, a triplet or a n-plet of events takes place.

A sequence consists of a given amount of information related not only to the presence of a number of events but also on their mutual location on a time axis. The need for introducing stochastic functions is imposed by the search of capability of describing the sequence.

Five stochastic functions with hazard implications have been introduced by Pacilio et al.(1980a,b,c) on the base of earlier investigations by Schenkova (1972,1973) and Basili et al.(1976). Two of them, P(0,t) and P(1,t) are probabilities, three of them , $\pi(t)dt$ p(t)dt AC(t)dt,are conditional probabilities. P(0,t) and P(1,t) are defined in the event domain,i.e. they represent the probability of occurrence of no and one event, respectively, in a given time base t. The number of data to be handled is then established by the time sampling and is practically uninfluenced by the number of events. $\pi(t)$,p(t) and AC(t) are defined in the time domain,i.e. they represent the probability of occurrence of a time interval t between each random origin and the next event, between each event and the next one, between each event and all the following ones, respectively. The number of data to be handled is then established by the number of events, say N, of the sequence under analysis. This number is $N_o \leq N$ for $\pi(t)dt$, N-1 for p(t)dt and $\frac{1}{2}N(N-1)$ for AC(t)dt.

The lot of these five probabilities covers a wide range of interest. In the case of sporadic events the role of P(0,t) and P(1,t) is preferential with a prevalence for P(0,t) if $\lambda t<1$. In the case of numerous events $\pi(t)dt$, p(t)dt and AC(t)dt are preferential. P(0,t) and P(1,t) offer clues on the presence or less of events and ignore mutual time-spacing of events; p(t)dt offers information about contiguous events only, $\pi(t)dt$ has a hybrid role in between, only AC(t)dt gives the entire amount of information needed in order to describe the sequence in a univocally determined fashion. But AC(t)dt is sometimes difficult to interpret and the number of data is always finite: it comes out that a good diagnostic procedure is constituted by the adoption of the whole set of five probabilities.

From their definitions it can be understood the fundamental role of the origin and the end of time interval inwhich the earthquake must occur and the different hazard implications that this choice determines.

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APPENDIX

The term $\pi(t)$ is strictly related to $P(0,t)$ under fairly general assumptions. If the definition of $\pi(t)$ and $P(0,t)$ are recalled, then

$$\int_0^t \pi(\tau)\,d\tau$$

expresses the probability that after a time t=0 chosen at random, the first event occurs between 0 and t, whereas $P(0,t)$ is the probability that the same event occurs after t. Therefore

$$P(0,t) + \int_0^t \pi(\tau)\,d\tau = 1$$

which can be used to formulate (t) as a function of $P(0,t)$, i.e.

$$\pi(t) = -\frac{d}{dt} P(0,t)$$

The quantity $p(t)$ is closely related to $\pi(t)$ and therefore to $P(0,t)$ under slightly more restrictive conditions. If the earthquake occurrence process is assumed to be symmetrical in time, $\pi(t)$ is also the probability that after an event occurs between 0 and dt, an empty interval of duration t follows. The same probability can be expressed by a product. The first factor is the probability of one event between 0 and dt which is λdt. The second factor is the probability that the next event occurs any time later than t. This factor can be computed as

$$1 - \int_0^t p(\tau)\,d\tau$$

the integral being the probability that after an event at time t=0 the next event occurs between 0 and t. In conclusion, one obtains

$$\pi(t)dt = \lambda dt\left[1 - \int_0^t p(\tau)\,d\tau\right]$$

which can be used to formulate $p(t)$ as a function of $\pi(t)$ and therefore of $P(0,t)$

$$p(t) = -\frac{1}{\lambda}\frac{d}{dt} = \frac{1}{\lambda}\frac{d^2 P(0,t)}{dt^2}$$

The expected number of pairs of events in a time interval 0--t can be defined as

$$\tfrac{1}{2}\overline{N(N-1)} = \int_0^t dt_2 \int_0^{t_2} \varphi(t_1,t_2)\,\lambda\, dt_1 \qquad 0 \leq t_1 \leq t_2 \leq t$$

where $\varphi(t_1,t_2)$ is the conditional probability that the second event occurs at t_2 given a first event at t_1. The function $\lambda\varphi(t_1,t_2)$ is defined as autocorrelation function $AC(t_1,t_2)$ of variable N. Under the ergodic hypothesis, AC does not depend on the choice of t_1 but only on the time elapse $t_2-t_1=t$. In conclusion $AC(t)$ can also be given in its differential formulation, i.e.

$$AC(t) = \tfrac{1}{2}\frac{d^2}{dt^2}\overline{N(N-1)}$$