THE MINIMAX, VERSUS THE STATISTICAL, PREDICTION OF EARTHQUAKE RESISTANCE

by

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SYNOPSIS

The paper deals with the utilization of statistical data towards the assessment of the earthquake resistance of structures, especially relative to a recently proposed method which minimizes the need for such data. This method, called the "minimax" method here, is first reviewed. It is then shown how probabilistic information could be combined with the method and that considerable design economies could be expected from doing so. However, evidence is finally presented which indicates that this information must be very reliable and very accurate, if it is to be useful, and that serious errors may be expected otherwise.

INTRODUCTION

One of the main problems of earthquake engineering is the prediction of the earthquake resistance of existent or proposed structures. The solution to this problem is complicated by many factors, among them one which has been called the "factor of ignorance" [1], a term intended to say that not nearly as much is known concerning the incidence, intensity, and other characteristics of strong ground motion, than is needed to make such predictions reliably.

There is an apparent way out of the various uncertainties that beset the assessment of earthquake resistance. It seems a natural one and it has in fact been adopted in a large number of studies. This is to assume that an earthquake is a random phenomenon whose incidence and nature are governed by certain probability distributions. However, one can question whether or not this is a legitimate way out of the difficulty, for it assumes that those probability distributions are fully known. The fact is that these are hardly known, either. It is quite appropriate to ask, therefore, whether use of probabilities really has overcome the factor of ignorance or whether it has not merely shifted it from one set of parameters to another.

Recently [2], a method was proposed which minimizes the amount of probabilistic information that is needed in making an assessment of structural earthquake resistance. Its basic idea is that, in view of the various uncertainties, that surround the problem, the resistance be assessed by studying the response of the structure to an excitation which is the most "critical" one for it, among those that can reasonably be expected at that location. Ideas of this kind are not new to engineering. They lead to the so-called "worst-case analyses" and "minimax methods." The techniques are quite satisfying conceptually but they often suffer from a fatal defect: in an effort to be prepared for the worst among all forseeable eventualities, they are frequently too pessimistic to be practical. In the present instance, there is fortunately considerable evidence to the contrary.

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Nevertheless, several questions present themselves, for instance:

(a) Can probabilistic information be used to reduce the conservatism of the minimax method if that should be desirable and, if so, how should it be used?

(b) If available probabilistic information is not considered very reliable, should it be used or would it better to proceed without it?

In this paper, an attempt is made to provide at least partial answers to these two questions. The next section gives a brief outline of the minimax method, and the sections that follows provides an answer to Question (a). Evidence is presented there which indicates that substantially more optimistic assessments are likely from the minimax method when the underlying probability distributions are completely known. In fact, there is good reason to question whether that method should be used in the first place under these circumstances. Question (b) is taken up in the last section, and it is shown that the probabilistic prediction of earthquake resistance appears to be highly sensitive to the nature of the probability distributions: unless these distributions are known very precisely, large errors may result from their use. Hence, considerable caution may be necessary in how statistical data are used and in what interpretations are attached to the results to which they lead. In many cases, one may in fact be better off without them.

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RECAPITULATION OF MINIMAX METHOD

The assessment of the earthquake resistance of a structure is a problem of decision-making under uncertainty. The decision is whether or not to rate a structure as being earthquake-resistant or, perhaps more generally, what level of earthquake-resistance to assign to it. The uncertainty arises from the fact that very little is known concerning the nature of the earthquakes that should be anticipated at a particular location, the interaction of a structure at the location with the ground, and similar factors.

In such decision-making problems, the so-called "minimax" approach (or also the "worst-case analysis") has sometimes been successful. This approach rests on a simple notion, namely this: in the presence of uncertainty, the decision-maker should assume that the conditions confronting him are the least advantageous which are compatible with what he knows of the situation. In such a way, he will be prepared for the worst and can only gain if the situation actually turns out to be better.

In the context of earthquake resistance, the minimax approach more specifically assumes that the person charged with the assessment of the earthquake resistance of a structure can set an upper limit M on the intensity of earthquakes which the structure should be able to withstand at its particular location. Beyond that, however, no information is assumed

to be available to him, or to be reliable enough to be used by him. This includes information of statistical as well as non-statistical nature. (In practice, the person may in fact wish to choose several limits M depending on what kind of damage he is willing to tolerate and on what the consequences are of a structure's exceeding that tolerance.)

To be more specific, let x denote the excitation to the structure, for instance the ground acceleration, and $\|x\|$ its "intensity". The inequality

$$\|x\| \leq M \tag{1}$$

defines the class of earthquakes which is being considered by the investigator of earthquake resistance.

Suppose next that y is one of the structural variables of interest. Thus, y might be the relative displacement between two floors of a building, the compressive stress in one column, or the bending moment in another. Whichever it is, it is assumed that a certain tolerance level has been set for this variable. This tolerance would then be in the form of an inequality similar to (1), namely

$$\|y\| \leq L \tag{2}$$

which says that "severity" $\|y\|$ of the response of the variable y to the excitation x should not exceed the limit L.

The idea of the method then is simply this [2]: one searches for the excitation x^* among all those satisfying (1) which produces the most severe response y^*. (These will be called the "critical excitation" and the "critical response" of the structure.) If the severity $\|y^*\|$ of this response does not exceed L in (2), the structure can justifiably be regarded as being resistant to all earthquakes under consideration.

The analysis which is based on this idea patently depends on what interpretations are attached to the quantities $\|x\|$ and $\|y\|$, the "intensity" of the excitation and the "severity" of the response. A number on interpretations are possible. The one that has proven most useful for $\|x\|$ is

$$\|x\|^2 = \int_{-\infty}^{\infty} x^2(t)\, dt \tag{3}$$

which was suggested by Housner and Jennings [3] and which is often called the "energy" of the excitation. For the severity $\|y\|$ of the reponse, the response peak

$$\|y\| = \max_t |y(t)| \tag{4}$$

seems to be the best choice in general. It is found [2] that, with these measures, the critical excitation and the peak of the critical response are related in a very simple way to the characteristics of the structure, at least if the structure is elastic. If h is the impulse response of the structure, then

$$x^*(t) = \pm \frac{M}{N} h(-t) \quad , \quad \|y^*\| = MN \tag{5}$$

where

$$N^2 = \int_{-\infty}^{\infty} h^2(t)\, dt \ . \tag{6}$$

In the particular case of a structure with a single degree of freedom

$$h(t) = \omega_n^{-1} \exp(-\zeta\omega_o t) \sin\omega_n t \,, \qquad \omega_n^2 = \omega_o^2(1-\zeta^2) \tag{7a}$$

$$N^2 = 4\omega_n^3 \zeta \,. \tag{7b}$$

In this paper, the assumption will be made throughout that the structure responds elastically and that the measures for the intensity of the excitation, and of the severity of the response are those in eq.'s (3) and (4).

THE UTILIZATION OF STATISTICAL INFORMATION

The idea of assessing the earthquake resistance of a structure by way of its critical excitation, is an attempt at minimizing the amount of statistical information used in the assessment. This seems desirable because the kind of information that is really needed for this purpose will no doubt be difficult to obtain. One can nevertheless ask the question which was raised in the Introduction, namely, whether statistical data can be put to good use in conjunction with the minimax method, or in fact whether the minimax method should be resorted to in the first place when such data are available.

In this section, a partial answer is given to that question. It will more specifically be assumed that all probability distributions governing ground motion are completely known. It will develop that in such case the minimax method is really inappropriate. However, if one insists on using it anyway one can do so and, by all indications, substantially reduce the tolerances on one's design.

In the preceding section, a criterion for the earthquake resistance of a structure was set down: according to it, a structure would be considered safe against a given excitation if the peak response $\|y\|$ of one (or more) of its structural variables does not exceed a certain tolerance level L. In a probabilistic formulation, the event to be guarded against is therefore an exceedance of L by the response peak $\|y\|$,

$$\|y\| > L \tag{8}$$

and the structure should presumably be so designed that the probability $P\{\|y\| > L\}$ of this event is acceptably small.

Now, if the probabilistic nature of the earthquakes to be expected at a certain location really is completely known, as is being assumed in this section, the probability $P\{\|y\| > L\}$ can in principle be calculated directly and the question therefore be answered of whether it is, or is not, acceptably small. This has in fact been done [3] [4] [5] [6], either by Monte Carlo methods or analytically. The analytical approach which will be needed here, proceeds as follows. It assumes that

(a) the most important phase of an earthquake is a segment of a sample function drawn from a stationary Gaussian random process

(b) the duration T_s of this phase is in the order of 10 to 15 sec.

(c) the auto-covariance of the random process is one suggested by Kanai [7], namely

$$R_x(\tau) = C e^{-\alpha|\tau|} \cos(\beta|\tau| - \psi) \tag{9a}$$

with

$$C = 6.14 \times 10^{-3} \text{ ft}^2 \text{ sec}^{-3},$$

$$\alpha = 9.3 \text{ sec}^{-1}, \quad \beta = 12.5 \text{ rad sec}^{-1}, \quad \tan\psi = .75 \tag{9b}$$

(d) and the structure under consideration is elastic, with a single degree of freedom, so that (7) holds.

The auto-covariance R_y of the response y of the structure is then made up of two terms of the same form as (9a), viz.

$$R_y(t) = C_1 e^{-\alpha|\tau|} \cos(\beta|\tau| - \psi_1) + C_2 e^{-\alpha_2|\tau|} \cos(\beta_2|\tau| - \psi_2) \tag{10a}$$

in which α and β are the same as in (9b), and

$$\alpha_2 = \zeta\omega_n, \quad \beta_2^2 = \omega_n^2(1-\zeta^2) \tag{10b}$$

Under these assumptions, the probability of the event of interest can be calculated. This is the probability $P\{\|y\| > L\}$ of the response peaks exceeding the tolerance level L. A formula for this exceedance probability, due to Pickands [9] and valid for large L and large γT_s, is

$$P\{\|y\| > L\} = \exp[-\exp(-L\sqrt{2\log 2\gamma T_s} + \eta)] \tag{11a}$$

where the constant η is

$$\eta = 2\log 2\gamma T_s + \frac{1}{2}\log\pi - \log\log 2\gamma T_s \quad . \tag{11b}$$

and γ is the first coefficient in the Maclaurin series for $R_y(\tau)$,

$$R_y(\tau) = R_y(0)[1 - \gamma|\tau| + o(\tau)].$$

This probability can therefore be calculated quite easily.

In view of the formula (11), if the probabilistic nature of the ground motion is really known completely (or at least adequately), there is no real incentive for using the minimax method in the first place. If one insists on doing so, anyway, one can perhaps invent the following "scenario" for the circumstances under which one might wish to combine the minimax and the probabilistic methods.

Suppose that the designer of a structure has completed his design. He knows the safety limit L which one of the structural variables, namely y, should not exceed, and he further knows the impulse response h which relates y to the ground acceleration x. In particular, therefore, he knows the quantity N in (6) or its approximation (7b). He can then conclude immediately that the safety limit L cannot possibly be exceeded as long as the intensity $\|x\|$ of the excitation does not exceed L/N. In

symbols, he can be sure that

$$\|y\| \leq L$$

i.e., that his structure will be absolutely safe, provided that

$$\|x\| \leq L/N \tag{12}$$

Suppose now that the designer made the disturbing discovery of L/N being rather small. This might be disquieting to him since it would imply that the chances of an earthquake with an intensity

$$\|x\| > L/N \tag{13}$$

are fairly high. Fortunately for the designer, the mere fact that the intensity $\|x\|$ exceeds (L/N), does not also imply an exceedance of the structural limit L, and hence of structural damage. On the contrary, there are many excitations x whose intensities $\|x\|$ do exceed (L/N) but which do not generate response peaks $\|y\|$ in excess of L. The quantity, therefore, that might be of interest to the designer at this point is the probability that $\|y\| > L$ when it is known that $\|x\| > L/N$. In symbols, the designer might be interested in the conditional probability

$$P\{\|y\| > L \mid \|x\| > L/N\}, \tag{14}$$

for it will tell him what risks he incurs from the fairly strong earthquakes, with intensities (13). This probability can in fact be estimated in a way to be explained presently. It will be seen that it can be quite low. The designer accordingly may incur little risk from those earthquakes.

To arrive at an estimate of (14), one can reason as follows. Consider the probability $P\{\|y\| > L\}$ of the exceedance of the safety limit L. This probability can be written

$$P\{\|y\| > L\} = P\{\|y\| > L \mid \|x\| \leq L/N\} \cdot P\{\|x\| \leq L/N\}$$
$$+ P\{\|y\| > L \mid \|x\| > L/N\} \cdot P\{\|x\| > L/N\}.$$

But if $\|x\| \leq L/N$, the response peak $\|y\|$ cannot possibly be greater than L. That is,

$$P\{\|y\| > L \mid \|x\| \leq L/N\} = 0.$$

The probability (14) of interest therefore is

$$P(\|y\| > L \mid \|x\| > L/N) = \frac{P\{\|y\| > L\}}{P\{\|x\| > L/N\}} \quad . \tag{15}$$

and it can be calculated if the probabilities on the right can be. The one in the numerator can in fact calculated from (11), at least under certain assumptions concerning ground motion and structural response. Under the same assumptions, however, the one in the denominator can be calculated as well. One can more particularly show that

$$P\{\|x\| > \frac{L}{N}\} = \exp\left(-L^2/2S_m N^2 + A\right) \tag{16}$$

where A is a constant and S_m is the maximum of the spectral density of the ground motion. The formula is valid for large L and large γT_s. In the special case of ground motion characterized by (9), $S_m = 8.9 \times 10^{-4}$ ft^2 sec^{-3}, as was determined by Shinozuka [10], and

$$A = 2\alpha T_s \quad .$$

Combining (11) and (16), one obtains for the conditional probability of interest

$$P\{\|y\|>L \mid \|x\|>L\} = \exp\{L^2/2S_m N^2 - A - \exp[-L\sqrt{2\log 2\gamma T_s} + \eta]\} \qquad (17)$$

The first two-terms in the exponent are due to the denominator in (15), and the third to the numerator. It is clear that for reasonably large L, the third will dominate the other two. In other words, the conditional probability of interest will be rather weakly influenced by the incidence of the strong earthquakes which obey (13). The designer, therefore, will often commit little error if he ignores the denominator, or else, he will gain considerably if he bases his design on the probability of the event (8) rather than of (13).

EFFECT OF INACCURACY IN THE STATISTICAL INFORMATION

The analysis presented in the preceding section indicates that design economies can be expected from the utilization of probabilistic data concerning the nature of the earthquakes at a given locality. The analysis was, however, based on the assumption that the statistics of the earthquakes are of a very special kind and furthermore that they are perfectly known. This section is devoted to a critical review of this assumption, of how well justified it may be, and of what the consequences might be of its failure.

The particular issue to be discussed in this section is the sensitivity of the critical exceedance probability $P\{\|y\|>L\}$ to the assumption that the strong ground motions at a certain location form a Gaussian random process. There seems to be relatively little basis for this assumption, yet even very small departures from the Gaussian (as will be shown below) can produce large variations in that probability.

A Gaussian random process, speaking now very qualitatively, will typically result from the linear superposition of the effects of many elementary disturbances of some kind which are random but which are generated by a source of constant mean intensity. (The text book examples of this kind of phenomenon are the shot noise in vacuum tubes which is due to randomness of electron emission from a heated cathode, and the thermal noise generated by the Brownian motion of the molecules in resistors.) The population of earthquakes, however, is not obviously generated in any such way. It is, first of all, difficult to conceive of a seismic source which somehow underlies the generation of the earthquakes in a given location and which produces a stream of elementary shocks of a fixed mean intensity. Furthermore, there is little reason to think that the disturbance observed on the earth's surface is the result of a linear superposition of those elementary shocks since the medium through which the seismic waves travel is highly nonlinear.

One can perhaps hope that the probability distribution which governs the ground acceleration x is roughly Gaussian at least in the vicinity of $x = 0$. Kobori et al [11] have investigated this but have had to conclude

that even that is not clearly so. Unfortunately, in the present contest, it is the tails of the distribution which matter, and there is no evidence, observational or other, to indicate Gaussian behavior there.

In what follows, a semi-quantitative analysis is presented which shows how sensitive the results of a probabilistic analysis of structural safety, such as the one in the preceding section, are to the behavior of the probability distribution along its tails. An exact analysis unfortunately is greatly complicated by the fact that there exists virtually no theory concerning the exceedance probabilities $P\{\|y\| > L\}$ of non-Gaussian random processes and hence no formulae of the type (11) which hold for the Gaussian ones. In order to overcome this difficulty, one can perhaps reason as follows. Compare formula (11) with the well-known expression for the probability that at least one among n independent Gaussian random variables $y_1, y_2, \ldots y_n$ exceeds a limit $\pm L$. This expression is [12, p. 374]

$$P(L) = \exp[-\exp(-L\sqrt{2\log n}) + \eta'] \tag{18a}$$

with

$$\eta' = 2\log 2n - \frac{1}{2}\log 4\pi - \log\log 2n \ . \tag{18b}$$

The comparison shows that the exceedance probability of a Gaussian random process is essentially the same as that of $n = \gamma T$ independent Gaussian random variables. It is conceivable that a similar rule holds also for non-Gaussian processes. In such a case, the analysis of a random process could be reduced to that of so many independent variables with a certain non-Gaussian distribution. This kind of equivalence, at any rate, will be assumed to be valid in what follows.

The question to be discussed here than is this. Suppose that n independent samples $y_1, y_2, \ldots y_n$ of a structural random variable with the probability density p(y) have been observed. What is the probability that at least one of them exceeds a given threshold ($\pm L$)? We shall seek an answer under the following additional assumptions:

(1) p(y) is Gaussian between two given limits, (-Y) and (+Y) say, but outside these limits, it belongs to one of the distributions of the "exponential type". Thus,

$$p(y) = \begin{cases} A_1 \exp(-y^2/2\sigma^2)/\sqrt{2\pi\sigma} & |y| \leq Y \\ A_2 |y|^m \exp(-B|y|^r/r) & |y| \geq Y \end{cases} \tag{19}$$

where $r > 0$. p(y) is assumed continuous at $y = \pm Y$.

(2) n is large.

(3) L is large and $L > Y$.

Assumptions (2) and (3) are fairly traditional in the theory of the statistics of extremes [12; p. 376]; the terminology in (1) is similar to Gumbel's [8, p. 120]. The parameters A_1 and A_2 in (19) are determined by the requirement

$$\int_{-\infty}^{\infty} p(y)\, dy = 1$$

and by a second one, namely that p(y) be continuous at $y=\pm Y$. For r=2, m=0, $A_1 = A_2 = 1/\sqrt{2\pi}\sigma$, and $B = 1/\sigma^2$, p(y) is the same Gaussian distribution beyone $\pm Y$, as within these limits. Therefore, if the parameters of (19) are close to these, the distribution p(y) is only slightly non-Gaussian. Nevertheless, the values one obtains for the exceedance probability $P\{\|y\|>L\}$ can be greatly affected. In fact, one finds that under the above assumptions,

$$P\{\|y\|>L\} = \exp\{-\exp\} - \{[LB^{\frac{1}{r}}(r\log n)^{\frac{r-1}{n}} + \eta''\ ]\} \qquad (20)$$

where η'' is a constant similar to η' in (18b), and depending on n and on the various parameters in (19). Eq. (20) is valid for large L and large n (terms that are $O(\log n)^{-1}$ and $O(L^{-1})$ are disregarded in it). The exceedance probability in (20) is of the familiar double-exponential form (18) which holds when the variants are normal and in fact reduces to that form when p(y) is Gaussian. However, since all parameters of the distribution enter into the exponent, and some even into an exponential in the exponent, the distribution itself is extremely sensitive to any departure from the Gaussian in the direction of one of the other distributions of the exponential type. For that matter, it is very sensitive even if it remains Gaussian but if its parameters are varied.

A numerical example may illustrate the difficulty. Suppose that a safety limit for a structure had been set on the assumption that the earthquakes at a particular location formed a Gaussian random process and that the latter were equivalent to $n = \gamma T_s = 5$ independent normal variates, as far as the exceedance probability is concerned. Suppose further that the desired exceedance probability were 2% and that the tolerance limit L had been set accordingly, but on the assumption that the underlying random process is Gaussian. Suppose finally that the true nature of the random process were not Gaussian (i.e., m=0) beyond $y = \sigma$ but exponential with m=1 in (19). This would introduce a very slight departure from the Gaussian along the tails of the distribution, probably much too slight to be detected from available statistical data. Nevertheless, the exceedance probability (20) is found to be 14.1% in this case, that is seven times as high as anticipated from the Gaussian distribution.

One can therefore conclude that any probabilistic statements concerning the earthquake resistance of structure may be quite sensitive to the assumptions concerning the underlying probabilities, and may be quite unreliable unless those assumptions are well founded.

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NOTE ADDED IN PROOF.

Professors L. H. Koopmans and C. Qualls have kindly read a preprint of the above paper and have made several comments on it, including the following which is particularly important. They point out that eq. (11) of the paper fails to take into account the fact that T_s is a function of L, and in fact that it increases rather rapidly with L. A formula which takes this increase into account is given by C. Qualls and H. Watanable in their paper on Asymptotic properties of Gaussian process (Ann. Math. Stat. 43, 1972, pp. 580-596). The formula would replace the expression (11) for the probability $P\{\|y\| > L\}$ by one which is proportional to $L \exp(-L^2/2\sigma^2)$, and the expression for the conditional probability in (17) by one which is proportional to $\exp \alpha L^2$ where α can be positive or negative depending on whether or not

$$S_m N^2 < \sigma^2.$$

The conclusion stated after eq. (17) could then be much too optimistic.