DYNAMIC ANALYSIS OF CYLINDRICAL SHELLS CONTAINING LIQUID

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SYNOPSIS

The free oscillations of thin cantilever cylindrical shells partially filled with liquid are studied. The shells considered are comparatively long and the assumed displacement functions are of beam type mode in longitudinal direction. Donnell's equilibrium equations are used. The equations for the natural frequencies are derived from the variational equation by the Rayleigh-Ritz method and the effect of the internal liquid on the shell mode shapes is considered by the added hydrodynamic mass. The results of some computations are presented.

GLOSSARY OF TERMS

a	= radius of the shell
U,V,W	= shell displacements u,v,w, nondimensionalized by the radius a
$F_n(X)$	= nth axial mode shape of an empty shell $f_n(x)$, nondimensionalized by the radius a
H	= h/a, nondimensional liquid depth
H_s	= h_s/a, nondimensional thickness of shell
L	= l/a, nondimensional length of shell
E	= modulus of elasticity
P_r	= p_r/E, nondimensional pressure loading on shell
ρ_L	= mass density of liquid
ρ_s	= mass density of shell
β	= $\rho_L a/\rho_s h_s$, density parameter
Ω^2	= designated frequency, nondimensionalized by $\frac{(1-\nu^2)a^2\omega^2}{E}$
τ	= ωt, nondimensional time
X,θ,R	= cylindrical coordinates, origin at the bottom center. X and R are nondimensionalized by a

1. INTRODUCTION

The interaction of liquid and the elastic containers plays an important role in the dynamic response under earthquake excitations. In general, since the frequencies of the first few dominant modes of sloshing liquid are usually much smaller than the coupled natural frequency, the shell has been assumed as rigid. However in the case of comparatively long cylindrical shells or in the case of nonlinear problems, the containers must be considered as thin elastic shells which interact with liquid. In this interaction problem many theories have been studied through the breathing vibration analysis for the fuel tanks of rockets. Lindholm et al. reported the distortion of axial patterns from empty shell modes for various liquid -shell height ratio in the experiment[1]. Keeping in mind this distortion, Leory has presented that the displacements of shells partially filled with liquid as linear combinations of all the natural modes for empty shells are in good agreement with the experiment[2]. However, as these investigations were related with the subjects of fuel tanks, the boundary conditions were considered as freely supported ends and breathing vibrations at high

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frequencies were considered. Among others, Baron et al. presented the equations of vibrations for free-simple supported shells partially filled with liquid by use of Lagrange 's equation.[3] Arya et al. studied the coupled system for free-fixed cylindrical shells by the approximate application of additional virtual mass given by Baron et al.[4]

The purpose of this paper is to present the coupling phenomena of shells and liquid. Displacements for the axial displacement mode are assumed as the same for cantilever beams. The accurate potential theory which is compatible with wetted surface condition of free-fixed shells, are used to obtain the additional liquid mass for the system. Taking into consideration these, a variational equation which yields frequency equations is used and the distortions of axial displacement modes from empty shell modes are discussed. Since linear shell theories and linear potential flow theories are assumed in this paper, nonlinear effects are neglected and shells are assumed as comparatively long.

2. NATURAL FREQUENCIES

For containers, thin cylindrical shells are considered and the following dynamic equations are derived by adding the inertia force to the Donnell's equilibrium equations.

$$F_1 = L_{11}U + L_{12}V + L_{13}W - \Omega^2\frac{\partial^2 U}{\partial \tau^2} = 0 \qquad (1.a)$$

$$F_2 = L_{21}U + L_{22}V + L_{23}W - \Omega^2\frac{\partial^2 V}{\partial \tau^2} = 0 \qquad (1.b)$$

$$F_3 = L_{31}U + L_{32}V + L_{33}W - \Omega^2\frac{\partial^2 W}{\partial \tau^2} + \frac{(1-\nu^2)}{H_s}P_r$$

where

$$L_{11}=\frac{\partial^2}{\partial X^2}+\frac{(1-\nu)}{2}\frac{\partial^2}{\partial\theta^2},\quad L_{12}=\nu\frac{\partial^2}{\partial X\partial\theta}+\frac{(1-\nu)}{2}\frac{\partial^2}{\partial X\partial\theta},\quad L_{13}=\nu\frac{\partial}{\partial X}$$

$$L_{21}=\nu\frac{\partial^2}{\partial X\partial\theta}+\frac{(1-\nu)}{2}\frac{\partial^2}{\partial X\partial\theta},\quad L_{22}=\frac{\partial^2}{\partial\theta^2}+\frac{(1-\nu)}{2}\frac{\partial^2}{\partial X^2},\quad L_{23}=\frac{\partial}{\partial\theta}$$

$$L_{31}=-\nu\frac{\partial}{\partial X},\quad L_{32}=-\frac{\partial}{\partial\theta},\quad L_{33}=-(\mu\nabla^4+1)$$

$$U=u/a,\ V=v/a,\ W=w/a,\ X=x/a,\ R=r/a,\ H_s=h_s/a,\ P_r=p_r/E,\ \tau=\omega t,\ \mu=H_s^2/12$$

$$\beta=\rho_L a/(\rho_s h_s),\ \Omega^2=(1-\nu^2)a^2\omega^2\rho_s/E,\ \nabla^4=\frac{\partial^4}{\partial X^4}+2\frac{\partial^4}{\partial X^2\partial\theta^2}+\frac{\partial^4}{\partial\theta^4}$$

The above equations are nondimensionalized. U,V,W(positive outward) are the displacements in the cylindrical coordinates X(positive upward),θ,R respectively and the shell model used in this paper is illustrated in Fig. 1; t is the time; h_s $_s$ are the thickness and density of the shell, respectively; E, are the modulus of elasticity and Poisson's ratio; a is the radius of the middle surface of the shell; P_r is the nondimensional hydrodynamic loading which appears only in the radial equilibrium equation because of the assumptions of neglecting the viscousity of liquid and considering the lateral vibrations of shells. This pressure will be derived later by using additional liquid mass M_{mnn} as follows:

$$\frac{(1-\nu^2)}{H_s}P_r = \sum_m\sum_n\sum_{\hat{n}}\Omega^2\beta M_{mn\hat{n}}F_n(X)\cos(m\theta)W_{mn} \qquad (2)$$

If the boundary conditions for the shell are assumed free-fixed ends and the shell is comparatively long, the following conditions are obtained

$$U = V = W = \frac{\partial W}{\partial X}=0 \qquad \text{at } X=0 \qquad (3.a)$$

$$N_x=N_x=\frac{\partial^2 W}{\partial X^2}=\frac{\partial^3 W}{\partial X^3}=0 \qquad \text{at } X=L \qquad (3.b)$$

where N_x,N_x , are stress resultants and nondimensionalized by $\frac{Eh_s}{(1-\nu^2)}$; L is nondimensional height of the shell.

If the condition $|\lambda_i|a/m^2l^2 \ll 1$ by Yu are assumed, the empty shell oscillates in cantilever beam modes in longitudinal direction, so that the displacements for the shell satisfying (3.a),(3.b) and partially filled with liquid are represented by the linear combinations of the empty shell modes as follows assuming simple harmonic motion:

$$U(X,\theta,\tau)=\sum_m\sum_n \bar{U}_{mn}(\tau)F_n'(X)\cos(m\theta)=\sum_m\sum_n U_{mn}F_n'(X)\cos(m\theta)\cos(\tau) \quad (4.a)$$

$$V(X,\theta,\tau)=\sum_m\sum_n \bar{V}_{mn}(\tau)F_n(X)\sin(m\theta)=\sum_m\sum_n V_{mn}F_n(X)\sin(m\theta)\cos(\tau) \quad (4.b)$$

$$W(X,\theta,\tau)=\sum_m\sum_n \bar{W}_{mn}(\tau)F_n(X)\cos(m\theta)=\sum_m\sum_n W_{mn}F_n(X)\cos(m\theta)\cos(\tau) \quad (4.c)$$

where

$$F_n(X)=\cosh(\bar{n}_n aX)-\cos(\bar{n}_n aX)-k_n(\sinh(\bar{n}_n aX)-\sin(\bar{n}_n aX)),\quad F_n'(X)=\frac{dF_n(X)}{\bar{n}_n a\, dX}$$

$$k_n(X)=\cosh(\bar{n}_n l)+\cos(\bar{n}_n l)/(\sinh(\bar{n}_n l)+\sin(\bar{n}_n l))$$

the values of $n_n l$ are given by $\cos(\bar{n}_n l)\cosh(\bar{n}_n l)=-1$, whose roots are

$$\bar{n}_n l=1.8751, 4.6941, 7.8548, 10.995, 14.137, 17.279.20.420, 23.562, \cdots\cdots$$

To derive the frequency equations, variational equations which obtained by integrations of the equations of motion with respect to time, circumferential and axial coordinatea are used:

$$\int_{t_0}^{t_1}\int_0^{2\pi}\int_0^{l}(\ F_1\delta U + F_2\delta V + F_3\delta W\ )dXd\theta dt = 0 \quad (6)$$

where U, V, W are the variations of the displacement equations (4.a),(4.b) (4.c). Substituting the assumed displacement functions and their variations and integrating with respect to X and θ, the following equations are obtained.

$$\Big[(-K_{n'}^2H_{1n'n'} + \frac{(1-\nu)}{2}m^2)U_{mn'} + (-\frac{(1+\nu)}{2}mK_{n'})V_{mn'} + (-\nu K_n)W_{mn'} - \Omega^2U_{mn'}\Big]\delta U_{mn'} +$$

$$+\Big[(\frac{(1+\nu)}{2}mK_{n'}H_{2n'n'})U_{mn'} + (m^2-\frac{(1-\nu)}{2}K_{n'}^2H_{2n'n'})V_{mn'} + mW_{mn'} - \Omega^2V_{mn'}\Big]\ \delta V_{mn'} +$$

$$+\Big\{(\nu K_{n'}H_{2n'n'})U_{mn'} + mV_{mn'} + \Big[1+\frac{H_s^2}{12}(K_{n'}^4+m^4-2m^2K_{n'}^2H_{2n'n'})\Big]W_{mn'} -$$

$$-\sum_n\Omega^2(\delta_{nn'}+ M_{mn'n})W_{mn'}\Big\}\delta W_{mn'} = 0 \qquad (N\geq n;\ n'=1 \text{ to } N) \quad (7)$$

where $K_{n'}=\bar{n}_{n'}a$, $H_{1n'n'}=\int_0^l F_{n'}(X)F_{n'}''(X)dX/\int_0^l(F_{n'}(X))^2dX$,

$$F_{n'}^{(n)}=\frac{d^nF_{n'}(X)}{dX^n}/(\bar{n}_{n'}a)^n\ ,\quad H_{2n'n'}=\int_0^l F_{n'}'''(X)F_{n'}'(X)dX/\int_0^l(F_{n'}(X))^2dX$$

Since the variations U_{mn}, V_{mn}, W_{mn}, are arbitrary, the equations in the braces must be zero indivisually, so that the homogeneous, linear frequency equations are obtained. These equations can be expressed in the following matrix form.

$$\begin{bmatrix}[A_1] & [B_1] & [C_1]\\ [A_2] & [B_2] & [C_2]\\ [A_3] & [B_3] & [C_3]\end{bmatrix}\begin{Bmatrix}\{U_{mn}\}\\ \{V_{mn}\}\\ \{W_{mn}\}\end{Bmatrix} - \Omega^2\begin{bmatrix}[I] & [0] & [0]\\ [0] & [I] & [0]\\ [0] & [0] & [M_3]\end{bmatrix}\begin{Bmatrix}\{U_{mn}\}\\ \{V_{mn}\}\\ \{W_{mn}\}\end{Bmatrix} = 0 \quad (8)$$

If shell's displacements are considered in N terms of empty shell modes, the coefficients matrix of the unknown U_{mn}, V_{mn}, W_{mn}, are 3N x 3N matrix indivisually and each submatrix are given by(7) directly. The first coefficent matrix of equation (8) is obtained from the corresponding static equilibrium equations and the second coefficent matrix is obtained from the inertia forces. The additional liquid mass given by liquid pressure is considered in matrix M_3 which contains nondiagonal elements. Finally the equation (7) can be written in the following equation.

$$\Big[[K]\ - \Omega^2\ [M]\Big]\cdot\{Q\}\ = 0 \quad (9)$$

This is the frequency equation of the coupling motion of shells and contained liquid. Substituting the displacement equations (3),we obtain the vibration modes.

3. GENERALIZED ADDED MASS OF LIQUID

The fluid is assumed to be nonviscous, irrotational and incompressible. The velocity potential corresponding to the mn-th component of shell motion must satisfy the Laplace equation.

$$\nabla^2\phi_{mn} = (\partial^2/\partial r^2 + \partial/r\partial r + \partial^2/r^2\partial\theta^2 + \partial^2/\partial x^2)\phi_{mn} = 0 \qquad (10)$$

Three of the boundary conditions associated with this element are

$$\partial\phi_{mn}/\partial r = \partial W_{mn}/\partial t = f_n(x)\cos(m\theta)\partial W_{omn}/\partial t \qquad \text{at } r=a \qquad (11.a)$$

$$\partial\phi_{mn}/\partial x = 0 \qquad \text{at } x=0 \qquad (11.b)$$

$$\partial^2\phi_{mn}/\partial t^2 + g\,\partial\phi_{mn}/\partial x = 0 \qquad \text{at } x=h \qquad (11.c)$$

where g is acceleration of gravity. The solution of equation (10) are separated into the particular solution ϕ_p, which satisfies the nonhomogeneous boundary condition (11.a), and the complementary solution ϕ_c, which satisfies homogeneous boundary condition on the wetted surface which is the same for rigid containers.

$$\phi_{mn} = \phi_p + \phi_c = \sum_{k=0}^{\infty} D_{kn}(t) I_m(k\pi r/h)\cos(k\pi x/h)\cos(m\theta) - \sum_{m'}\sum_{j=0} E_{m'jn}(t) J_{m'}(\lambda_{m'j} r/a)\cosh(\lambda_{m'j} x/a)\cos(m'\theta) \qquad (12)$$

where $I_m(k\pi r/h)$, $J_{m'}(\lambda_{m'j} r/a)$ are modified and unmodified Bessel functions of the first kind, respectively; $\lambda_{m'j}$ is j-th root of the equation $J'_{m'}(\lambda_{m'j})=0$. The general solution of this equation has been given by Chu.[5]

$$\phi_{mn} \sum_{k=0} D_{kn} R_{mk}\cos(k\pi x/h)\cos(m\theta)(a\,dW_{omn}/dt) - \sum_{j=1}\left(\frac{\omega^2}{\omega^2-\omega_{mj}^2}\right)\frac{\sum_{k=0}(-1)^k D_{kn}E_{kj}}{\cosh(\lambda_{mj}h/a)} J_m(\lambda_{mj} r/a)\cosh(\lambda_{mj} x/a)\cos(m\theta)\left(a\frac{dW_{omn}}{dt}\right) \qquad (13)$$

where

$$D_{kn} = \int_0^h f_n(x)\cos(k\pi x/h)\,dx\,\frac{2}{h(1+\delta_{ok})}, \qquad R_{mk} = \frac{I_m(\xi_k r/a)}{\xi_k I'_m(\xi_k)}$$

$$E_{kj} = \frac{\int_0^1 R_{mk} J_m(\lambda_{mj}\frac{r}{a})\frac{r}{a}\,d(\frac{r}{a})}{\int_0^1 J_m^2(\lambda_{mj}\frac{r}{a})\frac{r}{a}\,d(\frac{r}{a})}\ (j>0); \quad = 2/\xi_k \quad (j=0, k>0); \quad =0 \quad (j=0, k=0)$$

$$\xi_k = k\pi a/h, \qquad \omega^2_{mj} = g\lambda_{mj}\tanh(\lambda_{mj}h/a)/a$$

where ω_{mj} is j-th natural frequency of $\cos(m\theta)$ mode of liquid. Here the dimensionalized equation(4.d) which is the displacement function of the cantilever shell is used in $f_n(x)$ of the equation(13).

The linearlized liquid pressure which corresponds to mn-th displacement mode is given by Bernoulli equation.

$$p_{mn} = -\rho_L \frac{\partial\phi_{mn}}{\partial t} \qquad \text{at } r=a \qquad (14)$$

Generalizing the pressure loading p_{mn} in the longitudinal direction with the weighting function $f_n(x)$, the equivalent added mass M_{mnn} are calculated from the following equation.

$$p_{mnn} = \frac{\int_0^l f_{n'}(x) p_{mn}\,dx}{\int_0^l f_{n'}(x)^2\,dx} = -\rho_L a M_{mnn'}\frac{d^2 W_{mn}}{dt^2} \qquad (15)$$

where M_{mnn} has the same form as the Chu et al.'s equation except the integrations with the dimensionalized function (4.d). (see eq.16 of ref.5) From this general mass, the liquid pressure p_r can be calculated as follows:

$$p_r = \sum_m\sum_n\sum_{n'} p_{mnn} = \sum_m\sum_n\sum_{n'} \rho_L a\omega^2 W_{mn} M_{mnn'} F_{n'}(X)\cos(m\theta) \qquad (16)$$

Finally, the load in the equation (1.c) takes the following form.

$$\frac{(1-\nu^2)}{Hs}p_r = \frac{1-\nu^2}{Hs}\left(\frac{p_r}{E}\right) = \sum_m\sum_n\sum_{n'} \Omega^2 M_{mnn'}\beta F_{n'}(X)\cos(m\theta)\, W_{mn} \qquad (17)$$

Since the particular solution ϕ_p of the potential ϕ_{mn} are sought in the form $\cos(k\pi x/h)$, orthogonal property is not satisfied, so that the additional mass $M_{m\bar{n}}$ matrix becomes the nondiagonal matrix and the influence of internal liquid to the shell modes are observed.

4. NUMERICAL RESULTS AND DISCUSSION

At first to check the accuracy of the present method the frequencies of an empty shell, which has the values $a/h=173$, $l/a=4.38$, $\nu=0.3$, $E=7.2\times10^5$ kg/cm^2, $\rho_s=8.1633\times10^{-6}$kg.sec^2/cm^4, are compared with Weingarten's results for various values of the circumferential and the longitudinal wave numbers m and n. Fig.2 shows that the solutions by Donnell's method give good agreement with the Weingarten's results. Fig.3 and Fig.4 show frequency-depth curves for the same shell used above. Both figures show that the frequencies decrease monotonically with increasing liquid depth. In Fig.3 and Table. 1, the comparisons of the present results with those by Arya's method, in which the virtual liquid mass is derived for a hinged-free cylinder and the displacement functions of the coupled system are assumed as the same for a empty cylinder, is shown. The difference of the two results are seen and the former and the latter results show the tendencies of flexural and shear types, respectively. The influence of the assumed displacement functions for the shell partially filled with liquid is shown by curves of n:1×1 and 8×8 matrix in Fig.3 and Fig.4 respectively, where the circumferential wave number m=3 is taken. In the case of the first longitudinal mode i.e. n=1, the frequency-depth curves show no difference between n:1×1 and 8×8 matrix. In other words, the displacement mode of a shell is as the same as for an empty shell. This fact is also explained in Fig.5-a, where the influence of internal liquid to the shell mode is little except the shell almost filled with liquid. In the case of the second longitudinal mode, n=2, the frequency-depth curvespossesses a kink induced by the distortion of the axial pattern from the empty shell mode. This tendency is also shown in the result by Chu for simply supported shells. In Fig.5-b,5-c the distortions of axial patterns from empty shell mode are shown for various liquid depth, where longitudinal wave numbers are taken as n=2 and n=3 respectively. In both cases the influences of liquid to shell mode are observed, but clear relations between liquid depth and shell modes are not observed from these examples.

5. CONCLUSIONS

For cantilever shells partially filled with liquid, the influences of distortions of axial patterns from empty shell modes are discussed in the frequency-depth relation and in the shell modes in coupled motion with contained liquid by obtaining the added mass of liquid applying the linear combination of natural modes for empty shells. As the result the following conclusions are obtained. The influence of internal liquid to shell mode is little for mode, n=1, but for modes, n>1, those for empty shells can not represent the modes of shells partially filled with liquid.

REFFERENCE

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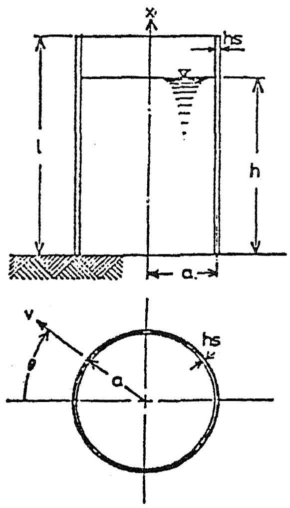


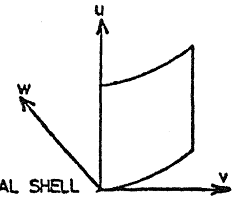
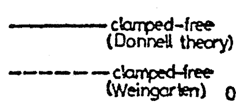


Fig.1 NOMECLATURE FOR CYLINDRICAL SHELL

Water depth ratio, h/L	Frequency of vibration with free-clamped φ	Frequency of vibration by A.S.Arya
0	86.937	83.65
0.332	85.6780	58.150
0.535	74.767	40.450
0.93	38.276	33.120

Table.1 FREQUENCY OF LATERAL VIBRATION
n=1 m=3 (L/a=4.38, a/hs)



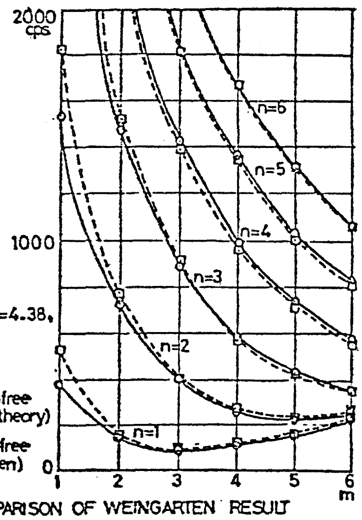


Fig.2 GRAPHICAL COMPARISON OF WEINGARTEN RESULT
(EMPTY SHELL FREQUENCY : l/a=4.38 a/h=173)

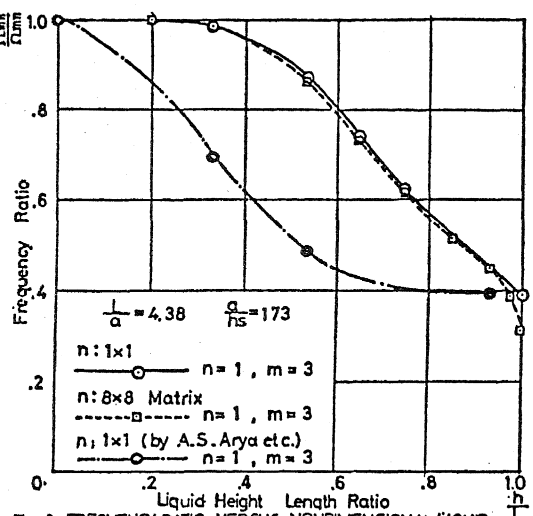


Fig.3 FREQUENCY RATIO VERSUS NONDIMENSIONAL LIQUID DEPTH (n= 1 , m= 3)

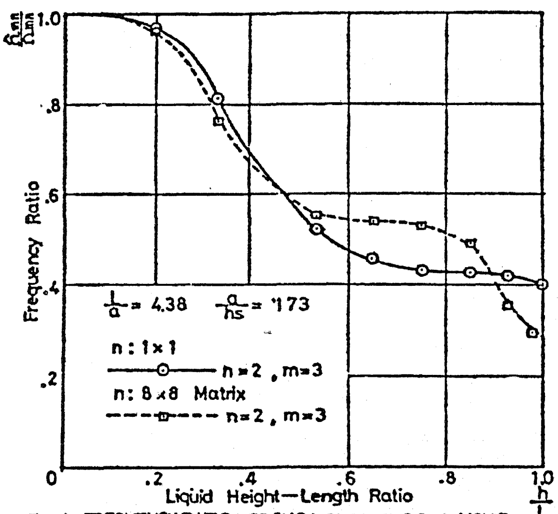


Fig.4. FREQUENCY RATIO VERSUS NONDIMENSIONAL LIQUID DEPTH (n=2 , m=3)

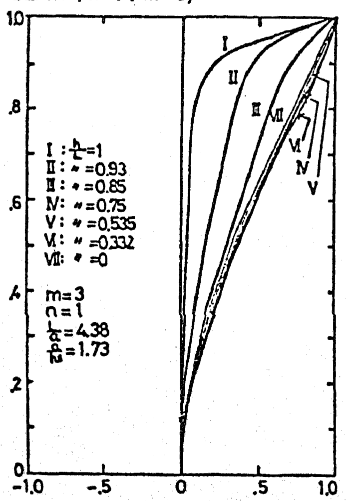


Fig. 5-a FIRST RADIAL MODE SHAPES

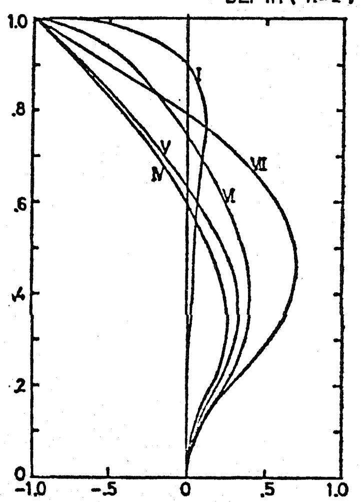


Fig. 5-b SECOND RADIAL MODE SHAPES

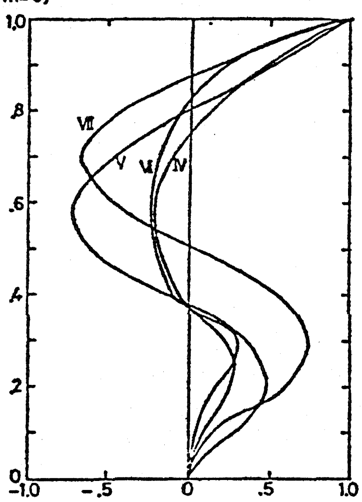


Fig.5-c THIRD RADIAL MODE SHAPES