STOCHASTIC RESPONSE OF STRUCTURE DUE TO SPATIALLY VARIANT EARTHQUAKE EXCITATIONS

by

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SYNOPSIS

The cross spectral density function between spatially variant earthquake excitations is assumed such that the statistic correlation decreases in an exponential manner as the distance increases. The system, where a rigid slab is supported by many columns, bottom ends of which are subjected to multiple inputs which have the above-mentioned characteristic, is supposed. Frequency response functions of the absolute acceleration and of the relative displacement are mathematically derived. Stochastic responses are analytically evaluated when excitations are band limited white noises with such spatial variation. Responses due to noises given concretely as the simulation of typical earthquakes are numerically estimated.

INTRODUCTION

The movement of a ground surface during an earthquake will not be identical even in the relatively limited plane, but possible to have the spatial variation. Such seismic waves can be recognized indirectly by the comparison of an accelerogram recorded in a building foundation with that obtained simultaneously on a ground surface in its vicinity. Through the analysis of those observations, the mathematical form of the cross spectral density function between earthquake motions, where the statistic correlation decreases in an exponential manner as the distance increases, can be regarded as the most realistic idealization.[1)] The purpose of this paper is to investigate theoretically the stochastic response of the structure subjected to multiple excitations which have the above-mentioned characteristic. In the earthquake response analysis of the structure, the foundation is usually assumed to translate uniformly. The spatially variant ground motions, however, should be introduced if the foundation is not rigid enough, since such variation is possible to have a favorable influence upon the aseismic design of the structure.

FREQUENCY RESPONSE FUNCTIONS

Consider an idealized single-story-structure shown in Fig. 1. Massless columns, n in number, which have the same structural properties one another, are located at equal intervals. A massive rigid slab is mounted on them. Bottom ends of columns are subjected to mutually correlated multiple inputs. Dealing only with the horizontal vibration in the longitudinal direction of the structure, this is a single-degree-of-freedom system with a natural circular frequency, ω_b. Now supposed that each foundation has the simultaneous forced displacement, $X_j (j=1, 2, \cdots, n)$, the total displacement, X_t, of the slab is equal to the sum of the static displacement, X_s, which is exactly same as the arithmetical mean of X_j and the dynamical displacement, X_d, governed by the following equation of

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motion, where the viscous damping proportional to $\dot{X}_d$ with a corresponding damping ratio, h, is assumed.

$$\ddot{X}_d + 2h\omega_b\dot{X}_d + \omega_b{}^2X_d = -\frac{1}{n}\sum_{j=1}^{n}\ddot{X}_j. \quad \cdots\cdots (1)$$

If only the j-th foundation is subjected to an unit harmonic acceleration, $\ddot{X}_j = \exp[i\omega t]$ where i is an imaginary unit and t is time, then the frequency response function of $\ddot{X}_t$, ${}_AH_j$, will be given, with $\beta = \omega/\omega_b$, by

$${}_AH_j = B(\beta,h)/n, \quad \cdots\cdots (2)$$

where $$B(\beta,h) = \frac{1 + 2h\beta i}{1 - \beta^2 + 2h\beta i}. \quad \cdots\cdots (3)$$

When an unit harmonic displacement, $X_j = \exp[i\omega t]$, is supposed as an input of the j-th foundation, the frequency response function of the k-th story displacement, ${}_DH_{jk}$, is written, with Kronecker delta, δ_{jk}, as

$${}_DH_{jk} = \frac{1}{n}\cdot B(\beta,h) - \delta_{jk}, \quad \cdots\cdots (4)$$

Under the condition of the stationary random process along with the linear system, the cross spectral density matrix of outputs, $\underline{S}(\omega)$, can be calculated through the frequency response function matrix of the system, $\underline{H}(\omega)$, by the following relationship, when the cross spectral density matrix of n multiple inputs, $\underline{G}(\omega)$, is given.

$$\underline{S}(\omega) = \underline{H}(\omega)\cdot\underline{G}(\omega)\cdot\underline{H}(\omega)^{*T}, \quad \cdots\cdots (5)$$

where $\star$ and T represent a conjugate complex and transposition of matrix, respectively. Now the cross spectral density function between the j-th and the k-th ground motions, $G_{jk}(\omega)$, is assumed as[1)]

$$G_{jk}(\omega) = \exp[-\rho|\xi_{jk}|]\cdot G(\omega), \quad \cdots\cdots (6)$$

where $G(\omega)$ is the power spectral density function common to the excitation of each foundation and $|\xi_{jk}|$ means the distance between the j-th and the k-th points. ρ, which represents the degree of correlation, is a non-negative constant with an unit of reciprocal of length. $\rho = 0$ corresponds to perfect correlation, whereas $\rho = \infty$ to absence of correlation.

Here defining a non-dimensional quantity called "space correlation index" as $\gamma = \rho L$, where L is the total length of a structure, the power spectral density function of the absolute acceleration response, ${}_AS(\omega)$, is obtained, substituting Eqs. (2) and (6) into (5), as

$${}_AS(\omega) = |{}_AF(n,\gamma,\beta,h)|^2\cdot{}_AG(\omega), \quad \cdots\cdots (7)$$

where $$|{}_AF(n,\gamma,\beta,h)| = {}_AR(n,\gamma)\cdot|B(\beta,h)|, \quad \cdots\cdots (8)$$

$${}_AR(n,\gamma) = \sqrt{\frac{1}{n} + \frac{2e^{-\gamma/(n-1)}}{n^2(1 - e^{-\gamma/(n-1)})}\left\{ n - 1 - \frac{e^{-\gamma/(n-1)} - e^{-n\gamma/(n-1)}}{1 - e^{-\gamma/(n-1)}} \right\}}, \quad \cdots\cdots (9)$$

and ${}_AG(\omega)$ is the power spectral density function common to the ground acceleration of each foundation. If, in particular, $n \to \infty$, then ${}_AR(n,\gamma)$ given by Eq. (9) becomes of much more simple form as

$${}_AR(\gamma) = \frac{1}{\gamma}\sqrt{2(e^{-\gamma} + \gamma - 1)}. \quad \cdots\cdots (10)$$

${}_AR(\gamma)$ is also equal to the absolute value of the transfer function of the soil-rigid foundation system without a flexible superstructure.[1)] The relationship between this revised coefficient, ${}_AR(n,\gamma)$, and the space correlation index, γ, with the number of columns, n, is displayed in Fig. 2. ${}_AR(n,\gamma)$ monotonously decreases to the limiting value $1/\sqrt{n}$, as γ increases. Since γ is considered to be less than, say, 3, in general,[1)] ${}_AR(n,\gamma)$ does not depend so much upon n, and therefore can be replaced nearly by ${}_AR(\gamma)$ given by Eq. (10). In addition to that, when n is greater than merely 3, ${}_AR(n,\gamma)$ is almost equivalent to ${}_AR(\gamma)$ even if γ is rather high. After all the revised coefficient can be approximated simply by ${}_AR(\gamma)$ regardless of the number of columns.

Now introduce a non-dimensional quantity, λ, which represents the horizontal coordinate normalized by L as shown in Fig. 1. λ is related to the column-number as $\lambda = (k-1)/(n-1)$. Substituting Eqs. (4) and (6) into (5), the power spectral density function of the k-th story displacement response, ${}_DS_k(\omega)$, then becomes

$$ {}_DS_k(\omega) = \left|{}_DF(n,\lambda,\gamma,\beta,h)\right|^2 \cdot {}_DG(\omega), \quad \cdots\cdots (11) $$

where

$$ \left|{}_DF(n,\lambda,\gamma,\beta,h)\right| = \sqrt{1 + {}_AR(n,\gamma)^2 \cdot |B(\beta,h)|^2 - Q(n,\lambda,\gamma)\cdot \mathrm{Re}[B(\beta,h)]}, \quad \cdots\cdots (12) $$

$$ Q(n,\lambda,\gamma) = \frac{2}{n} + \frac{2e^{-\gamma/(n-1)}}{n}\left[\frac{2 - e^{-\lambda\gamma} - e^{-(1-\lambda)\gamma}}{1 - e^{-\gamma/(n-1)}}\right], \quad \cdots\cdots (13) $$

Re[ ] represents the real part of [ ] and ${}_DG(\omega)$ is the power spectral density function common to the ground displacement of each foundation. The value of $|{}_DF|$ slightly changes with a variable n and is almost equivalent to that when $n = \infty$ if n is greater than around 3. Besides the effect of n when $n > 1$ is limited mostly to the range of $\beta < 1$. If $\beta = 0$, in particular, $|{}_DF|$ rapidly converges, as n increases, to

$$ {}_DR(\lambda,\gamma) = \frac{1}{\gamma}\sqrt{2(e^{-\gamma} + \gamma e^{-\lambda\gamma} + \gamma e^{-(1-\lambda)\gamma} - \gamma - 1) + \gamma^2}, \quad \cdots\cdots (14) $$

which is displayed in Fig. 3. If $\gamma \to \infty$, ${}_DR(\lambda,\gamma) \to 1$ with the rather fast convergence. The effect of λ is also limited to the range of $\beta < 1$, and as found from this figure, ${}_DR(\lambda,\gamma)$ is minimum at the center of the structure, while maximum at its edge. The latter is, as the case may be, more than 1.5 times as high as the former in usual values of γ. Fig. 4 shows $|{}_DF|$ with a parameter γ, when $n = \infty$, $\lambda = 0$ and $h = 0.1$. The considerable difference is recognized as γ changes. Especially the increase where $\beta < 1$ together with the decrease when $\beta = 1$ is remarkable. The both approach unity as $\gamma \to \infty$, which means the magnification factor becomes unity all over the frequencies if inputs are uncorrelated. γ acts as a damper, and with the increasing γ, the dynamic effect tends to vanish.

MEAN SQUARE RESPONSES

The power spectral density function common to each ground acceleration, ${}_AG(\omega)$, is assumed to be a band limited white noise with a constant density W over $\omega_l \le |\omega| \le \omega_u$. The mean square response of the absolute acceleration of a rigid slab, ${}_A\sigma^2$, is written from Eq. (7) as

$$ {}_A\sigma^2 = \int_{-\infty}^{\infty} \left|{}_AF\right|^2 \cdot {}_AG(\omega)\,d\omega. \quad \cdots\cdots (15) $$

Utilizing Eqs. (3), (8) and (9) to integrate, the response due to the above-mentioned disturbance, becomes, with $\beta_u = \omega_u/\omega_b$ and $\beta_l = \omega_l/\omega_b$, as

$$ {}_A\sigma^2 = {}_Af(\beta_u) - {}_Af(\beta_l), \quad \cdots\cdots (16) $$

where $$ {}_Af(\beta) = {}_AR(n,\gamma)^2\omega_b W \left[\frac{1-4h^2}{4\sqrt{1-h^2}} \log \frac{\beta^2 + 2\sqrt{1-h^2}\beta + 1}{\beta^2 - 2\sqrt{1-h^2}\beta + 1} + \frac{1+4h^2}{2h}\left(\tan^{-1}\frac{\beta+\sqrt{1-h^2}}{h} + \tan^{-1}\frac{\beta-\sqrt{1-h^2}}{h}\right)\right]. \cdots (17) $$

Especially when $n = 1$, $\beta_l = 0$ and $\beta_u \to \infty$, then Eq. (17) is reduced to

$$ {}_A\sigma^2 = (1 + 4h^2)\pi\omega_b W/(2h), \quad \cdots\cdots (18) $$

which exactly coincides with the conventional result for the perfectly correlated ideal white noise.

On the other hand, the mean square response of the story displacement, ${}_D\sigma^2$, is expressed from Eq. (11) as

$$ {}_D\sigma^2 = \int_{-\infty}^{\infty} |{}_DF|^2 \cdot {}_DG(\omega)\,d\omega. \quad \cdots\cdots (19) $$

The evaluation of this integral using Eqs. (3), (12) and (13) finally gives

$$ {}_D\sigma^2 = {}_Df(\beta_u) - {}_Df(\beta_l), \quad \cdots\cdots (20) $$

where $$ {}_Df(\beta) = \frac{2W}{\omega_b^3}\left[\frac{Q(n,\lambda,\gamma) - 2{}_AR(n,\gamma)^2}{\beta} + \frac{Q(n,\lambda,\gamma) - {}_AR(n,\gamma)^2 - 1}{3\beta^3} - \frac{2(1-2h^2)Q(n,\lambda,\gamma) - (5-8h^2){}_AR(n,\gamma)^2}{8\sqrt{1-h^2}} \log\frac{\beta^2 + 2\sqrt{1-h^2}\beta + 1}{\beta^2 - 2\sqrt{1-h^2}\beta + 1} + \frac{4h^2Q(n,\lambda,\gamma) + (1-8h^2){}_AR(n,\gamma)^2}{4h}\left(\tan^{-1}\frac{\beta+\sqrt{1-h^2}}{h} + \tan^{-1}\frac{\beta-\sqrt{1-h^2}}{h}\right)\right]. $$

$$ \cdots\cdots (21) $$

If particularly $n = 1$, $\beta_l = 0$ and $\beta_u \to \infty$ in Eq. (21), then

$$ {}_D\sigma^2 = \pi W/(2h\omega_b^3), \quad \cdots\cdots (22) $$

which agrees with the result for the perfectly correlated white noise.

NUMERICAL EXAMPLES

Two different band limited white noises shown in Fig. 5 are dealt with as examples. The noise 1 indicated by a solid line lies in a relatively low frequency region comparing with the noise 2 shown by a dotted line which is situated slightly toward the high frequency region. The both have the identical mean square value of acceleration[2], ${}_A\sigma_g^2$, as $1.2 \times 10^4 \mathrm{gal}^2$. As shown in Table 1, however, the mean square value of velocity and of displacement, ${}_V\sigma_g^2$ and ${}_D\sigma_g^2$, respectively, extremely differ each other. ω_b is set to be $6\pi\ \mathrm{sec}^{-1}$ which is located nearly at the center of the bandwidth of each excitation. h is assumed 0.1. n is fixed to be infinity as its representative. The value of λ is taken zero, which makes the displacement response maximum. After all γ is chosen as a variable.

Fig. 6 illustrates the root mean square (R.M.S.) response of the absolute acceleration, ${}_A\sigma$, versus γ. A solid line represents responses due to two noises, which are approximately identical. This is expressed as the product of the response due to the perfectly correlated input, ${}_A\sigma_0$, by the revised coefficient, ${}_{A}R(\gamma)$. Besides ${}_{A}\sigma_{o}$ due to the ideal white noise, computed from Eq. (18) agrees with associated values considered here with only 1% error. Now a chain line in this figure indicates the R.M.S. of the ground acceleration, ${}_{A}\sigma_{g}$, for reference.

Fig. 7 shows the R.M.S. response of the story displacement, ${}_{D}\sigma$, against γ. Two solid lines correspond to responses due to two noises, which nearly agree with each other only when $\gamma = 0$, but considerably discord as γ increases, to approach associated R.M.S. values of the ground displacements, ${}_{D}\sigma_{g}$. Since wide difference exists between ${}_{D}\sigma_{g}$ of two noises, responses have the corresponding discrepancy as such.

As is found from Eq. (21), it is impossible to abstract exactly the revised coefficient as in case of the acceleration response. However, if, as this example, β_1 is relatively small comparing with unity, and if γ has some value with rather large n, then Eq. (20) is approximately reduced to the following simple form.

$$ {}_{D}\sigma \approx {}_{D}R(\lambda,\gamma)\cdot {}_{D}\sigma_{g}. \qquad (23) $$

Two dotted lines in Fig. 7 represent Eq. (23) where $\lambda = 0$. The both well agree with associated solid lines except when γ is quite small.

SUMMARY AND CONCLUSIONS

When the earthquake motion on a free surface spatially varies in such a manner as expressed by Eq. (6), the acceleration induced to the structure decreases by ${}_{A}R(\gamma)$ given by Eq. (10), if the foundation is infinitely rigid. Responses are calculated as traditional outputs due to thus decreased inputs. On the other hand, if each foundation is completely seperated, the story displacement response is influenced by ${}_{D}R(\lambda,\gamma)$ in Eq. (14) together with the displacement of inputs. Therefore the quite different standpoint should be taken in its estimation, although the acceleration response is approximately equal to that in case of the rigid foundation. Most of actual structures belong to the category between these two extreme cases. The space correlation index of the free surface motion, γ, is considered to be divided into following two terms; the space correlation index which contributes to the transfer characteristic from the ground to the foundation, γ_e, and the space correlation index of the input induced to the structure from the foundation, γ_f, which depends upon the rigidity of the foundation. γ_f is close to zero if its rigidity is high enough, whereas to γ if quite low. The acceleration response will not differ so much from that in case of the rigid foundation, but the story displacement response is governed also by the peculiar mechanism with the application of γ_f to Eq. (23). Transfer functions as well as R.M.S. responses are plainly summarized in Table 2. In conclusion, the spatial variation of earthquake excitations is possible to have a great influence upon the response of structures.

REFERENCES

1) Yutaka Matsushima, "Spectra of Spatially Variant Ground Motions and Associated Transfer Functions of Soil-Foundation System," Proceedings of the Fourth Japan Earthquake Engineering Symposium, Nov. 1975
2) Yutaka Matsushima, "Stochastic Response of Structure due to Three-dimensional Earthquake Excitations," Building Research Institute, Research Paper No. 58, Mar. 1974

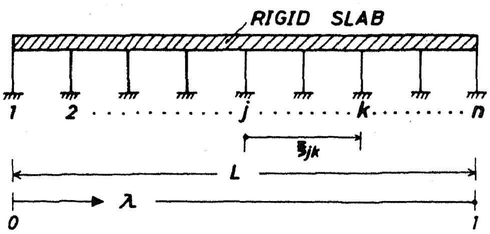

Fig. 1 Idealized structure

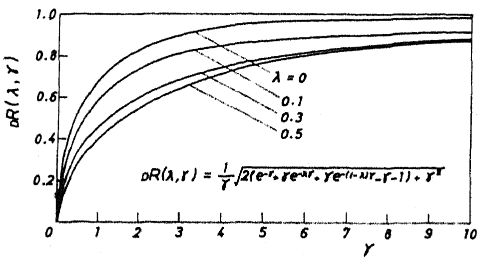

Fig. 3 Revised coefficient of displacement

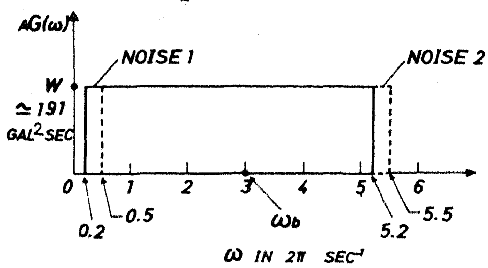

Fig. 5 Two different band limited white noises

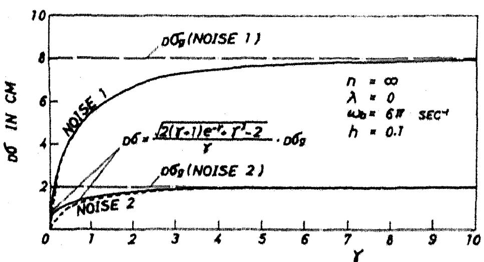

Fig. 7 Root mean square response of displacement

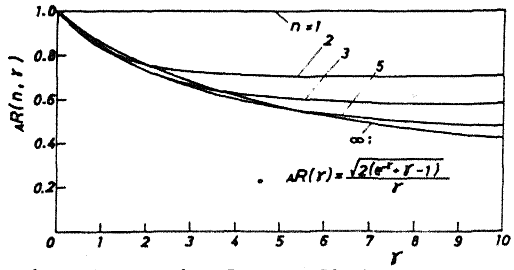

Fig. 2 Revised coefficient of acceleration

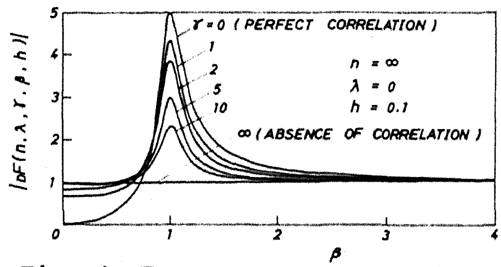

Fig. 4 Frequency response function of displacement

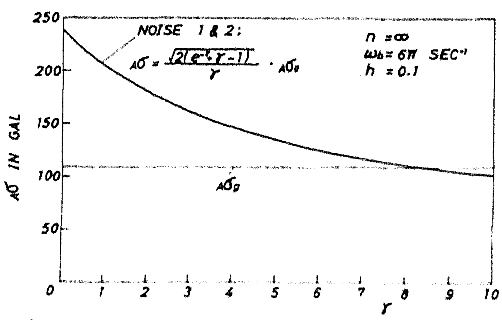

Fig. 6 Root mean square response of acceleration

Table 1. Root mean square values of excitations

	$_A\sigma_g$ gal	$_V\sigma_g$ kine	$_D\sigma_g$ cm
Noise 1	110	17.1	8.01
Noise 2	110	10.5	2.03

Table 2 Transfer function and root mean square response

Type of foundation		Transfer function: Ground → Foundation	Transfer function: Foundation → Structure	Root mean square response
Rigid	Acc.	$_AR(\gamma)$	1	$_AR(\gamma)\cdot{}_A\sigma_o$
	Disp.			$_AR(\gamma)\cdot{}_D\sigma_o$
Flexible	Acc.	$_AR(\gamma e)$	$_AR(\gamma_f)$	$_AR(\gamma e)\cdot{}_AR(\gamma_f)\cdot{}_A\sigma_o$
	Disp.		$_DR(\lambda,\gamma_f)$	$_AR(\gamma e)\cdot{}_DR(\lambda,\gamma_f)\cdot{}_D\sigma_g$
Seperated	Acc.	1	$_AR(\gamma)$	$_AR(\gamma)\cdot{}_A\sigma_o$
	Disp.		$_DR(\lambda,\gamma)$	$_DR(\lambda,\gamma)\cdot{}_D\sigma_g$