VERTICAL LOAD EFFECT ON STRUCTURAL DYNAMICS

By

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SYNOPSIS

The object of this investigation is to examine P-Δ effect of gravity or the effect of vertical component of the earthquake excitation in the dynamics of beam-yielding steel structures with bi-linear force-displacement relationship. In the analysis, a single-degree-of-freedom shear model is used. Statistical mean square response of that system which is subjected to white noise excitation at the base in both the horizontal and the vertical directions is expressed by Voltera's type Integral Equation and is solved by applying Laplace Transformation Theory. In order to examine the dynamic behaviour of elastic-plastic structures, the authors developed an original method of equivalent linearization.

INTRODUCTION

Structural members made of steel have advantages that they are relatively light and ductile. So, steel members are widely used for designing rather tall structures where some plastic deformation are permitted. In these steel structures, when subjected to large external force in both the horizontal and the vertical directions, it might be expected that P-Δ effect due to the increase of horizontal displacement might become not to be ignored. In this paper, this P-Δ effect is examined. There are several papers about this problem. Jennings and Husid,[1] using an ensemble of pseudo-earthquakes, reported that the P-Δ effect of gravity on the collapse of structures can not be ignored, while that of vertical component of earthquake excitation is negligible. C.K.Sun et al.[2] proposed new design criteria to supplement the present design code requirements.

In this paper, by using statistical method, mean square (MS.) transient response of a bi-linear elastic-plastic structure, which is subjected to excitation in both the horizontal and the vertical directions, is calculated by applying equivalent linearization method and the P-Δ effect in its dynamic behaviour is examined. The relation between P-Δ effect and structural parameter such as height, initial stiffness, yield displacement or stiffness ratio is also discussed.

ASSUMPTIONS FOR THE ANALYSIS

The structure is assumed to be a beam-yielding steel one, and is transformed into a single-degree-of-freedom shear model shown in Fig.1. Bi-linear hysteresis with positive stiffness ratio, shown in Fig.2, is used to represent the force-displacement relationship of the structure, where such effects as stress hardening and stiffness degrading are neglected. Vertical vibration is taken to remain elastic when the plastic deformation begins to take place in the horizontal direction. It is assumed that the horizontal excitation is a quasi-nonstationary white noise and the vertical excitation is a

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stationary one. In the analysis, maximum response is estimated at the level of 3σ.

EQUATION OF MOTION AND MEAN SQUARE TRANSIENT RESPONSE

Equation of motion of the system shown in Fig.1 is expressed as:

$$\ddot{x} + 2h\omega\dot{x} + (\omega^2 - G/H)x = -\ddot{Z}_H(t) - \ddot{y}x/H \tag{1}$$

$$\ddot{y} + 2\eta p\dot{y} + p^2 y = -\ddot{Z}_V(t) , \tag{2}$$

where

- x - relative displacement in the horizontal direction,
- y - relative displacement in the vertical direction,
- ω - natural circular frequency of horizontal vibration,
- p - natural circular frequency of vertical vibration,
- h - damping factor in horizontal vibration,
- η - damping factor in vertical vibration,
- H - height of system,
- G - gravity acceleration.

As the horizontal excitation is non-stationary and the system is non-linear in the horizontal direction, Eq.(1) must be solved by applying step-by-step algorism. Let the horizontal excitation be quasi-nonstationary and be expressed as $\ddot{Z}_H(t)=\alpha(t)\times\xi(t)$, where $\alpha(t)$ is a stationary white noise and $\xi(t)$ a shape function. Then, denoting the n-th step algorism by subscript n, and putting a_1, a_0, ξ_n as

$$a_1 = 2h_{eq.n}\,\omega_{eq.n} , \quad a_0 = \omega_{eq.n}^2 - G/H , \quad \xi_n = \xi(t_n)$$

equation (2) becomes

$$L\ddot{x} = \ddot{x} + a_1\dot{x} + a_0 x = -\alpha(t)\xi_n - \ddot{y}x/H , \tag{3}$$

where

- L - differential calculus,
- h_{eq} - equivalent damping factor,
- ω_{eq} - equivalent circular frequency.

Eq.(3) is transformed into

$$\underset{(0\le t\le \Delta t_n)}{x(t)_n} = -\xi_n\int_0^t g(t-\tau)\alpha(\tau)\,d\tau - (1/H)\int_0^t g(t-\tau)\ddot{y}(\tau)x(\tau)\,d\tau + \varphi(t) , \tag{4}$$

where Δt_n is the time increment for the n-th step and

$$Lg(t) = \delta(t) , \quad L\varphi(t) = 0 .$$

Assuming that $\alpha(t)$ and $\ddot{Z}_V(t)$ are Gaussian white noise with the following conditions,

$$\langle\alpha(t)\rangle = \langle\ddot{Z}_V(t)\rangle = 0 , \quad \langle\alpha(t)\alpha(t+\tau)\rangle = 2\pi S_\alpha\delta(\tau) , \quad \langle\ddot{Z}_V(t)\ddot{Z}_V(t+\tau)\rangle\, 2\pi S_V\delta(\tau) ,$$

where $\langle\cdot\rangle$ denotes the average value, δ the Dirac's delta function, S_α and S_V the power spectral densities. Then, from Eq.(4), MS response of x and $\dot{x}$ in the n-th step are calculated as:

$$\langle x^2(t)\rangle_n = 2\pi S_\alpha\xi_n^2\int_0^t g^2(t-\tau)\,d\tau + 2\pi\bar{S}_{\ddot{y}n}H^{-2}\int_0^t g^2(t-\tau)\langle x^2(\tau)\rangle_n\,d\tau + \langle\varphi^2(t)\rangle \tag{5}$$

$$\langle \dot{x}^2(t)\rangle_n = 2\pi S_\alpha\xi_n^2\int_0^t \dot{g}^2(t-\tau)\,d\tau + 2\pi\bar{S}_{\ddot{y}n}H^{-2}\int_0^t \dot{g}^2(t-\tau)\langle x^2(\tau)\rangle_n\,d\tau + \langle\dot{\varphi}^2(t)\rangle \tag{6}$$

$$\langle x(t)\dot{x}(t)\rangle_n = 2\pi S_\alpha\xi_n^2\int_0^t g(t-\tau)\dot{g}(t-\tau)\,d\tau + 2\pi\bar{S}_{\ddot{y}n}H^{-2}\int_0^t g(t-\tau)\dot{g}(t-\tau)\times \langle x^2(\tau)\rangle_n\,d\tau + \langle\varphi(t)\dot{\varphi}(t)\rangle , \tag{7}$$

where

$$\bar{S}_{\ddot{y}n} = S_{\ddot{y}}(\omega_{eq.n}, t_n) \doteqdot \frac{\langle y^2(t_n)\rangle}{\bar{y}^2} S_V \frac{1 + 4\eta^2(\omega_{eq.n}/p)^2}{\{1-(\omega_{eq.n}/p)^2\}^2 + 4\eta^2(\omega_{eq.n}/p)^2}$$

$$\langle y^2(t)\rangle = \frac{\pi S_V}{2\eta p^3}\left\{1.0 - \frac{1}{1-\eta^2}e^{-2\eta pt}(1.0 - \eta^2\cos 2\bar{p}t + \eta\sqrt{1-\eta^2}\sin 2\bar{p}t)\right\}$$

$$\bar{y}^2 = \langle y^2(t)\rangle\big|_{t=\infty} , \quad \bar{p}^2 = p^2(1-\eta^2) .$$

Further, by applying Laplace Transformation Theory, Eqs.(5), (6), (7) become:

$$\langle x^2(t)\rangle_n = \mathcal{L}^{-1}\{Y(s)\}$$

$$\langle \dot{x}^2(t)\rangle_n = \pi S_a \xi_n^2 \{1/a_1 + e^{-a_1 t}(-\frac{a_1}{a_1 \omega_d^2}\cos 2\omega_d t + \frac{1}{2\omega_d}\sin 2\omega_d t)\}$$
$$+ e^{-a_1 t}(A_2 \cos^2\omega_d t + B_2 \sin\omega_d t \cos\omega_d t + C_2 \sin^2\omega_d t) + 2\pi \bar{S}_{\ddot{y}n} H^{-2}\mathcal{L}^{-1}\{K_2(s)Y(s)\}$$

$$\langle x(t)\dot{x}(t)\rangle_n = \pi S_a \xi_n^2 \frac{1}{2\omega_d^2}(1-\cos 2\omega_d t) + e^{-a_1 t}(A_3 \cos^2\omega_d t + B_3 \cos\omega_d t \sin\omega_d t$$
$$+ C_3 \sin^2\omega_d t) + 2\pi \bar{S}_{\ddot{y}n} H^{-2}\mathcal{L}^{-1}\{K_3(s)Y(s)\} \quad ,$$

where

$$Y(S) = \{2\pi S_a \xi_n^2 K_1(S)/S + \Phi(S)\}/\{1 - 2\pi \bar{S}_{\ddot{y}n} H^{-2} K_1(S)\}, \quad \Phi(S) = \mathcal{L}\{\varphi^2(t)\}$$
$$K_1(S) = \mathcal{L}\{g^2(t)\}, \quad K_2(S) = \mathcal{L}\{\dot{g}^2(t)\}, \quad K_3(S) = \mathcal{L}\{g(t)\dot{g}(t)\}, \quad A_2 = \langle \dot{x}_0^2\rangle$$
$$B_2(S) = -(1/\omega_d)\{2a_0\langle x_0 \dot{x}_0\rangle + a_1\langle \dot{x}_0^2\rangle\}, \quad C_2 = (1/\omega_d^2)(a_0\langle x_0^2\rangle + a_0 a_1\langle x_0 \dot{x}_0\rangle + a_1^2\langle \dot{x}_0^2\rangle/4)$$
$$A_3 = \langle x_0 \dot{x}_0\rangle, \quad B_3 = (1/\omega_d)(\langle \dot{x}_0^2\rangle - a_0\langle x_0^2\rangle), \quad C_3 = -\{a_0 a_1\langle x_0^2\rangle/2 + (\omega_d^2 + a_1^2/2)\langle x_0 \dot{x}_0\rangle +$$
$$a_1\langle \dot{x}_0^2\rangle/2\}/\omega_d^2, \quad \langle x_0^2\rangle = \langle x^2(t)\rangle_{n-1}\big|_{t=\Delta t_{n-1}}, \quad \langle \dot{x}_0^2\rangle = \langle \dot{x}^2(t)\rangle_{n-1}\big|_{t=\Delta t_{n-1}},$$
$$\langle x_0 \dot{x}_0\rangle = \langle x(t)\dot{x}(t)\rangle_{n-1}\big|_{t=\Delta t_{n-1}}, \quad \omega_d^2 = \omega_{eq}^2(1 - h_{eq}^2) .$$

EUIVALENT LINEARIZATION METHOD

To calculate theoretically the MS transient reponse of the elastic-plastic system with bi-linear hysteresis, random fluctuation of the response is transformed again into the bi-linear loop having the amplitude which slowly varies with time. When considering that the system vibrates, even in random vibration, with a period near the natural period of the system, and assuming that the transformed bi-linear loop also vibrates with the same period, this period is almost equivalent to that of the bi-linear system, subjected to a sinusoidal excitation, with the same ductility factor and the same stiffness ratio as those of transformed bi-linear loop. The hysteresis damping factor and the circular frequency of this stationary resonant response is assumed to be of the equivalent viscous damping and the equivalent circular frequency of the system which is undergoing elastic-plastic vibration. The method to calculate these coefficients is as follows.

Equation of motion of an elastic vibration from point 1 to point 2 in Fig.2 and that of a plastic vibration from point 2 to point 3, when subjected to a sinusoidal excitation are expressed as:

$$\ddot{x} + \omega_0^2\{x - \bar{X}_Y(\bar{\mu} - 1)(1 - r)\} = -\bar{\alpha}\sin\bar{\omega}t$$
$$\ddot{x} - \omega_0^2\{-rx + \bar{X}_Y(1 - r)\} = -\bar{\alpha}\sin\bar{\omega}t$$

where $\bar{\mu} = X_0/\bar{X}_Y$ and

$\bar{\omega}$ - circular frequency of the sinusoidal excitation,

$\bar{\alpha}$ - acceleration amplitude of the excitation.

From these equation with due boundary conditions at points 1, 2, 3 and with the conditon of that the response is in resonance, following transcendential simultaneous equations are obtained to calculate the equivalent viscous damping factor h_{eq} and the equivalent circular frequency ω_{eq}.

$$F_1(\eta, \upsilon) = G_1(\eta, \upsilon)\{G_2(\eta, \upsilon) + G_3(\eta, \upsilon)\} + G_4(\eta, \upsilon)G_5(\eta, \upsilon) = 0$$
$$F_2(\eta, \upsilon) = H_1(\eta, \upsilon)H_2(\eta, \upsilon) + H_3(\eta, \upsilon)H_4(\eta, \upsilon) = 0$$

In these equations, $r > 0$ and

$$G_1(\eta,\theta)=A+B\cos\theta\ ,\ G_2(\eta,\theta)=\eta(1-\eta^2)(\cos\theta-\cos\eta\theta)\ ,$$
$$G_3(\eta,\theta)=\eta(1-\eta^2)\{\cos(\sqrt{r}\pi/\eta-\sqrt{r}\theta)+\cos\eta\theta\}\ ,\ G_4(\eta,\theta)=(\eta\sin\theta-\sin\eta\theta)(r-\eta^2)$$
$$G_5(\eta,\theta)=B\sin\eta\theta-C\sqrt{r}\sin(\sqrt{r}\pi/\eta-\sqrt{r}\theta)\ ,\ H_1(\eta,\theta)=(A+B\cos\theta)(1-\eta^2)(r-\eta^2)$$
$$H_2(\eta,\theta)=\sin\eta\theta-(\eta/\sqrt{r})\sin(\sqrt{r}\pi/\eta-\sqrt{r}\theta)\ ,\ H_3(\eta,\theta)=\eta\sin\theta-\sin\eta\theta$$
$$H_4(\eta,\theta)=C\{1-\cos(\sqrt{r}\pi/\eta-\sqrt{r}\theta)\}-2(\bar{\mu}-1)\ ,\ a=H_1(\eta,\theta)(r-\eta^2)/H_3(\eta,\theta)\ ,$$
$$A=1+r-\bar{\mu}r\ ,\ B=1-r+\bar{\mu}r\ .\ C=(1-r)/r+\bar{\mu}\ ,\ \eta=\bar{\omega}/\omega_0\ ,\ \theta=\omega_0 t_2\ ,$$
$$\alpha_e=\bar{X}_Y\omega_0^2\ ,\ \bar{\alpha}=X_s\omega_0^2\ ,\ a=\bar{\alpha}/\alpha_e\ ,\ h_{eq}=1/2(X_s/X_0)\ ,\ \omega_{eq}=\bar{\omega}\ .$$

where t_2 denotes the time required for the motion from point 1 to point 2. The calculated results are shown in Fig.3.

In this analysis, root mean square (RMS) value of the response is used for checking plastic deformation. If the RMS displacement increases over the value 1/2.5 of the yield displacement of the system Xy, it is regarded that the plastic deformation has taken place. This assumption must be correct when considering the fact that, in the disital simulation of a random vibration of an elastic system with 200 simulated functions, the ratio of the maximum displacement to RMS displacement at each step is 2.0~3.0 and their expected value is about 2.5. In this analysis, ductility factor is calculated from $\mu=3\sigma_X/(2.5\bar{X}_Y)$, and the result of this analysis is found to be quite accurate for $\mu<4.0$.

EXAMPLES

Models used for the analysis are classified as shown in Table 1. Each model is so designed that 2.5 times the value of its RMS statinary response (i.e., the expected value of the maximum displacement), when subjected to stationary white noise with $S_\alpha=20.0$ in the horizontal direction, might be the yield displacement Xy. This yield displacement is set equal to 1/200 of the height of the structure for models A(A'),B(B'),C(C'), to 1/100 for model D(D') and to 1/300 for model E(E'). Models A,B,C,D,E have the stiffness ratio r= 0.1 and models A',B',C',D',E' have r=0.5. The circular frequency of the vertical vibration is set equal to 10 times the corresponding value of the horizontal vibration.

Power spectral density of $\alpha(t)$ is $S_\alpha=20$, $S_\alpha=80$, $S_\alpha=180$ and $S_\alpha=320$ [cm^2/sec^3], which respectively has its maximum acceleration 150, 300, 450 and 600 [cm/sec^2]. Power spectral density of $\ddot{Z}_V(t)$ is set equal to 1/4 of that of the value S_α. Shape function $\xi(t)$ is

$$\xi(t)=4.0\times\{\exp(-0.35\tfrac{\omega}{2\pi}t)-\exp(-0.70\tfrac{\omega}{2\pi}t)\}$$

This function has its maximum balue 1.0 at $t=2.0\times(2\pi/\omega)$.

In Table 2, the maximum ductility factors μ of the 10 models against various combinations of external forces are shown. In case 1, horizontal load only is considered, in case 2 gravity as well as horizontal load are considered and in case 3 are considered the both with the vertical component of the earthquake excitation. In Table 2, $\Delta\mu$ means the increase of μ for case 2 when compared to case 1, or for case 3 to case 2.

From Table 2, it is clear that the P-Δ effect of the vertical component is quite negligible. As the difference between the circular frequency of horizontal vibration and that of the vertical vibration is large, it seems that the parametric vibration or the like does not occur. It may be pointed out that, if the ductility factor is small, the stiffness ratio affects little the RMS displacement and also the P-Δ effect of gravity; while, if the ductility factor is considerably large (far under the collapse limit), stiffness ratio affects much. In this case, small stiffness ratio causes rapid increase of both the RMS displacement and the P-Δ effect of gravity. This gravity effect is larger in taller structures when constructed based on the same design data as shown in the examples above. So it is important for taller structures to control the yield displacement under the proper value. However, as lower structures genarally have smaller stiffness and relatively large yield displacement, cares must be taken for selecting proper yield displacement values in disigning even lower structures.

CONCLUSIONS

The results of the above analysis may be summarized as follows:

1. P-Δ effect of the vertical component of the earthquake excitation can be neglected.
2. P-Δ effect of gravity may increase the horizontal displacement more than 10% and the effect must be taken into consideration even if the structure is far from the collapse. This is important especially for structures with small stiffness ratio.
3. It is important to control the yield displacement under the proper value especially in taller structures.
4. If ductility factor becomes larger than about 2.0, stiffness ratio affects horizontal displacement and also the gravity effect so much that special cares must be taken in evaluating the stiffness beyond yield displacement.

REFERENCIES

1) P.C.Jennings and R.Husid, ASCE, Vol.94, EM5, pp.1045,(1968).
2) C.K.Sun, G.V.Berg and R.D.Hanson, ASCE, Vol.99, EM1, pp.184, (1973).

Fig.1 Analytical model

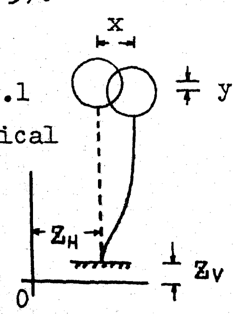
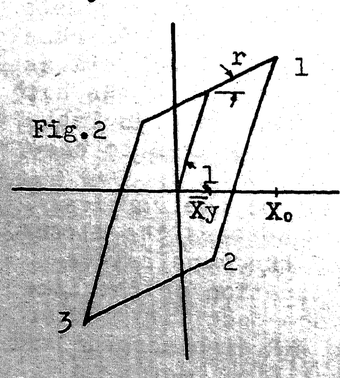


Fig.2



Bi-linear hysteresis

Fig.3

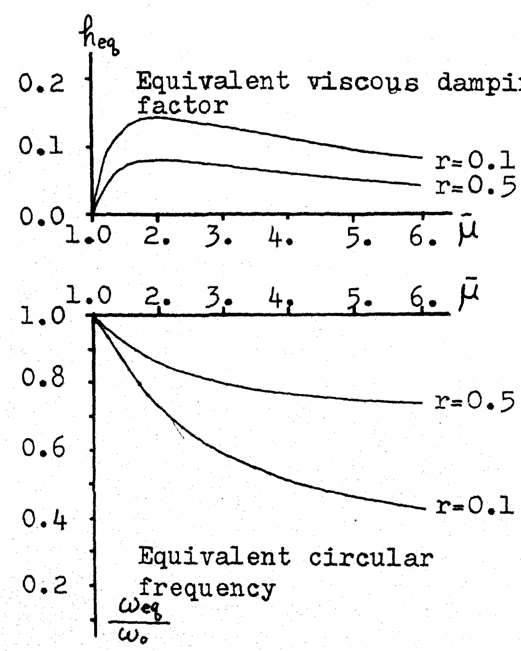


TABLE-1

MODEL	H(cm)	Xy(cm)	Tx(sec)	Ty(sec)	h(%)	η(%)	R
A (A')	400	H/200	0.46	0.046	2.0	10.0	0.1(0.5)
B (B')	2000	H/200	1.36	0.136	2.0	10.0	0.1(0.5)
C (C')	4000	H/200	2.16	0.216	2.0	10.0	0.1(0.5)
~~D (D')~~	400	H/100	0.73	0.073	2.0	10.0	0.1(0.5)
E (E')	4000	H/300	1.64	0.164	2.0	10.0	0.1(0.5)

TABLE-2

TABLE-2-A', model A'

	S_α=20		S_α=80		S_α=180		S_α=320	
case	μ	$\Delta\mu$%	μ	$\Delta\mu$%	μ	$\Delta\mu$%	μ	$\Delta\mu$%
1	.822	—	1.36	—	1.77	—	2.42	—
2	.829	0.85	1.38	1.02	1.80	1.63	2.47	2.39
3	.829	0.	1.38	0.	1.80	0.	2.47	0.

TABLE-2-B', model B'

	S_α=20		S_α=80		S_α=180		S_α=320	
case	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$
1	.824	—	1.36	—	1.77	—	2.42	—
2	.836	1.45	1.39	1.97	1.82	3.21	2.53	4.20
3	.836	0.	1.39	0.	1.82	0.	253	0.

TABLE-2-C', model C'

	S_α=20		S_α=80		S_α=180		S_α=320	
case	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$
1	.824	—	1.36	—	1.77	—	2.42	—
2	.839	1.82	1.40	2.48	1.84	3.65	2.55	5.39
3	.839	0.	1.40	0.	1.84	0.	2.55	0.

TABLE-2-D', model D'

	S_α=20		S_α=80		S_α=180		S_α=320	
case	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$
1	.824	—	1.36	—	1.77	—	2.42	—
2	.841	2.06	1.40	2.77	1.85	4.33	2.58	6.39
3	.841	0.	1.40	0.	1.85	0.	2.58	0.

TABLE-2-E', model E'

	S_α=20		S_α=80		S_α=180		S_α=320	
case	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$
1	.824	—	1.36	—	1.77	—	2.42	—
2	.832	0.97	1.38	1.31	1.81	2.02	2.49	3.00
3	.832	0.	1.38	0.	1.81	0.	2.49	0.

TABLE-2-A, model A

	S_α=20		S_α=80		S_α=180		S_α=320	
case	μ	$\Delta\mu$%	μ	$\Delta\mu$%	μ	$\Delta\mu$%	μ	$\Delta\mu$%
1	.822	—	1.32	—	1.60	—	2.44	—
2	.829	0.85	1.33	1.06	1.62	1.50	2.59	5.84
3	.829	0.	1.33	0.	1.62	0.	2.59	0.

TABLE-2-B, model B

	S_α=20		S_α=80		S_α=180		S_α=320	
case	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$
1	.824	—	1.32	—	1.60	—	2.46	—
2	.836	1.45	1.34	1.81	1.64	2.74	2.73	11.0
3	.836	0.	1.34	0.	1.64	0.	2.73	0.

TABLE-2-C, model C

	S_α=20		S_α=80		S_α=180		S_α=320	
case	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$
1	.824	—	1.32	—	1.60	—	2.46	—
2	.839	1.82	1.35	2.34	1.66	3.55	2.82	14.7
3	.839	0.	1.35	0.	1.66	0.	2.82	0.

TABLE-2-D, model D

	S_α=20		S_α=80		S_α=180		S_α=320	
case	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$
1	.824	—	1.32	—	1.60	—	2.45	—
2	.841	2.06	1.35	2.80	1.67	4.24	2.90	18.0
3	.841	0.	1.35	0.	1.67	0.	2.90	0.

TABLE-2-E, model E

	S_α=20		S_α=80		S_α=180		S_α=320	
case	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$	μ	$\Delta\mu$
1	.824	—	1.32	—	1.60	—	2.45	—
2	.832	0.97	1.34	1.36	1.63	1.93	2.64	7.61
3	.832	0.	1.34	0.	1.63	0.	2.64	0.