# STATIC ANALYSIS OF ASYMMETRIC MULTISTORY STRUCTURES

*by*

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## SYNOPSIS

A simplified criterion for the computation of dynamic amplification factors for twisting moments in multistory systems is proposed. By means of a generalized-coordinates approach and proper approximations to the first translational, torsional and combined modes, applicability of results valid for single-story systems is extended to multi-story structures. The accuracy of the approximate procedure is calibrated by comparing its results with those produced by standard modal analysis.

## INTRODUCTION

Seismic static analysis of multistory structures is usually performed under the assumptions that their mass is lumped at floor levels and that the effects of the design earthquake are represented by a set of lateral forces, each acting at center of gravity of one of the masses. Implicit in the criterion is the assumption that torsional oscillations are accounted for by the static torsional eccentricities at each story (that is, the lever arm of the static story shear with respect to the twist center) multiplied by suitable amplification factors. The criterion breaks down, for instance, in buildings where nominal static eccentricities differ drastically among stories, as in that case torsional oscillations induced by the largest static eccentricities give place to inertia forces that may produce significant dynamic eccentricities throughout the structure. This paper presents and calibrates a method for determining a set of dynamic torsional couples, assumed to act the floor masses, and such that story torsional moments computed by static equilibrium of the portion of building above a given story, under the action of floor lateral forces and torsional couples, provide an approximate value of the actual dynamic story torsion. The criterion may be applied when story shears and eccentricities are obtained either from a static analysis or from a dynamic analysis that neglects torsional degrees of freedom.

## STATIC TORSION

A definition of static torsion that differs from that ordinarily used is introduced as follows. Let $Q_i$ be the lateral force acting at the center of gravity of mass $M_i$, and $J_i$ the moment of inertia of $M_i$ with respect to a vertical axis going through its center of gravity. Let $\theta_i$ and $\delta_i$ be respectively the rotation of mass $M_i$ in a horizontal plane and the translation of its center of gravity in the direction parallel to $Q_i$. *By definition*, the static torsional couple $T_i$, assumed acting on mass $M_i$, is related to $Q_i$ by the following proportionality:

$$\frac{T_i}{Q_i} = \frac{J_i \theta_i}{M_i \delta_i} \qquad (1)$$

## DYNAMIC TORSION

An approximate estimate of dynamic torsion can be obtained if $T_i$ as given by eq 1, is multiplied by an adequate amplification factor. A criterion for defining such a factor is derived in the following, by extension of the results presented by Newmark and Rosenblueth (1971) for simple story systems.

Let us start by defining a reference system, as shown either in fig 1a or 1b. It is an asymmetric single story structure with mass M, assumed concentrated at the top, center of gravity g, and moment of inertia J with respect to a vertical going through g. Let the story shear stiffness in the assumed direction of seismic forces be denoted by $K_{11}$, the twist center by t and the story torsional stiffness with respect to g by $K_{22}$; e is the nominal eccentricity. The static torsion is defined according to eq 1. From $K_{11}$, $K_{22}$ and e it is easy to obtain the off-diagonal stiffness $K_{12}$.

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Let $Z_i$, i= 1,2, be the characteristic vectors of the system and $z_1$, $z_2$ the components of either of them. The conditions for free vibration can be expressed as follows:

$$-\omega^2 \begin{bmatrix} M & \\ & J \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = 0 \tag{2}$$

after premultiplying by $Z^T = [z_1 \ z_2]$, one obtains, after making $z_1 = 1$, $z_2 = \alpha$,

$$-\omega^2 (M + J\alpha^2) + K_{11} + (K_{12} + K_{21})\alpha + K_{22}\alpha^2 = 0 \tag{3}$$

In order to define a correspondence of the generalized representation of a multistory system and a reference system, denote by $Z_1$ and $Z_2$ the first two characteristic vectors, and by $Z_\delta$ and $Z_\theta$ respectively the shapes (or approximations thereof) of the fundamental modes in pure shear and pure torsion (assuming all centers of gravity linearly fixed) of the multistory system under study. If the approximation $Z^T = [Z_\delta^T \ \alpha Z_\theta^T]$ is introduced, the free vibration conditions can be written as

$$-\omega^2 \begin{bmatrix} \widetilde{M} & \\ & \widetilde{J} \end{bmatrix} \begin{Bmatrix} Z_\delta \\ \alpha Z_\theta \end{Bmatrix} + \begin{bmatrix} \widetilde{K}_{11} & \widetilde{K}_{12} \\ \widetilde{K}_{21} & \widetilde{K}_{22} \end{bmatrix} \begin{Bmatrix} Z_\delta \\ \alpha Z_\theta \end{Bmatrix} = 0 \tag{4}$$

where $\widetilde{M}, \widetilde{J}, \widetilde{K}_{11}, \widetilde{K}_{12}, \widetilde{K}_{21}$ and $\widetilde{K}_{22}$ are matrices.

Premultiplying by $Z^T$

$$-\omega^2 (Z_\delta^T \widetilde{M} Z_\delta + \alpha^2 Z_\theta^T \widetilde{J} Z_\theta) + Z_\delta^T \widetilde{K}_{11} Z_\delta + (Z_\theta^T \widetilde{K}_{21} Z_\delta + Z_\delta^T \widetilde{K}_{12} Z_\theta)\alpha + Z_\theta^T \widetilde{K}_{22} Z_\theta \alpha^2 = 0 \tag{5}$$

The correspondence with the reference system is obvious. Hence, amplification results available for the single-story system can be extended to the multistory system.

Because a simplified procedure is aimed for, where no conventional computation of modes $Z_\delta$ and $Z_\theta$ is required, a reasonable shape must be assumed for them: perhaps linear, or proportional to the static displacements under the $Q_i$'s. Determination of the generalized masses $Z_\delta^T \widetilde{M} Z_\delta$ and $Z_\theta^T \widetilde{J} Z_\theta$ is straightforward and does not make use of quantities not obtained in ordinary practice. Computation of generalized stiffnesses is expressed in terms of stiffness submatrices, but can easily be expressed interms of virtual works: take for instance $Z_\theta^T \widetilde{K}_{21} Z_\delta = Z_\theta^T Q_\delta$, where $Q_\delta$ is the vector of forces associated with the approximation to the translational mode. The corresponding virtual work can be determined in terms of either internal or external forces.

The method proposed is as follows:

a) Lateral forces $Q_i$ are applied; $\delta_i$, $\theta_i$ and $T_i$ are computed.

b) Generalized mass and stiffness matrices, as used in eq 5, are obtained.

c) Dynamic amplification factor of static torsional couple $T_E$ with respect to the center gravity of the floor mass in a single story structure and the corresponding ratio of dynamic to static shears are determined, either by means of approximate expressions or through application of figures such as 2-6, to be discussed.

d) Factors determined as in the previous paragraph are applied to all $T_i$'s and $Q_i$'s, respectively.

## DYNAMIC AMPLIFICATION OF TORSION FOR DESIGN PURPOSES

Attention is focused again on fig 1. The dynamic amplification factor of torsion is traditionally computed as the ratio of the dynamic eccentricity (dynamic torsional moment $T_D'$ with respect to the twist center divided by dynamic shear $V_D$) to the static eccentricity e. For this purpose, $T_D'$ and $V_D$ are obtained by superposition of the corresponding modal contributions according to the following equation (Newmark and Rosenblueth, 1971):

$$R^2 = \sum_i R_i^2 + \sum_{i \neq j} \sum \frac{R_i R_j}{1 + \epsilon_{ij}^2} \qquad (6)$$

Here, R is the response of interest, $R_i$ the corresponding contribution of the i-th mode (with a properly chosen sign) and $\epsilon_{ij}$ a quantity that accounts for the stochastic correlation between the rimes at which $R_i$ and $R_j$ occur. Figures 2 and 3 were obtained in accordance with the above criterion, both for a hyperbolic and for constant acceleration spectrum, taking R equal successively to the dynamic lateral force $Q_D$ and the dynamic torsional couple $T_D'$ with respect to the twist center. However, in order to compare these figures with figs 4-6, they are plotted in terms of $Q_D$ and $T_D$, referred to the center of mass. It is implied, for instance, that design shear forces $V_1$ and $V_2$ for the respective diaphragms in fig 1a should be obtained from considerations of static equilibrium in terms of $Q_D$ and $T_D$. Results clearly differ from the case when R in eq 6 is either $V_1$ or $V_2$, that is, when modal superposition is directly performed for internal forces at the diaphragms, because "maximum probable" values of $Q_D$ and $T_D'$ or $T_D$ do not occur simultaneously and because, in addition, $V_1$ and $V_2$ do not necessarily take place simultaneously with either of them. Figs 4 and 5 show what is obtained when $V_1$ and $V_2$ are directly computed with eq 6 and $Q_D$ and $T_D$ are obtained from them by statics.

Discrepancies are obvious. Take for instance the case when $\eta = 1$ and $e^2\ M/J = 0.03$. From figs 2 and 3 obtains $V_1 = 0.28$, $V_2 = 0.61$, whereas from figs 4 and 5, $V_1 = 0.5$ and $V_2 = 0.5$. Suppose that stiffnesses were given by the elements shown in fig 1b, instead of those in fig 1a, and that $Q_D$ and $T_D$ are obtained by statics from the values of the shears acting on each diaphragm determined directly by means of eq 6. The ratio of dynamic to static shear is in this case equal to that given by fig 2. The ratio of dynamic to static torsional couples, however, differs from fig 3; it is shown in fig 6. Discrepancies observed call the attention to the need for revising usual criteria for the specification of design torsional eccentricities. This, however, does not affect the validity of the comparison made in the sequel between "exact" and approximate values of the response of asymmetric multistory structures.

## RESPONSE OF MULTISTORY SYSTEMS

The approximate procedure described above was calibrated by comparing its results with the "exact" values obtained through application of modal analysis (with one translational and one torsional degrees of freedom per floor) to a number of multistory systems, covering a reasonably wide range of mechanical properties. Only the constant acceleration spectrum was considered. All stories in all systems were supposed to have a plan similar to fig 1a, with story stiffness and eccentricity (defined as the distance from the center of figure to the center of stiffness) as given in table 1. All masses were equal in a given system and b/a was taken as 2 in all cases.

"Exact" values of dynamic lateral forces and torsional couples acting at the centers of mass of all floors were obtained by statics from the "exact" design shears at both walls at all stories, directly computed by means of eq 6. Ratios of "exact" to approximate values are shown in fig 7. It is seen that the proposed method leads to design shears which are in general greater than those given by exact modal analysis, but deviations are clearly within tolerable limits for practical applications.

## CONCLUSIONS

Dynamic torsional response of multistory systems can be predicted by extrapolation of results applicable to single-story systems. Definitions of generalized masses, moments of inertia, stiffnesses and static torsions that permit such extrapolation are proposed. Story shears and twisting moments for design must be defined in such a manner as to lead, through conventional methods of structural analysis, to adequate design values of forces acting on the resisting elements.

## REFERENCE


1. Newmark, N M and Rosenblueth, E, *Fundamentals of earthquake engineering*, Prentice Hall, Englewood Cliffs, N. J. (1971)


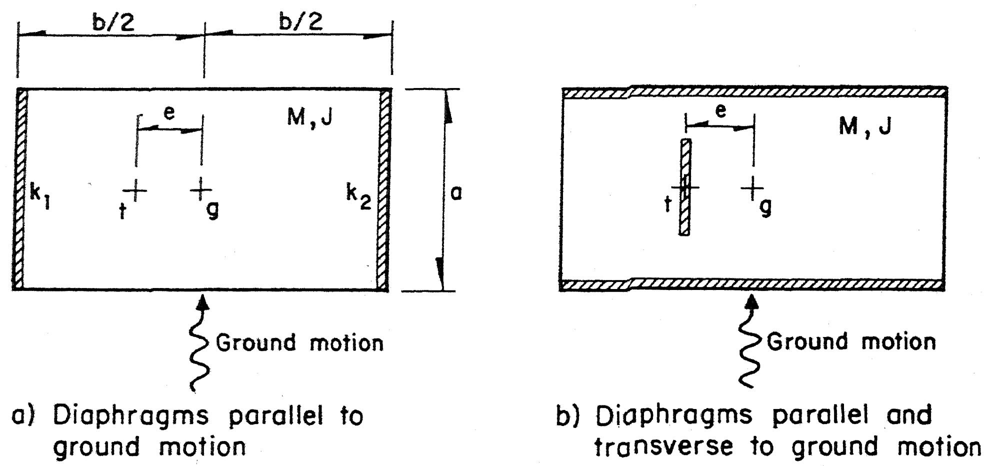


Fig 1. Reference single-story systems

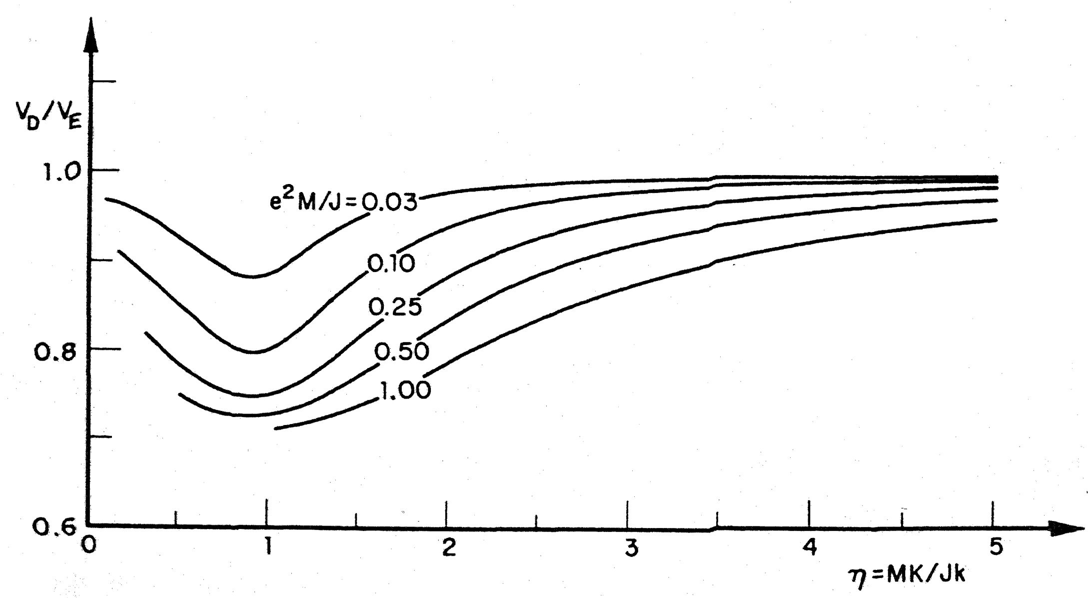


Fig 2. Ratio of dynamic to static shear. System Ib. Constant acceleration spectrum

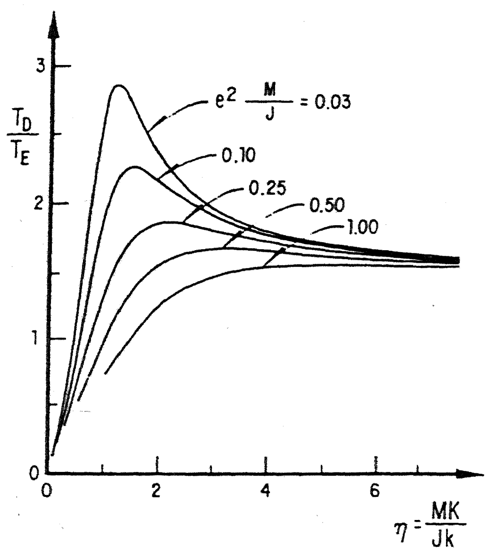


Fig 3. Ratio of dynamic to static torsion. System Ib. Constant acceleration spectrum

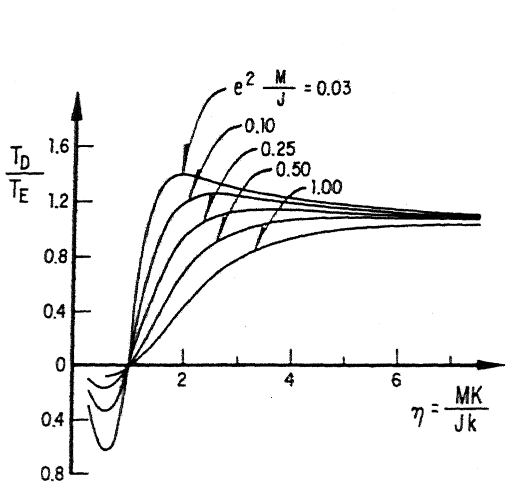


Fig 4. Ratio of dynamic to static torsion according to proposed mode-superposition criterion System Ia. Constant acceleration spectrum

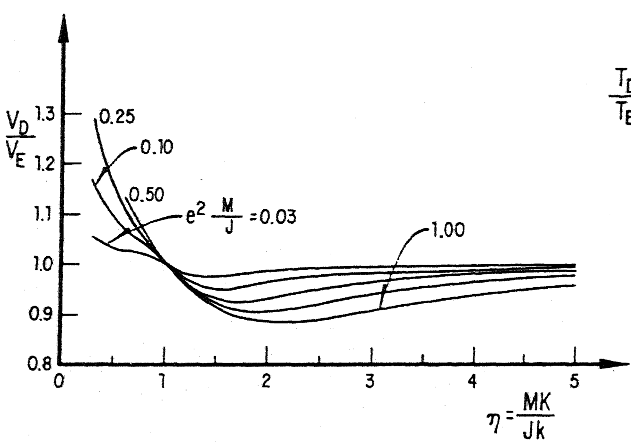


Fig 5. Ratio of dynamic to static shear according to proposed mode – superposition criterion. System Ia Constant acceleration spectrum

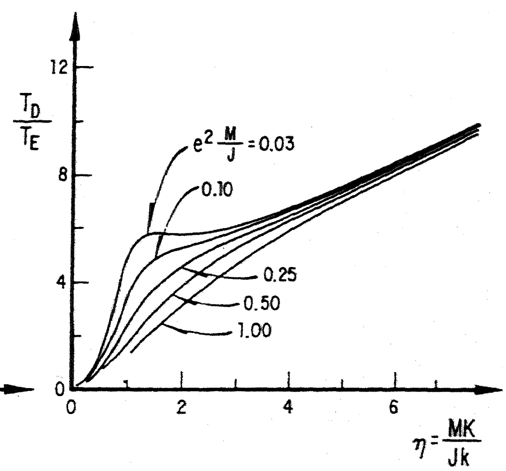


Fig 6 Ratio of dynamic to static torsion according to proposed mode-superposition criterion. System Ib. Constant acceleration spectrum

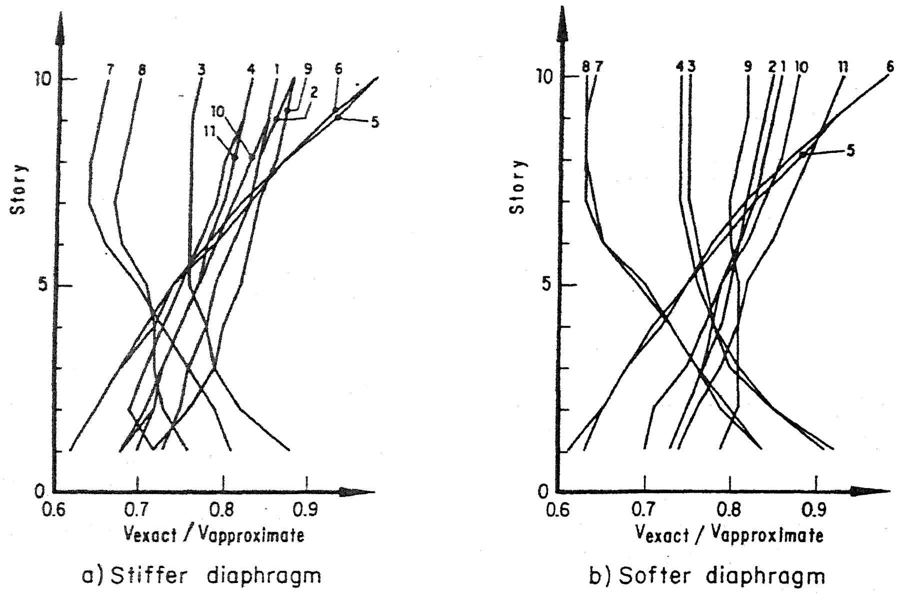


Fig 7. Ratio of exact to approximate diaphragm shear

**TABLE 1. SYSTEMS STUDIED**

| system / story | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 9 | | 10 | | 11 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | K | e/b | K | e/b | K | e/b | K | e/b | K | e/b | K | e/b | K | e/b | K | e/b | K | e/b | K | e/b | K | e/b |
| 1 | 1.0 | 0.1 | 1.0 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 0.1 | 0.1 | 0.1 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 1.0 | 0.0 | 1.0 | 0.3 | 1.0 | 0.3 |
| 2 | 1.0 | 0.1 | 1.0 | 0.3 | 0.9 | 0.1 | 0.9 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 1.0 | 0.0 | 1.0 | 0.3 | 1.0 | 0.0 |
| 3 | 1.0 | 0.1 | 1.0 | 0.3 | 0.8 | 0.1 | 0.8 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 1.0 | 0.0 | 1.0 | 0.3 | 1.0 | 0.0 |
| 4 | 1.0 | 0.1 | 1.0 | 0.3 | 0.7 | 0.1 | 0.7 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 1.0 | 0.0 | 1.0 | 0.3 | 1.0 | 0.0 |
| 5 | 1.0 | 0.1 | 1.0 | 0.3 | 0.6 | 0.1 | 0.6 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 1.0 | 0.0 | 1.0 | 0.3 | 1.0 | 0.0 |
| 6 | 1.0 | 0.1 | 1.0 | 0.3 | 0.5 | 0.1 | 0.5 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 0.1 | 0.1 | 0.1 | 0.3 | 1.0 | 0.3 | 1.0 | 0.0 | 1.0 | 0.0 |
| 7 | 1.0 | 0.1 | 1.0 | 0.3 | 0.4 | 0.1 | 0.4 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 0.1 | 0.1 | 0.1 | 0.3 | 1.0 | 0.3 | 1.0 | 0.0 | 1.0 | 0.0 |
| 8 | 1.0 | 0.1 | 1.0 | 0.3 | 0.3 | 0.1 | 0.3 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 0.1 | 0.1 | 0.1 | 0.3 | 1.0 | 0.3 | 1.0 | 0.0 | 1.0 | 0.0 |
| 9 | 1.0 | 0.1 | 1.0 | 0.3 | 0.2 | 0.1 | 0.2 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 0.1 | 0.1 | 0.1 | 0.3 | 1.0 | 0.3 | 1.0 | 0.0 | 1.0 | 0.0 |
| 10 | 1.0 | 0.1 | 1.0 | 0.3 | 0.1 | 0.1 | 0.1 | 0.3 | 1.0 | 0.1 | 1.0 | 0.3 | 0.1 | 0.1 | 0.1 | 0.3 | 1.0 | 0.3 | 1.0 | 0.0 | 1.0 | 0.0 |