A SIMPLE PROCEDURE FOR PREDICTING AMPLIFIED RESPONSE SPECTRA AND EQUIPMENT RESPONSE

by

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SYNOPSIS

This paper presents a new procedure to evaluate secondary system response directly from the specified ground response spectra. It is assumed that a modal analysis of the primary structure has been made. The maximum acceleration of a one-degree secondary system is expressed as the square root of a sum (over all significant primary system modes) of contributions which depend on (i) S_{Ak} = the pseudo-acceleration (ground) response spectrum for the period and damping of primary system mode k, and (ii) S_{Ae} = the pseudo-acceleration (ground) response spectrum for the equipment period and damping. The amplification factors multiplying S_{Ak} and S_{Ae} are easy to evaluate and have been derived from a nonstationary random vibration study of secondary system response.

INTRODUCTION

In a number of important problem situations arising in earthquake engineering, the dynamic model of interest consists of a primary system whose response provides the input to a secondary system. The primary system is often a linear lumped mass system and the secondary system is a simple linear oscillator. One application of this model is in the area of soil amplification where the soil layers constitute the primary system and the secondary system is a one-degree structure. The second application arises in the seismic analysis of equipment and appendages in buildings. The response quantity of interest is the displacement of the equipment relative to the motion of the structure at the point of support for the equipment.

These combined (primary-secondary) systems can, in principle, be modeled as one multi-degree-of-freedom system. It is common, however, to separate the seismic analysis of the secondary system from that of the primary system, and to view the acceleration response at the point of support (of the secondary system) as the input for the dynamic analysis of the secondary system. This uncoupled dynamic model is suitable when the inertial properties of the (massive) primary and (light) secondary system are very different, and has great practical value when an entire spectrum of secondary systems needs to be examined.

The first step in the analysis is aimed at obtaining the characteristics of the accelerogram at the support point of the secondary system. These will depend on the base acceleration and on the primary system dynamic properties. The second step consists of the evaluation of the secondary system response to the motion at its support. The output may be, for example, the secondary system's maximum acceleration which can be converted into equivalent static loads for stress analysis. Finally, if the secondary system is itself a multi-degree-of-freedom system, the second step in the procedure will have to be repeated for each secondary system mode.

RANDOM VIBRATION ANALYSIS OF MULTI-DEGREE-OF-FREEDOM-SYSTEMS

A major step in the seismic analysis of a structure or a soil-structure system is to construct a dynamic model, frequently a lumped-parameter model

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whose parameters are the elements of the mass and stiffness matrices. These may be determined by any one of several conventional procedures (see, for example, Biggs, 1964). The natural frequencies and shapes of the normal modes can then be determined by solving the eigenvalue problem. In the normal mode method, the n-degree system response at a point on the structure is expressed in terms of the modal coefficients c_k and the generalized modal coordinates $y_k(t)$, $k = 1, 2, \ldots, n$. In particular, the displacement at point B relative to the ground is:

$$y(t) = \sum_{k=1}^{n} \phi_{kB} \Gamma_k y_k(t) = \sum c_k y_k(t) \qquad (1)$$

Each component, $y_k(t)$, is the response of a one-degree system characterized by the (undamped) natural frequency ω_k and an assigned percentage of critical damping ζ_k. Also, $c_k = \phi_{kB}\Gamma_k$, where ϕ_{kB} = the characteristic shape ordinate for mode k at point B, and Γ_k = the participation factor in the kth mode. A response quantity closely related to y(t) is the absolute acceleration response

$$z(t) = x(t) + \ddot{y}(t) = x(t) + \sum_{k=1}^{n} c_k \ddot{y}_k(t) \qquad (2)$$

where $\ddot{y}(t)$ = the second derivative of $y(t)$, $\ddot{y}_k(t)$ = the second derivative of the kth modal coordinate, and $x(t)$ = the ground acceleration.

Relative Displacement Response

Assume that the ground motion is characterized by its spectral density function $G(\omega)$ and strong-motion duration s. The system relating input acceleration $x(t)$ and output relative displacement $y(t)$ has the impulse response function:

$$h(t) = \sum_{k=1}^{n} c_k h_k(t) \qquad (3)$$

where $h_k(t)$ is the impulse response function of a one-degree system with parameters ω_k and ζ_k. The truncated Fourier transform of h(t), or the time-dependent transfer function is:

$$H(\omega, t) = \int_0^t h(t-\tau) e^{-i\omega\tau} d\tau = \sum_{k=1}^{n} c_k H_k(\omega, t) \qquad (4)$$

and the time-dependent spectral density function of y(t) equals:

$$G_y(\omega, t) = G(\omega) |H(\omega, t)|^2 = G(\omega) \sum_{k=1}^{n} \sum_{j=1}^{n} c_k c_j H_k(\omega, t) H_j^*(\omega, t) \qquad (5)$$

in which $H_j^*(\omega, t)$ = the complex conjugate of $H_j(\omega, t)$. Although the expression $G_y(\omega, t)$ has the form of a double summation, significant contributions to its moments with respect to frequency (spectral moments) usually come from the terms for which j=k. This is particularly true when modal frequencies are well-separated and when damping values are low. Algebraic manipulation allows one to express Equation 5 approximately as follows (Vanmarcke, 1972 and 1976)(*):

$$G_y(\omega, t) \simeq G(\omega) \sum_{k=1}^{n} |H_k(\omega, t)|^2 \{c_k^2 + \sum_{j \neq k} c_j c_k A_{kjt}\} \qquad (6)$$

(*)The use of Eq. 6 to evaluate the spectral moments results in a percentage error which is of the order of the square of the damping factor.

in which A_{kjt} is a factor which depends on the ratio of modal frequencies $r = \omega_j/\omega_k$ and on the equivalent damping values $\zeta_{kt} = \zeta_k(1 - e^{-2\zeta_k\omega_k t})-1$ and $\zeta_{jt} = \zeta_j(1 - e^{-2\zeta_j\omega_j t})-1$

$$A_{kjt} = \frac{8r\, \zeta_{kt}\,(\zeta_{jt} + \zeta_{kt}\, r)[(1 - r^2)^2 - 4r(\zeta_{kt} - \zeta_{jt}\, r)(\zeta_{jt} - \zeta_{kt}\, r)]}{8r^2[(\zeta_{kt}^2 + \zeta_{jt}^2)(1 - r^2)^2 - 2(\zeta_{jt}^2 - \zeta_{kt}^2 r^2)(\zeta_{kt}^2 - \zeta_{jt}^2 r^2)] + (1 - r^2)^4}$$

(7)

At $r = 1$, $A_{kjt} = 2\zeta_{kt}/(\zeta_{kt} + \zeta_{jt})$, which is equal to one if $\zeta_{jt} = \zeta_{kt}$ [*]. A_{kjt} vanishes when r is either very small or very large. Integrating Equation 6 over all frequencies gives the time-dependent variance of the multi-degree system response $y(t)$:

$$\sigma_y^2(t) = \int_0^\infty G_y(\omega,t)d\omega \simeq \sum_{k=1}^{n}(c_k^2 + \sum_{j\neq k} c_j c_k A_{kjt})\,\sigma_{y_k}^2(t) = \sum_{k=1}^{n}\alpha_{kt}c_k^2\sigma_{y_k}^2(t)$$

(8)

where

$$\alpha_{kt} = 1 + \sum_{j\neq k}(c_j/c_k)\,A_{kjt}$$

(9)

Absolute Acceleration Response

Very similar results can be obtained for the absolute acceleration response $z(t)$ at a point in an n-degree structure. The time-dependent spectral density function $G_z(\omega,t)$ of the absolute acceleration response $z(t)$ has an approximate form similar to that of $G_y(\omega,t)$ in Equation 6:

$$G_z(\omega,t) \simeq G(\omega)\left|1 + \omega^2 \sum_{k=1}^{n} c_k H_k(\omega,t)\right|^2 \simeq G(\omega)\left\{1 + \omega^4 \sum_{k=1}^{n}\alpha_{kt}c_k^2|H_k(\omega,t)|^2\right\}$$

(10)

with α_{kt} defined by Equation 9. As before, the understanding is that the time-dependent transfer function $|H_k(\omega,t)|$ is, for practical purposes, equivalent to a complete transfer function $|H_k(\omega)|$ with an equivalent time-dependent damping parameter ζ_{kt}. The approximation on the right side of Equation 10 is suitable for determining the spectral moments of $z(t)$ and for use in random vibration analysis leading to response statistics of "secondary systems" dealt with below.

RANDOM VIBRATION ANALYSIS OF SECONDARY SYSTEMS

When an earthquake strikes, the secondary system will respond to the motion $z(t)$ generated at the support point on the primary system. The frequency content of the motion $z(t)$ is described by the (time-dependent) spectral density function $G_z(\omega,t)$. If the primary system is not too lightly damped[**], the variation of the frequency content of $z(t)$ <u>with time</u> will often

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[*] Also, $A_{kjt} = r(\zeta_{jt}/\zeta_{kt})A_{kjt}$, so that $A_{jkt} + A_{kjt} = 2$ if $r = 1$.

[**] For very lightly damped or very long period primary systems, the function $G_z(\omega,s)$ must be viewed as an approximate representation of the frequency content during the most intense part of the motion $z(t)$ (which will occur toward the end of the input motion and just thereafter, during the early part of the motion decay period).

not be important, and $G_z(\omega,s)$ will provide a reasonable representation of the input spectral density function for the random vibration analysis of the secondary system. The time-dependent spectral density function $G_e(\omega,t)$ of the secondary system response can be expressed by

$$G_e(\omega,t) = G_z(\omega,s)|H_e(\omega,t)|^2 \qquad (11)$$

where $|H_e(\omega,t)|^2$ = the transient squared amplification function for the secondary system, the exact form of which can be approximated by

$$|H_e(\omega,t)|^2 = [(\omega_e^2 - \omega^2)^2 + 4\zeta_{et}^2\omega_e^2\omega^2]^{-1} \qquad (12)$$

where $\zeta_{et} = \zeta_e(1 - e^{-2\zeta_e\omega_e t})^{-1}$ and ζ_e = the secondary system damping. Attention focuses on the variance of the response at its peak value, when $t = s$. Working in terms of the pseudo-acceleration response variance $\sigma_e^2(s)$ can be approximated by

$$\sigma_e^2(s) = \omega_e^4 \int_0^\infty G_e(\omega,s)d\omega \simeq \int_0^{\omega_e} G_z(\omega,s)d\omega + G_z(\omega_e,s)\ \omega_e\left(\frac{\pi}{4\zeta_{es}} - 1\right) \qquad (13)$$

Inserting the expression for $G_z(\omega,s)$ given by Eq. 10 yields:

$$\sigma_e^2(s) \simeq \sum_{k=1}^{n'} \alpha_{ks}c_k^2\sigma_k^2(s) + \left[1 + \sum_{k=1}^{n} \alpha_{ks}c_k^2\omega_e^4|H_k(\omega_e,s)|^2\right]\left[\sigma_{eg}^2(s) - \sigma^2F^*(\omega_e)\right] \qquad (14)$$

where $\sigma_{eg}^2(s)$ and $\sigma_k^2(s)$ are the pseudo-acceleration response variances of one-degree systems supported on the ground, corresponding, respectively, to the secondary system and to mode k of the primary system; also:

$$\alpha_{ks} = \left[1 + \sum_{j \neq k} (c_j/c_k)A_{kjs}\right]$$

n = the number of significant primary system modes and n' = the number of primary system modes for which the frequency ratio ω_k/ω_e is less than one; also σ^2 is the ground acceleration variance and $F^*(\omega_e) = \sigma^{-2}\int_0^{\omega_e} G(\omega)d\omega$ is the normalized cumulative spectral density of the ground acceleration which increases monotonically from 0 to 1 as ω_e increases from 0 to ∞. The value of the product of ω_e and the kth modal transfer function $H_k(\omega_e,s)$ vanishes when $\omega_e \ll \omega_k$ and tends to one if $\omega_e \gg \omega_k$. If $\omega_e \ll \omega_k$, Equation 14 yields desired result $\sigma_e^2(s) = \sigma_{eg}^2(s)$, and if $\omega_e \gg \omega_k$, $\sigma_e^2(s)$ approaches the variance of the primary system response acceleration (as $\sigma_{eg}^2(s) \to \sigma^2$ and $F^*(\omega_e) \to 1$).

The random vibration result, Equation 14, now provides the basis for the following new proposal to evaluate secondary system response directly from the specified ground response spectra:

$$A_e = \left\{\sum_{k=1}^{n'} \alpha_{ks}c_k^2S_{Ak}^2 + \left[1 + \sum_{k=1}^{n} \alpha_{ks}c_k^2\omega_e^4|H_k(\omega_e,s)|^2\right](S_{Ae}^2 - A_g^2F^*(\omega_e))\right\}^{\frac{1}{2}} \qquad (15)$$

in which A_e = maximum acceleration of a one-degree secondary system, S_{Ak} = the pseudo-acceleration (ground) response spectrum for the period and damping of mode k, S_{Ae} = the pseudo-acceleration (ground) response spectrum for the equipment period and damping, A_g = maximum acceleration of the ground; α_{ks}, n and n'

are as defined in Equation 14. Amplified response spectra can be estimated directly using Equation 15.

If the secondary system has several significant modes, Equation 15 can be used to estimate each modal contribution. The computations proposed for the one-degree case must be repeated for each secondary system mode, and the individual modal maxima must be combined, for example, by the widely used root-sum-square procedure.

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