AN EFFICIENT APPROACH FOR THE DYNAMIC SENSITIVITY ANALYSIS

Juan Cassis[I]

SYNOPSIS

A new efficient approach for the dynamic sensitivity analysis of structures is introduced. The method computes expressions of exact derivatives of structural response functions with respect to design variables. The proposed method is compared with the traditional approximate finite differences approach and with another exact method introduced previously. Examples applied to building frames subjected to a ground motion show the efficiency of the proposed solution.

INTRODUCTION

The necessity of performing sensitivity analyses in structural engineering; i e, the computation of partial derivates of the structural response functions with respect to the design variables, has significantly increased in the last years due mainly to the development of optimum structural design methods.

Until recently, the computation of derivatives of numerical functions used to be performed by numerical methods, e.g., finite differences. In the last years, exact and, at a same time, more efficient formulas for computing derivatives in structural mechanics have been developed.

In dynamic sensitivity analysis, the partial derivatives of the displacements, which are central in the whole analysis process, are currently determined in terms of the eigenvalue and eigenvector derivatives as well as in terms of the derivatives of other relevant quantities as modal participation factors and dynamic load factors. This exact approach is called here Method 1.

The method introduced in the paper, called Method 2, is also exact. However, its formulation is simpler than Method 1 and by-passes the computation of derivatives of eigenvalues, eigenvectors and other dynamic quantities.

DYNAMIC STRUCTURAL ANALYSIS

As a reference to describing both methods of dynamic sensitivity analysis, it is necessary to give a brief description of the dynamic structural analysis.

For linear elastic structures subjected to dynamic loads, the finite elements governing equation of motion is

$$[M]\,\ddot{\vec{u}} + [C]\,\dot{\vec{u}} + [K]\,\vec{u} = \vec{F}(t) \qquad (1)$$

where $[M]$, $[C]$ and $[K]$ are respectively the mass, viscous damping and stiffness matrices referred to the system coordinates in the structure;

I Professor of Civil Engineering, University of Chile, Santiago, Chile.

$\ddot{\vec{u}}$, $\dot{\vec{u}}$, $\vec{u}$ and $\vec{F}(t)$ are respectively the vectors representing the accelerations, velocities, displacements and applied dynamic loads, all in system coordinates; t is the time variable. If it is assumed that $[C]$ can be expanded in Caughey's series, then the damped system admits classical normal modes of vibration, and Eqs.(1) can be uncoupled by a transformation from nodal to normal coordinates $\vec{q}$; namely

$$\vec{u} = [\Phi]\,\vec{q} \tag{2}$$

in which $[\Phi]$ is the modal matrix.

The transformation in Eq.(2) applied to Eq.(1) leads to the following uncoupled system of equations

$$[m]\,\ddot{\vec{q}} + [c]\,\dot{\vec{q}} + [k]\,\vec{q} = [\Phi]^T\,\vec{F} \tag{3}$$

The ith equation of this system may be written as

$$\ddot{q}_i + 2\xi_i\omega_i\dot{q}_i + \omega_i^2 q_i = \frac{1}{m_i}\,\vec{\Phi}_i^T\,\vec{F} \tag{4}$$

where $\xi_i = \frac{c_i}{2m_i\omega_i}$ is the ith mode relative damping and $\omega_i^2 = \frac{k_i}{m_i}$ is the square of the ith mode frequency.

If F is separable in a product of a space and a time function,

$$\vec{F}(t) = \vec{p}\,f(t) \tag{5}$$

then

$$\vec{\Phi}_i^T\,\vec{F} = \vec{\Phi}_i^T\,\vec{p}\,f(t) = \Gamma_i\,f(t) \tag{6}$$

in which $\Gamma_i = \vec{\Phi}_i^T\,\vec{p}$ is the ith modal participation factor.

For zero initial conditions, the solution of Eq.(4) may be written as

$$q_i(t) = \frac{\Gamma_i}{m_i\omega_{di}}\int_0^t f(\tau)\,e^{-\xi_i\omega_i(t-\tau)}\,\text{sen}\,\omega_{di}(t-\tau)\,d\tau \tag{7}$$

where $\omega_{di} = \sqrt{1-\xi_i^2}\,\omega_i$ is the ith damped natural frequancy.

If the load is a plane ground motion $\ddot{u}_g(t)$ (horizontal acceleration), then

$$\vec{F}(t) = -[M]\,\vec{e}\,\ddot{u}_g(t) \tag{8}$$

$$\Gamma_i = \vec{\Phi}_i^T\,[M]\,\vec{e} \tag{9}$$

and $\vec{e}$ is a vector with components equal to 1 when they correspond to coordinates of horizontal displacements and zero otherwise.

DYNAMIC SENSITIVITY ANALYSIS, METHOD 1

For this derivation, the eigenvectors are assumed to be M-orthonormalized. From Eq.(2), take the derivative of $\vec{u}$ with respect to a design variable d_j

$$\frac{\partial\vec{u}}{\partial d_j} \equiv \vec{u}_{,j} = [\Phi_{,j}]\,\vec{q} + [\Phi]\,\vec{q}_{,j} \tag{10}$$

The derivatives of the eigenvectors are computed from[1]

$$\vec{\Phi}_{i,j} = \sum_{i=1}^{n} a_{ijk} \, \vec{\Phi}_k \tag{11}$$

where

$$\left.\begin{aligned} a_{ijk} &= \frac{\vec{\Phi}_k^T ([K,_j] - \lambda_i [M,_j]) \vec{\Phi}_i}{\lambda_i - \lambda_k} \qquad i \neq k \\ a_{iji} &= -\frac{1}{2} \cdot \vec{\Phi}_i^T [M,_j] \vec{\Phi}_i \end{aligned}\right\} \tag{12}$$

and n is the number of modes. The eigenvalue λ_i is equal to ω_i^2.

The matrices $[M,_j]$ and $[K,_j]$ may be efficiently obtained from the derivatives of the element mass (consistent formulation) and stiffness matrices [2].

To compute $\vec{q},_j$, write Eq.(7) in the form

$$q_i(t) = \Gamma_i \, R_i(t) \tag{13}$$

thus

$$q_{i,j} = \Gamma_{i,j} \, R_i + \Gamma_i \, R_{i,j} \tag{14}$$

Now, from Eq.(6)

$$\Gamma_{i,j} = \vec{\Phi}_{i,j} \, \vec{p} \tag{15}$$

Or from Eq.(9) for a plane ground motion

$$\Gamma_{i,j} = \vec{\Phi}_{i,j} \, [M] \, \vec{e} + \vec{\Phi}_i \, [M,_j] \, \vec{e} \tag{16}$$

on the other hand

$$R_{i,j} = \frac{\partial R_i}{\partial \omega_i} \omega_{i,j} + \frac{\partial R_i}{\partial \xi_i} \xi_{i,j} \tag{17}$$

But $\xi_{i,j} = 0$ because in practice the ξ_i are preassigned independently of the design variables d_j.

The derivative $\omega_{i,j}$ is given by [1]

$$\omega_{i,j} = \frac{\vec{\Phi}_i^T ([K,_j] - \omega_i^2 [M,_j]) \vec{\Phi}_i}{2\omega_i} \tag{18}$$

This concludes all the computations necessary to obtaining $\vec{u},_j$ by Method 1.

DYNAMIC SENSITIVITY ANALYSIS, METHOD 2

This approach is analog to one used in static sensitivity analysis [3]. Instead of differentiating the solution $\vec{u}$ of Eq.(1) as in Method 1, in this method implicit partial differentiation is carried out directly from Eq.(1).

$$[M,_j] \, \ddot{\vec{u}} + [M] \, \ddot{\vec{u}},_j + [C,_j] \, \dot{\vec{u}} + [C] \, \dot{\vec{u}},_j + [K,_j]\vec{u} + [K] \, \vec{u},_j = \vec{F},_j \tag{19}$$

This equation can be rearranged to yield

$$[M]\ddot{\vec{u}},_j + [C]\dot{\vec{u}},_j + [K]\vec{u},_j = \vec{F},_j - ([M,_j]\ddot{\vec{u}} + [C,_j]\dot{\vec{u}} + [K,_j]\vec{u}) \tag{20}$$

Comparison of Eqs. (1) and (20) shows that both have the same coefficients but different loading vectors. Therefore, each $\vec{u}_{,j}$ is solved by modal analysis using the same set of eigenvalues and eigenvectors, already determined in the computation of $\vec{u}$. This feature is specially remarkable when Method 2 is compared with the finite differences approach. In effect, to compute the $\vec{u}_{,j}$ by finite differences it is necessary to perform at least m+1 dynamic structural analyses (m being the number of design variables), each of these analyses including a different eigenproblem.

Application to building frames subjected to a ground motion.

In the case of building frames the elements of the lumped mass matrix $[M]$ are practically independent of the design variables, since the contribution of the slab, walls and other portions of the building is significantly greater than the own frame mass. Therefore $[M_{,j}]$ may be taken as a null matrix. With respect to the term $\vec{F}_{,j}$, in the case of a ground motion, Eq. (8) shows that the term $\vec{F}_{,j}$ is null provided that $[M_{,j}]$ is null. In consequence, for building frames subjected to ground motions the only terms that contribute to the pseudo loading vector in Eq.(20) are $-[C_{,j}]\,\dot{\vec{u}}$ and $-[K_{,j}]\,\vec{u}$.

EXAMPLES

Finite differences and methods 1 and 2 were applied to compute the derivatives of the horizontal displacements in the steel frames subjected to a horizontal ground motion shown in Figs. 1 and 2. Relevant data are included in both figures. Computations were carried out on the IBM 370/145 computer at University of Chile. The program was written in Fortran and compiled by a Fortran G compiler. The cross sectional moment of inertia was taken as the design variable in each member.

In the example of Fig. 1, two independent design variables were taken. One corresponds to all columns made identical by linking of design variables and the other to all beams made also identical. Total CPU run times for finite differences, Method 1 and Method 2 were, respectively, 6.12 sec, 6.44 sec and 6.38 sec.

Twenty independent design variables were selected in the example of Fig.2 by making the columns in each story identical. Total CPU run times for finite differences, Method 1 and Method 2 were, respectively, 1 m 44 sec, 1 m 30 sec, 1 m 24 sec.

DISCUSSION AND CONCLUSIONS

Example 1, being a small problem with 3 horizontal degrees of freedom and 2 independent design variables, shows irrelevant differences in computer time among the 3 methods in comparison. However, in Example 2, having 10 horizontal degrees of freedom and 20 independent design variables, computer times go more apart, showing that exact methods required less computing time than finite differences. Between exact methods 1 and 2, the proposed Method 2 resulted more efficient.

In conclusion, in dynamic sensitivity analysis, exact methods, in addition to have an exact formulation, are more efficient than finite differences in large problems. The dimension of the problem is measured by the number of degrees of freedom of motion, related to the order of the eigenproblem in the dynamic analysis, and by the number of independent design variables, which is equal to the number of partial derivatives to be computed for the response functions. Furthermore, the exact method introduced herein proved to have a much simpler formulation and to be more efficient than the current exact method of sensitivity analysis.

REFERENCES

1. Fox, R.L. and M.P. Kappor, "Rates of change of Eigenvalues and Eigenvectors", AIAA J. Vol. 6, N°12, pp.2426-2429, December 1968.

2. Cassis, J., "Optimum Design of Structures Subjected to Dynamic Loads", UCLA-Eng.-7451, Los Angeles, California, June 1974.

3. Fox, R.L., "Optimization Methods for Engineering Design", Addison Wesley, Reading, Massachusetts, 1971.

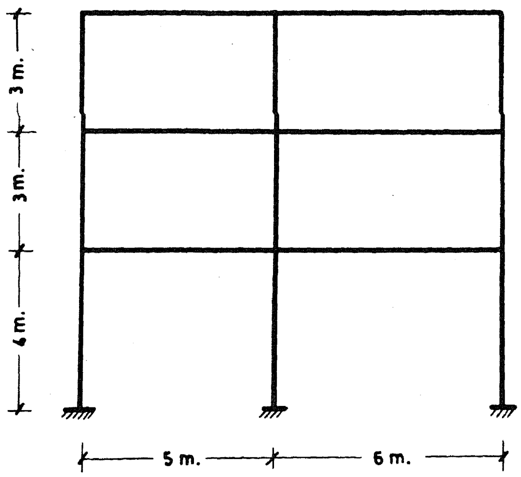

Fig. 1
3-story, 2-bay steel frame

All beams: 18 WF 60
All columns: 21 WF 73
Young mod. $E = 2.1 \times 10^6$ Kg/cm^2

Tributary weight per story: 50 T.

Ground motion:

$$u_g(t) = \begin{cases} 0 & t < 0 \\ u_{g0} \sin^2 pt & 0 \leqslant t \leqslant \pi/p \\ 0 & t > \pi/p \end{cases}$$

p = 30 rad/sec
u_{g0} = 3 cm
t = 0.1 sec
Zero damping

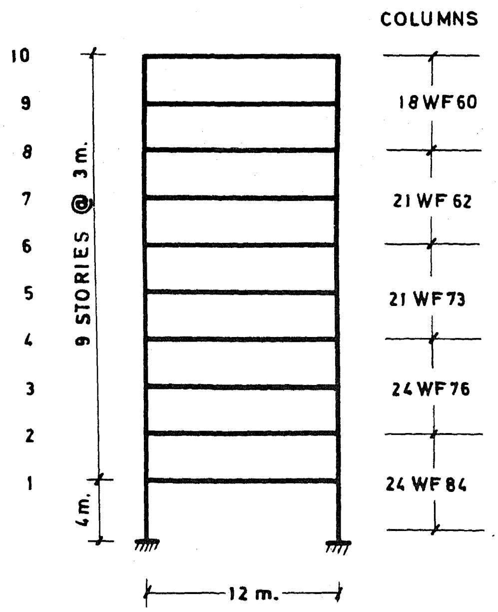

Fig. 2
10-story, 1-bay steel frame

All beams: 18 WF 60
Columns: As indicated in Fig. 2

Young modulus, tributary load per story and ground motion same as in Fig. 1.

Zero damping