APPLICATION OF STOCHASTIC DIFFERENTIAL EQUATIONS TO SEISMIC ANALYSES OF NONLINEAR STRUCTURES

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SYNOPSIS

This paper deals with the methods of solving the Fokker-Planck-Kolmogorov equations with an appropriate initial or end condition and a mixed type boundary condition on a bounded domain, particularly paying attention on the seismic reliability analyses of nonlinear structures. First, as the mathematical bases of this problem, a generalized Green's formula for nonself-adjoint partial differential systems and the associated stationary functional theorems are presented. Then, defining a reliability function in terms of Green's function of the Fokker-Planck-Kolmogorov partial differential system with a bounded boundary surface, some kinds of solution techniques based on the boundary value problem as well as the variational method are discussed.

INTRODUCTION

It is well-known that the dynamic behaviors of a soil-structure system to earthquake excitations are expressed by a set of the first order nonlinear ordinary differential equations including uncorrelated stochastic processes, which is called the stochastic differential equations, taking a sufficient number of state variables and enlarging the system so that the Gaussian white noise processes are reasonably assumed as the inputs to the system. It is also known that if the nonlinearities involved in the equations are of quasi-linear type, the solution is unique and belongs to the Markov vector processes and significant quantities describing the probabilistic state of the system such as the transition probability density function, the probability of exceeding a prescribed barrier surface are governed by the Fokker-Planck-Kolmogorov equations together with pertinent initial and boundary conditions 1], 2]. By considering appropriate additional state variables, a class of hysteretic nonlinear systems is expressed in terms of quasi-linear characteristics 3], 4].

This paper deals with the methods of solving the Fokker-Planck-Kolmogorov equations with an initial or end condition and a mixed type boundary condition on a bounded domain, especially in relation to the seismic reliability analyses of nonlinear structures. An analytical expression of the probability of aseismic safety of single-degree-of-freedom structures was first presented by E. Rosenblueth based on the diffusion equation, which is a special kind of the Fokker-Planck-Kolmogorov equations, together with pertinent initial and boundary conditions 5]. For a general class of linear dynamic systems such as multi-degree-of-freedom structures, however, it is difficult to solve analytically such the linear parabolic partial differential systems involving the boundary conditions on bounded domains. As far as nonlinear dynamic systems are concerned, only a few analytical solution is known even for the steady-state probability density functions 2]. From this aspect, the methods of solving approximately such the Fokker-Planck-Kolmogorov differential systems are discussed here. Prior to the discussions on the seismic reliability analyses based on the Fokker-Planck-Kolmogorov formulation, some mathematical bases concerning a general class of nonself-adjoint partial differential systems are presented.

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MATHEMATICAL BASES

Corresponding to a linear partial differential system governing a physical scalar quantity, there exists the adjoint linear partial differential system defined by the following generalized Green's formula:

$$(v,\mathcal{L}u)_\Omega - (\mathcal{L}^*v,u)_\Omega = C(v,u) = -C^*(u,v) = (\mathcal{B}v,\mathcal{B}u)_{\partial\Omega} - (\mathcal{B}^*v,\mathcal{D}^*u)_{\partial\Omega} \tag{1}$$

in which Ω is a domain included in an n-dimensional full space $\mathcal{R}^n$, $\partial\Omega$ is its boundary surface, u and v are scalar-valued functions defined on $\bar{\Omega}=\Omega\cup\partial\Omega$ and $(\ ,\ )_\Omega$, $(\ ,\ )_{\partial\Omega}$ are scalar products defined on Ω and $\partial\Omega$, respectively. The operators $\mathcal{L}$ and $\mathcal{B}$ are, in general, nonself-adjoint linear partial differential operators associated with equation and boundary condition, respectively, and $\mathcal{L}^*$ and $\mathcal{B}^*$ are their adjoint operators which are uniquely determined together with the relevant boundary operators $\mathcal{D}$ and $\mathcal{D}^*$ from (1). Hence, there exists a pair of nonself-adjoint inhomogeneous partial differential systems, namely

$$\begin{aligned} &\mathcal{L}u + f = 0 \quad \text{on } \Omega, \qquad \mathcal{L}^*v + g = 0 \quad \text{on } \Omega \\ &\mathcal{B}u + k = 0 \quad \text{on } \partial\Omega, \qquad \mathcal{B}^*v + \ell = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2}$$

Correspondingly, a pair of Green's functions, $h(x,\xi)$ and $h^*(x,\eta)$ are defined as the solutions to the systems which are obtained by substituting $f=\delta(x-\xi)$, $k=0$ and $g=\delta(x-\eta)$, $\ell=0$, where $x\in\bar{\Omega}$, $\xi,\eta\in\Omega$ and δ means n-dimensional Dirac's delta-function, into (2), respectively. The reciprocal property of Green's functions for nonself-adjoint systems is obtained by substituting $v=h$ and $u=h^*$ in (1) as follows:

$$h(\xi,\eta) = h^*(\eta,\xi), \qquad \xi,\eta\in\Omega \tag{3}$$

Similarly, by making use of (1) and (3), the integral representation of solution to (2) in terms of Green's functions is obtained as follows:

$$u = (h,f)_\Omega - (\mathcal{D}h,k)_{\partial\Omega}, \qquad v = (g,h^*)_\Omega - (\ell,\mathcal{B}^*h^*)_{\partial\Omega} \tag{4}$$

It should be noted that in the above equations the scalar products and differential operators are concerned with the latter vector variables of the Green's functions. From (4), Green's functions associated with the boundary conditions concerning the operators $\mathcal{B}$ and $\mathcal{B}^*$ are respectively determined by

$$\begin{aligned} &h_b(x,\xi) = -\mathcal{D}_\xi h(x,\xi), \qquad h_b^*(x,\eta) = -\mathcal{D}_\eta^* h^*(x,\eta) \\ &x\in\bar{\Omega}, \qquad \xi,\eta\in\partial\Omega \end{aligned} \tag{5}$$

In particular, in the case where $(\mathcal{L},\mathcal{B},\mathcal{D}) = (\mathcal{L}^*,\mathcal{B}^*,\mathcal{D}^*)$ the differential system is called self-adjoint, and its Green's function becomes symmetric.

For nonself-adjoint inhomogeneous partial differential systems given by (2), the associated variational principles are expressed by the following two stationary functional theorems:

Theorem 1. The stationary point of the functional defined by

$$\begin{aligned} J(v,u) = {} & (v,\mathcal{L}u)_\Omega + (v,f)_\Omega + (g,u)_\Omega \\ & - (\mathcal{D}v,\mathcal{B}u)_{\partial\Omega} - (\mathcal{D}v,k)_{\partial\Omega} - (\ell,\mathcal{B}^*u)_{\partial\Omega} = J^*(u,v) \end{aligned} \tag{6}$$

with respect to v and u is a pair of solutions to (2), and vice versa.

Theorem 2. The stationary points of the functionals defined by

$$J(v;u) = (v,\mathcal{L}u)_\Omega + (v,f)_\Omega - (\mathcal{D}v,\mathcal{B}u)_{\partial\Omega} - (\mathcal{D}v,k)_{\partial\Omega} \tag{7}$$

and

$$J^*(u;v) = (\mathcal{L}^*v,u)_\Omega + (g,u)_\Omega - (\mathcal{B}^*v,\mathcal{B}^*u)_{\partial\Omega} - (\ell,\mathcal{D}^*u)_{\partial\Omega} \tag{8}$$

with respect to v and u, respectively, are the solutions to the first and second systems in (2), and vice versa.

By making use of (1), the first and second variations of the functional defined by (6) are expressed as

$$\delta_{v,u}J(v,u) = \delta_{u,v}J^*(u,v) = (\delta v,\mathcal{L}u+f)_\Omega - (\mathcal{D}\delta v,\mathcal{B}u+k)_{\partial\Omega} \tag{9}$$

and

$$\begin{aligned} \delta^2_{v,u}J(v,u) = \delta^2_{u,v}J^*(u,v) = {} & (\delta v,\mathcal{L}\delta u)_\Omega - (\mathcal{D}\delta v,\mathcal{B}\delta u)_{\partial\Omega} \\ & + (\mathcal{L}^*v+g,\delta u)_\Omega - (\mathcal{B}^*v+\ell,\mathcal{D}^*\delta u)_{\partial\Omega} \\ & + (\mathcal{L}^*\delta v,\delta u)_\Omega - (\mathcal{B}^*\delta v,\mathcal{D}^*\delta u)_{\partial\Omega} \end{aligned} \tag{10}$$

Hence, the arbitrariness of the variations δv and δu in (9) yields the equivalence between (2) and $\delta_{v,u}J(v,u) = \delta_{u,v}J^*(u,v) = 0$. On the other hand, the first variations of the functionals defined by (7) and (8) are respectively given by

$$\delta_v J(v;u) = (\delta v, \mathcal{L}u+f)_\Omega - (\mathcal{B}\delta v, \mathcal{B}u+k)_{\partial\Omega} \tag{11}$$
$$\delta_u J^*(u;v) = (\mathcal{L}^*v+g, \delta u)_\Omega - (\mathcal{B}^*v+\ell, \mathcal{B}^*\delta u)_{\partial\Omega} \tag{12}$$

Hence, the first and second systems in (2) are equivalent to $\delta_v J(v;u)=0$ and $\delta_u J^*(u;v)=0$, respectively. It is noted that solutions to the first and second systems in (2) make not only all orders of variations of the functionals defined respectively by (7) and (8) but also the functionals themselves to be zero. When the inhomogeneous terms in (2) are prescribed, it is always possible to convert the inhomogeneous terms of the boundary conditions to those of equations by introducing appropriate boundary homogenizing functions. In particular, for self-adjoint systems *Theorem* 1 reduces to the following corollary:

Corollary. The stationary point of the functional defined by

$$J(u) = (u, \mathcal{L}u)_\Omega + 2(u,f)_\Omega - (\mathcal{B}u, \mathcal{B}u)_{\partial\Omega} - 2(\mathcal{B}u, k)_{\partial\Omega} \tag{13}$$

is the solution to the self-adjoint partial differential system, $\mathcal{L}u+f=0$ on Ω, $\mathcal{B}u+k=0$ on $\partial\Omega$, and vice versa. Particularly, if the operators $\mathcal{L}$ and $\mathcal{B}$ are definite, the solution to the differential system is the extremal point of (13).

In relation to the approximate direct solution techniques to variational formulations, the corollary corresponds to Ritz's method, the theorems 1 and 2 may correspond to Ritz-Galerkin's method. And, the method of setting directly the functional defined by (7) or (8) to zero corresponds to a modified Galerkin's method or the weighted residual method 4].

RELIABILITY ANALYSES

The Fokker-Planck-Kolmogorov formulation for the probabilistic states of nonlinear dynamic systems subjected to uncorrelated stochastic excitations is given by the following nonself-adjoint partial differential systems 1],2]:

$$\begin{aligned}
&\mathcal{L}p = \tfrac{\partial^T}{\partial \mathbf{x}}\left[\tfrac{1}{2}\left(\tfrac{\partial^T}{\partial \mathbf{x}}(\mathbf{b}^T p)\right)^T - \mathbf{a}p\right] - \tfrac{\partial}{\partial t}p = 0 && \text{on } S\times T\\
&\mathcal{B}_{s\alpha}p = \mathbf{n}^T\left[\mathbf{a}p - \tfrac{1}{2}\left(\tfrac{\partial^T}{\partial \mathbf{x}}(\mathbf{b}^T p)\right)^T\right] - \tfrac{\alpha}{1-\alpha}p = 0 && \text{for } 0\le\alpha<1,\ \text{on } \partial S\times\bar{T}\\
&\quad = p = 0 && \text{for } \alpha = 1\\
&\text{and } \mathcal{B}_t p = p = p_{t_\ell}(\mathbf{x}) && \text{on } \bar{S}\times t_\ell
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\mathcal{L}^*q = \left[\tfrac{1}{2}\left(\mathbf{b}\tfrac{\partial}{\partial \mathbf{x}}\right)^T + \mathbf{a}^T\right]\tfrac{\partial}{\partial \mathbf{x}}q + \tfrac{\partial}{\partial t}q = 0 && \text{on } S\times T\\
&\mathcal{B}^*_{s\alpha}q = -\mathbf{n}^T\tfrac{1}{2}\mathbf{b}^T\tfrac{\partial}{\partial \mathbf{x}}q - \tfrac{\alpha}{1-\alpha}q = 0 && \text{for } 0\le\alpha<1,\ \text{on } \partial S\times\bar{T}\\
&\quad = q = 0 && \text{for } \alpha = 1\\
&\mathcal{B}^*_t q = q = q_{t_u}(\mathbf{x}) && \text{on } \bar{S}\times t_u
\end{aligned} \tag{15}$$

In the above equations, $\mathbf{a}(\mathbf{x},t)$ and $\mathbf{b}(\mathbf{x},t)=\mathbf{b}^T(\mathbf{x},t)$, where $\mathbf{x}\in\bar{S}=S\cup\partial S$ and $t\in\bar{T}=[t_\ell, t_u]$, are the mean rate vector and covariance rate matrix with respect to time t of the state vector $\mathbf{x}$ at t, $\partial/\partial\mathbf{x}$ means the vector differential operator, and $(\ )^T$ denotes the transposed vector or matrix. The scalar quantities $p(\mathbf{x},t)$ and $q(\mathbf{x},t)$ represent appropriate probability functions defined on $\bar{S}\times\bar{T}$, and $\mathbf{n}$ and α are the unit outward normal vector and a leaking coefficient function which are defined on the boundary surface ∂S. And, the functions $p_{t_\ell}(\mathbf{x})$ and $q_{t_u}(\mathbf{x})$ are initial and end conditions with respect to t. In applying the results of the preceding section, the domain Ω and its boundary surface $\partial\Omega$ are simply replaced by $S\times T$ and $\partial(S\times T)=\partial S\times\bar{T}\cup\bar{S}\times\partial T$, respectively. By substituting the operators $\mathcal{L}$, $\mathcal{B}_{s\alpha}$, $\mathcal{B}_t$ and $u=p$, $v=q$ in (1), the adjoint operators $\mathcal{L}^*$, $\mathcal{B}^*_{s\alpha}$ and $\mathcal{B}^*_t$ in (15) as well as the relevant boundary operators $\mathcal{B}_{s\alpha}$, $\mathcal{B}_t$, $\mathcal{B}^*_{s\alpha}$ and $\mathcal{B}^*_t$ are determined, and the latter ones are given by

$$\begin{aligned}
&\mathcal{B}_{s\alpha}q = -q, \quad \mathcal{B}^*_{s\alpha}p = -p && \text{for } 0\le\alpha<1 && \text{on } \partial S\times\bar{T}\\
&\mathcal{B}_{s1}q = -\mathbf{n}^T\tfrac{1}{2}\mathbf{b}^T\partial/\partial\mathbf{x}\,q, \quad \mathcal{B}^*_{s1}p = \mathbf{n}^T\left[\mathbf{a}p - \tfrac{1}{2}(\partial/\partial\mathbf{x}(\mathbf{b}^T p))^T\right] && \text{for } \alpha=1 &&\\
&\mathcal{B}_t q = q, \quad \mathcal{B}^*_t p = 0 \quad \text{on } \bar{S}\times t_\ell, \quad \mathcal{B}_t q = 0, \quad \mathcal{B}^*_t p = p \quad \text{on } \bar{S}\times t_u &&&&
\end{aligned} \tag{16}$$

From (4), (5) and (16), the solutions to (14) and (15) are written in terms of the relevant Green's functions h and h^*, which are the solutions to $\mathcal{L}h + \delta(\mathbf{x}-\mathbf{x}_0, t-t_\ell)=0$, $\mathcal{B}_s h = \mathcal{B}_t h = 0$, and $\mathcal{L}^*h^* + \delta(\mathbf{x}-\mathbf{x}_0, t-t_u)=0$, $\mathcal{B}^*_s h^* = \mathcal{B}^*_t h^* = 0$, respectively, in the following forms:

$$\begin{aligned}
p(\mathbf{x},t) &= -(\mathcal{B}_t h, h_t)_{\bar{S}\times t_\ell} = \int_S ds\, h(\mathbf{x},t;\mathbf{x}',t_\ell)\,p_{t_\ell}(\mathbf{x}')\\
q(\mathbf{x},t) &= -(\ell_t, \mathcal{B}^*_t h^*)_{\bar{S}\times t_u} = \int_S ds\, q_{t_u}(\mathbf{x}')\,h^*(\mathbf{x},t;\mathbf{x}',t_u)
\end{aligned} \tag{17}$$

in which

$$h(\mathbf{x},t;\mathbf{x}',t') = h^*(\mathbf{x}',t';\mathbf{x},t) \quad \text{for } t>t', \quad h = h^* = 0 \quad \text{for } t \le t' \tag{18}$$

It is noted that $h(\mathbf{x},t;\mathbf{x}',t')$ defined by the differential system with respect to $\mathbf{x}$ and t satisfies the adjoint differential system with respect to $\mathbf{x}'$ and t'. In general, (17) can be considered to represent the Chapman-Kolmogorov equation, and the Green's function h means the transition probability density function associated with a closed domain $\bar{S}$.

The reliability function r, which means the probability to remain inside $\bar{S}$ at time t when started from a deterministic point $\mathbf{x}_0$ at time t_0, is expressed in terms of h and h^* as follows:

$$r(t\,|\,\mathbf{x}_0,t_0) = \int_S ds\, h(\mathbf{x},t;\mathbf{x}_0,t_0) = \int_S ds\, h^*(\mathbf{x}_0,t_0;\mathbf{x},t) \tag{19}$$

Transforming time variable t_0 to τ by $\tau = t-t_0$, and setting $R(\mathbf{x}_0,\tau;t) = r(t\,|\,\mathbf{x}_0,t-\tau)$, the reliability function $R(\mathbf{x}_0,\tau;t)$ is governed by

$$\begin{aligned}
&\mathcal{L}^* R = \left[\tfrac{1}{2}\left(b(\mathbf{x}_0,t-\tau)\frac{\partial}{\partial \mathbf{x}_0}\right)^T + a^T(\mathbf{x}_0,t-\tau)\right]\frac{\partial}{\partial \mathbf{x}_0} R - \frac{\partial}{\partial \tau} R = 0 && \text{on } S\times T\\
&\mathcal{B}^*_{s\alpha} R = -n^T \tfrac{1}{2} b^T(\mathbf{x}_0,t-\tau)\frac{\partial}{\partial \mathbf{x}_0} R - \frac{\alpha(\mathbf{x}_0,t-\tau)}{1-\alpha(\mathbf{x}_0,t-\tau)} R = 0 && \text{for } 0 \le \alpha < 1 \quad \text{on } \partial S\times \bar{T}\\
&\mathcal{B}^*_{s1} R = R = 0 && \text{for } \alpha = 1\\
&\mathcal{B}^*_{t} R = R = 1 && \text{on } \bar{S}\times T_\ell
\end{aligned} \tag{20}$$

where $\mathbf{x}_0 \in \bar{S} = S \cup \partial S$, $\tau \in \bar{T} = [\tau_\ell, \tau_u] = [0, t-t_{0\ell}]$.

The reliability function $r(t\,|\,\mathbf{x}_0,t_0)$ is obtained by making use of the solution to (20) as follows:

$$r(t\,|\,\mathbf{x}_0,t_0) = R(\mathbf{x}_0,t-t_0;t), \quad \mathbf{x}_0 \in \bar{S}, \quad t_0 \in \bar{T} = [t_{u},t] \tag{21}$$

In particular, when a, b and α are time-independent, the reliability function $r(t\,|\,\mathbf{x}_0,0)$ is expressed by $R(\mathbf{x}_0,t)$ [5],[6].

The failure probability f is defined as

$$f(t\,|\,\mathbf{x}_0,t_0) = 1 - r(t\,|\,\mathbf{x}_0,t_0) = 1 - \int_S ds\, h(\mathbf{x},t;\mathbf{x}_0,t_0) = \int_T d\tau \int_{\partial S} d\partial S\, w(\mathbf{x},\tau\,|\,\mathbf{x}_0,t_0) \tag{22}$$

where w represents the probability density function of the first passage time at which the exceeding ∂S occurs. By making use of (1), (14) through (16), (18), (19) and (22), the function w is expressed in terms of h as follows:

$$\begin{aligned}
w(\mathbf{x},t\,|\,\mathbf{x}_0,t_0) &= \frac{\alpha(\mathbf{x},t)}{1-\alpha(\mathbf{x},t)}\, h(\mathbf{x},t;\mathbf{x}_0,t_0) && \text{for } 0 \le \alpha < 1\\
&= n^T\left[a\,h - \tfrac{1}{2}\left(\frac{\partial^T}{\partial \mathbf{x}}(b^T h)\right)^T\right] && \text{for } \alpha = 1
\end{aligned} \tag{23}$$

Since the content in the square brackets is the probability flow, it is clear that the homogeneous boundary conditions on ∂S_0 and ∂S_1, which correspond to $\alpha = 0$ and 1, represent the reflecting and absorbing boundary conditions, respectively. By making use of h_0, which is the Green's function associated with R^n, and h in (1), the following equations are obtained:

$$\varepsilon\, h(\mathbf{x},t;\mathbf{x}_0,t_0) + \int_T d\tau \int_{\partial S} d\partial S\, \mathcal{B}^*_{s\alpha;\xi,\tau} h_0(\mathbf{x},t;\xi,\tau)\, \mathcal{B}^*_{s\alpha;\xi,\tau} h(\xi,\tau;\mathbf{x}_0,t_0) = h_0(\mathbf{x},t;\mathbf{x}_0,t_0)$$

where $\varepsilon = 0$ for $\mathbf{x} \in R^n - \bar{S}$, $\varepsilon = 1$ for $\mathbf{x} \in S$, and $\mathbf{x}_0 \in S$, $\tau \in T = (t_0,t)$. (24)

If h_0 is known, the equation corresponding to $\mathbf{x} \in R^n - \bar{S}$ represents the first kind of Fredholm type integral equation with respect to $\mathcal{B}^* h$ for arbitrary $\mathbf{x} \in R^n - \bar{S}$ and time t. The Green's function h associated with $\bar{S}$ is determined from the equation corresponding to $\mathbf{x} \in S$. In particular, if $\partial S = \partial S_1$, the operators in (24) take the following forms [7]:

$$\mathcal{B}^*_{s1;\xi,\tau} h_0 = h_0(\mathbf{x},t;\xi,\tau), \quad \mathcal{B}^*_{s1;\xi,\tau} h = w(\xi,\tau\,|\,\mathbf{x}_0,t_0) \tag{25}$$

From the above discussions, it is found that the reliability function is evaluated through (19) by knowing the Green's function h or h^* associated with (14) or (15), otherwise, by directly solving (20). In determining h, h^* and h_0 and solving (14) and (15) or (20), analytical methods such as eigenfunction expansions may be limited to the time-invariant systems, in which a, b and α are time-independent. If the steady-state probability density function p_s in R^n, which gives zero probability flow, is known, the transformations, $p = p_s p'$, $q = p_s q'$ and $\tau = t-t_0$ yield a pair of adjoint systems

given by

$$p_s^{-1}\mathcal{L}p' = p_s^{-1}L_s p' - \frac{\partial}{\partial\tau}p' = \frac{1}{2p_s}\frac{\partial^T}{\partial x}\left[p_s b'\frac{\partial}{\partial x}p'\right] - \frac{\partial}{\partial\tau}p' = 0 \qquad \text{on } S\times T$$

$$p_s^{-1}\mathcal{B}_s p' = -n^T\frac{1}{2}b'\frac{\partial}{\partial x}p' - \frac{\alpha}{1-\alpha}p' = 0 \quad \text{for } 0\le\alpha<1 \qquad \text{on } \partial S\times\bar{T} \tag{26}$$

$$= p' = 0 \quad \text{for } \alpha = 1$$

$$p_s^{-1}\mathcal{B}_t p' = p' = p_0/p_s \qquad \text{on } \bar{S}\times\tau_x$$

and

$$p_s^{-1}\mathcal{L}^*q' = p_s^{-1}L_s^*q' - \frac{\partial}{\partial\tau}q' = \frac{1}{2p_s}\frac{\partial^T}{\partial x}\left[p_s b'^T\frac{\partial}{\partial x}q'\right] - \frac{\partial}{\partial\tau}q' = 0 \qquad \text{on } S\times T$$

$$p_s^{-1}\mathcal{B}_s^*q' = -n^T\frac{1}{2}b'^T\frac{\partial}{\partial x}q' - \frac{\alpha}{1-\alpha}q' = 0 \quad \text{for } 0\le\alpha<1 \qquad \text{on } \partial S\times\bar{T} \tag{27}$$

$$= q' = 0 \quad \text{for } \alpha = 1$$

$$p_s^{-1}\mathcal{B}_t^*q' = q' = q_t/p_s \qquad \text{on } \bar{S}\times\tau_x$$

in which $T = (\tau_x, \tau_{t_0}) = (0, t-t_{t_0})$ and

$$b' = b + S\ ,\quad a' = a - \frac{1}{2}\left(\frac{\partial^T}{\partial x}b'^T\right)^T = a + \frac{1}{2}\left(\frac{\partial^T}{\partial x}S\right)^T\ ,\quad S = -S^T \tag{28}$$

If there exists a skew symmetric matrix S which makes b' nonsingular and $b'^{-1}a'$ irrotational, the function p_s is expressed as

$$p_s(x) = \exp\left(-U(x)\right)\Big/\int_{R^n} ds\,\exp\left(-U(\xi)\right)\ ,\quad \frac{\partial}{\partial x}U = -2b'^{-1}\cdot a' \tag{29}$$

For instance, when the damping force vector takes the form $D(x,\dot{x}) = g(E)\Gamma\dot{x}$, where x and $\dot{x}$ are displacement and velocity vectors, $g(E)$ is a nonnegative function of total energy E, and Γ is the spectral density matrix of the excitations, a, b, U and S are expressed as follows 2]:

$$a = -\begin{bmatrix} 0 & -I \\ I & g(E)\Gamma \end{bmatrix}\frac{\partial}{\partial x}E\ ,\quad b = \begin{bmatrix} 0 & 0 \\ 0 & \Gamma \end{bmatrix}\ ,\quad S = \begin{bmatrix} 0 & \Psi \\ -\Psi^T & 0 \end{bmatrix}\ ,\quad z = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} \tag{30}$$

$$U(E) = 2\int_0^E g(\eta)\,d\eta\ ,\quad \Psi = \exp\left(U(x)\right)\left[C + 2\int_0^E \exp\left(-U(\xi)\right)d\xi\, I\right]$$

in which I is unit matrix, and C is a constant matrix.

By making use of the eigenvalue matrix Λ, all elements of which have nonpositive real parts, and bi-orthonormal adjoint eigenfunctions ${}_n\Psi'$ and ${}_n\Psi'$ of the pair of adjoint elliptic partial differential systems, $(L_s - \lambda p_s)\phi' = 0$, $\mathcal{B}_s\phi' = 0$, and $(L_s^* - \lambda p_s)\psi' = 0$, $\mathcal{B}_s^*\psi' = 0$, the Green's function h is expressed in the form 2],8],

$$h(x,t;x_0,t_0) = {}_n\Psi^T(x)\exp\left(\Lambda(t-t_0)\right){}_n\Psi(x_0) = p_s^{\frac{1}{2}}(x)\,{}_n\Psi'^T(x)\exp\left(\Lambda(t-t_0)\right){}_n\Psi'(x_0)p_s^{\frac{1}{2}}(x_0) \tag{31}$$

where ${}_n\Psi$ and ${}_n\Psi'$ are the bi-orthonormal adjoint eigenfunctions associated with (14) and (15).

In the cases of time-variant systems in which a, b or α are time-dependent, approximate techniques based on variational method including the finite element method and Galerkin's method may be advantageously applicable. For the Fokker-Planck-Kolmogorov differential systems given by (14) and (15), the functionals defined by (7) and (8) are expressed as follows:

$$J(q;p) = J_s(q;p) + J_t(q;p)\ ,\quad J^*(p;q) = J_s^*(p;q) + J_t^*(p;q)$$

$$\begin{aligned} J_s(q;p) = J_s^*(p;q) = {} & \int_T dt\int_S ds\left[\frac{\partial^T}{\partial x}q\left\{ap - \frac{1}{2}\left(\frac{\partial^T}{\partial x}(b^Tp)\right)^T\right\}\right] \\ & - \int_T dt\int_{\partial S_\alpha} d\partial S_\alpha\frac{\alpha}{1-\alpha}qp \\ & + \int_T dt\int_{\partial S_1} d\partial S_1\, n^T\frac{1}{2}b^T\left(\frac{\partial}{\partial x}q\right)p - \int_T dt\int_{\partial S_1} d\partial S_1\, q\, n^T\left[ap - \frac{1}{2}\left(\frac{\partial^T}{\partial x}(b^Tp)\right)^T\right] \end{aligned} \tag{32}$$

$$J_t(q;p) = -\int_T dt\int_S ds\, q\frac{\partial}{\partial t}p - \int_S ds\, q_{t_x}p_{t_x} - \int_S ds\, q_{t_x}x_t$$

$$J_t^*(p;q) = \int_T dt\int_S ds\left(\frac{\partial}{\partial t}q\right)p - \int_S ds\, q_{t_u}p_{t_u} - \int_S ds\, x_t\, p_{t_u}$$

where $T = (t_x, t_u) = (t_0, t)$.

By making use of appropriate trial function systems Φ and Ψ and unknown vectors c and d, the functions p and q are approximated in the forms

$$p = \Phi^T(x,t)c(t,t_0)\ ,\quad q = d^T(t_0,t)\Psi(x_0,t_0) \tag{33}$$

Substituting (33) into (32) and through stationarizing with respect to d and c, respectively, and differentiating with respect to time, two sets of the first order ordinary differential equations for the unknown vectors are obtained. After all, the solutions are expressed as

$$p(x,t;t_0) = \Phi^T(x,t)\,G_T(t,t_0)\,P(t_0) \tag{34}$$

$$q(x_0,t_0;t) = Q^T(t)\,G_T^{*T}(t_0,t)\,\Psi(x_0,t_0)$$

where

$$P(t_0) = \int_S ds\, p_{t_0}\Psi\ ,\quad Q^T(t) = \int_S ds\, q_t\,\Phi^T \tag{35}$$

and $G_T(t,t_0)=G^{*T}(t_0,t)$ are solutions to the sets of equations,

$$\begin{aligned} &A(t)\frac{d}{dt}G_T - B(t)G_T = 0 \quad , \quad G_T(t_0,t_0)=A^{-1}(t_0) \quad , \\ &A^T(t_0)\frac{d}{dt_0}G_T^* + B^T(t_0)G_T^* = 0 \quad , \quad G_T^*(t,t)=A^{-T}(t) \quad , \end{aligned} \qquad A=\int_S dS\,\Psi\Phi^T \neq A^T \tag{36}$$

in which if the boundary consists of ∂S_0 and ∂S_1 and if p and q satisfy the relevant boundary conditions, the matrix B is given by

$$B=\int_S dS\left(\frac{\partial}{\partial\tau}\Psi^T\right)^T\left[a\Phi^T-\frac{1}{2}\left(\frac{\partial T}{\partial\tau}B^T\right)^T\Phi^T-\frac{1}{2}b\left(\frac{\partial}{\partial\tau}\Phi^T\right)\right]\neq B^T \tag{37}$$

In the finite element method Φ used to be equal to Ψ and at least continuous across the element boundaries as well as to be time-independent.

CONCLUDING REMARKS

As the mathematical bases of seismic reliability analyses of nonlinear structures based on the Fokker-Planck-Kolmogorov formulation, a generalized Green's formula and stationary functional theorems associated with nonself-adjoint partial differential systems were presented by introducing adjoint differential operators with respect to equation as well as initial, end and boundary conditions. By making use of the generalized Green's formula, the reliability function associated with a mixed type bounded boundary condition was expressed in terms of Green's function which would be interpreted as the transition probability density function in a bounded or partially bounded domain. Also, the Kolmogorov backward type differential system governing the reliability function of time-variant systems was derived. Based on the stationary functional theorems the variational formulation was presented, which might be applicable to space- and time-variant Fokker-Planck-Kolmogorov differential systems. As regards the method of determining Green's function, the boundary value problem for linear elliptic partial differential systems as well as the integral equation formulation were discussed.

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REFERENCES

1] Lin, Y. K., "Probability Theory of Structural Dynamics," Mc-Graw Hill, 1967, 143-150, 262-272.
2] Caughey, T. K., "Nonlinear Theory of Random Vibrations," Advances in Applied Mechanics, Vol.11, Academic Press, 1971, 209-253.
3] Kobori, T., R. Minai, and Y. Suzuki, "Nonstationary Random Response of Bilinear Hysteretic Systems," Proc. 24th Japan National Congress for Applied Mechanics, 1974, 143-152.
4] Wen, Y-K., "Method for Random Vibration of Hysteretic Systems, "Proc. ASCE, No.EM2, April, 1976, 249-263.
5] Rosenblueth, E., "Some Applications of Probability Theory in Aseismic Design," Proc. 1st WCEE, 1956, 8-1-18.
6] Kac, M., "Probability Theory as a Mathematical Discipline and as a Tool in Engineering and Science," Proc. 1st Symposium on Engineering Applications of Random Function Theory and Probability, John Wiley, 1963, 31-67.
7] Siegert, A. J. F., "On the First Passage Time Probability Problem," Physical Review, Vol.81, No.4, 1951, 617-623.
8] Atkinson, J. D., "Eigenfunction Expansions for Randomly Excited Non-Linear Systems," J. Sound and Vibration, Vol.30, No.2, 153-172.