MAXIMUM RESPONSE STATISTICS FOR A LINEAR STRUCTURE

by

L. D. Lutes[I] and N. C. Chokshi[II]

SYNOPSIS

Results are presented from an empirical study of the probability distributions of extreme values and first passage times of a linear oscillator subjected to random excitation. The empirical data is compared with the results of several approximate analytical methods for predicting the response statistics, and the advantages and limitations of the various methods are discussed. It is verified that the extreme value distribution is closely approximated by a form presented by Gumbel, and that the first passage time distribution for stationary response is approximately exponential in form.

PROBLEM FORMULATION

The structural response problem considered here has been modeled as the response of a linear single-degree-of-freedom oscillator to a "burst" of white noise. The response will be designated $x(t)$, and the parameters of the oscillator are ω_o = natural circular frequency and β_o = fraction of critical viscous damping. The excitation is nonzero only during the interval $0 \leq t \leq T$, and is a segment of stationary, Gaussian (normal) white noise during that interval. The system was modeled on an electronic analog computer in order to obtain empirical results. The details are included in Ref. (1). Statistical quantities were estimated by ensemble averages, where the ensemble size was variable, but never less than 1000.

Two types of initial conditions have been studied. The first is zero initial conditions $[x(0) = 0,\ \dot{x}(0) = 0]$. The second situation considered has random initial values of x and $\dot{x}$ such that the random response studied is the same as a segment from a stationary response process. The standard deviation of $x(t)$ in the stationary response situation will be denoted by σ_s. This value is also approached by the zero-start response for large T and t.

The extreme values used here are Y1(T) and Y2(T), which represent the largest values achieved during the interval of the excitation by $x(t)$ and $|x(t)|$, respectively. The value Y2 is essentially the same as the quantity typically used in plotting earthquake response spectra. The quantity used to present the results is

$$L_i(k, N) = \text{Prob.}\ \{Yi(2\pi N/\omega_o) \leq k\sigma_s\} \qquad (i = 1, 2) \qquad (1)$$

Note that this quantity is the probability of survival of an excitation which persists for N response cycles, if failure consists of exceeding the level $k\sigma_s$ Obviously a plot of $L_i(k, N)$ versus k is the usual form of the cumulative probability distribution of the random variable $Yi(2\pi N/\omega_o)$. A plot of $L_i(k, N)$ versus N, on the

[I] Associate Professor of Civil Engineering, Rice University, Houston, Texas.

[II] Structural Engineer, Caudill Rowlett Scott, Inc., Houston, Texas.

other hand, is closely related to the cumulative distribution of another physically significant random quantity called first passage time. In particular, let R1(b) represent the random time when x(t) first crosses the level b from below, and R2(b) be the time when $|x(t)|$ has its first crossing. Then

$$L_i(k, N) \approx L_i(k, 0) \text{ Prob. } \{ Ri(k\sigma_s) > 2\pi N/\omega_o \} \qquad (i = 1, 2) \qquad (2)$$

This equation is exact if either Ri(b) is statistically independent of whether or not the level b is exceeded at time zero, or if $L_i(k, 0) = 1$ (which is true for the zero-start situation).

EMPIRICAL AND ANALYTICAL RESULTS

In the analog investigation it was convenient to estimate the $L_i(k, N)$ values directly. For a given N value an ensemble of x(t) responses were produced, and the number of these responses for which x(t) or $|x(t)|$ ever exceeded $k\sigma_s$ was counted. This number divided by the ensemble size was used as the estimate of $\{1 - L_i(k, N)\}$. Figures 1 and 2 show the dependence on k and N of some such empirical estimates of $L_i(k, N)$. Figure 1 is plotted on the so-called Gumbel extreme probability paper. The ordinate on this paper is such that any distribution of the form

$$L = \exp [-\exp \{ -(\pi/\sqrt{6}) (k\sigma_s - \mu_Y)/\sigma_Y - \gamma \}] \qquad (3)$$

will plot as a straight line. The parameters μ_Y and σ_Y represent the mean and standard deviation of Y, and are related to the intercept and slope of the straight line. The quantity γ is Euler's constant (0.577213...). Gumbel developed eq. (3) in the course of studying the distribution of the largest value from a family of independent, identically distributed random variables (2).

It is immediately obvious from Fig. 1 that the empirical data is closely approximated by straight lines on the Gumbel paper. One implication of this close agreement is the fact that a $L_i(k, N)$ function is adequately known if the two functions $\mu_{Yi}(N)$ and $\sigma_{Yi}(N)$ are known. Figure 3 shows empirical values of these mean and standard deviation functions. These values were not measured directly, but rather were obtained from the slopes and intercepts of the "best fit" straight lines in plots like Fig. 1.

Note that for $\beta_o = 0.04$ Fig. 3 includes data for both the Y1 and Y2 extreme values in both zero-start and stationary response. Lesser amounts of information are included for $\beta_o = 0.01$ and 0.10. The data for $\beta_o = 0.10$ are reproduced from Ref. 3, and the other values were measured in the course of this study (1). Only selected portions of the data were included in Figs. 1 and 2, but the data shown are typical. In particular, all the situations were closely approximated by straight lines on Gumbel paper.

The number of available approximate analytical techniques for predicting or bounding $L_i(k, N)$ values is large (1) and constantly growing. Figures 2 and 4 include results from a few of the methods which appear to be most useful on the bases of simplicity and/or accuracy. Crandall (4) has previously presented a survey and comparison of many of the techniques developed prior to 1970, and his work is listed here as the reference for such techniques.

A common feature of a number of the approximate techniques has been the result that $L_i(k, N)$ for stationary response varies exponentially with N, so that the results plot as a straight line on the logarithmic paper used in Fig. 2 for first-passage time distributions. The simplest method assumes that crossings of the level $k\sigma_s$ are independent so that the number of crossings is a Poisson process(4). The clump-size approximation attempts to correct for the effect of this independence assumption(5). The techniques developed by Mark(4) and Vanmarcke (4, 6) use Markov process assumptions and Rice and Beer(4) use a renewal process assumption as alternatives to relax the independent crossings assumption. The renewal(4), first-order entropy and pseudo-gaussian(7) approaches all give non-exponential dependence on N. The Poisson and entropy results involve the least calculation, with Mark's and Vanmarcke's methods requiring relatively little more work. The other three methods involve rather cumbersome numerical integrations.

Judging on the bases of the form of the plots in both Figs. 2 and 3, the actual numerical values computed, and the ease of computation, Vanmarcke's method seems to be the best of those considered. However, for stationary response with $k = 2$, the Mark and clump-size results may be preferred. The form of the N dependence of the non-exponential methods seems to be better than the exponential form, but the k dependence of the entropy method is not very satisfactory, and the other non-exponential methods are very difficult to compute for smaller k values. It appears, rather surprisingly, that the k dependence of the Gumbel distribution agrees more closely with the empirical data than do any of the approximate theories derived especially for continuously parametered processes.

Note that two L values completely define the $L(k, n)$ function if one accepts the approximations of a Gumbel distribution and an exponential distribution with respect to k and N, respectively. For example, values of $L(k_1, N_1)$ and $L(k_2, N_1)$ give all values of $L(k, N_1)$ (Fig. 1), and these values with the initial condition values of $L(k, 0)$ give all values of $L(k, N)$ (Fig. 2).

ACKNOWLEDGMENTS

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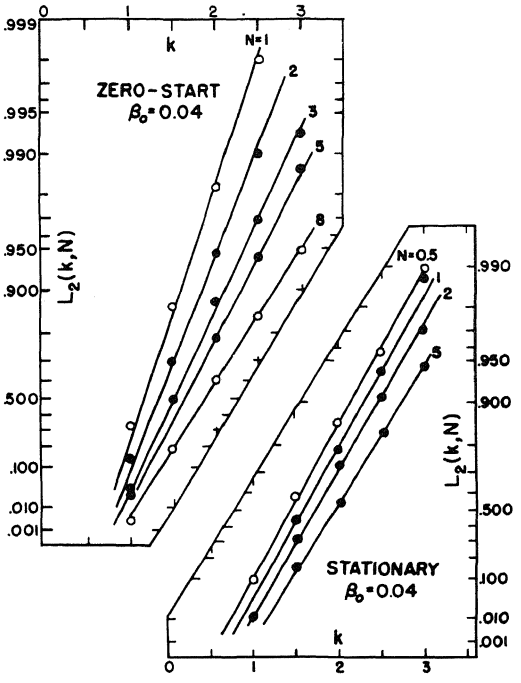

FIG. 1. EMPIRICAL EXTREME VALUE DISTRIBUTIONS

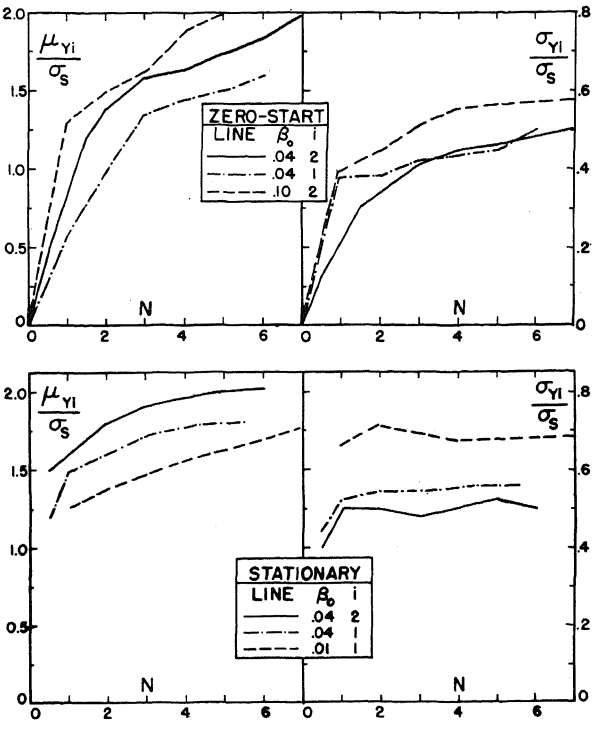

FIG. 3. EXTREME VALUE MEANS AND STANDARD DEVS.

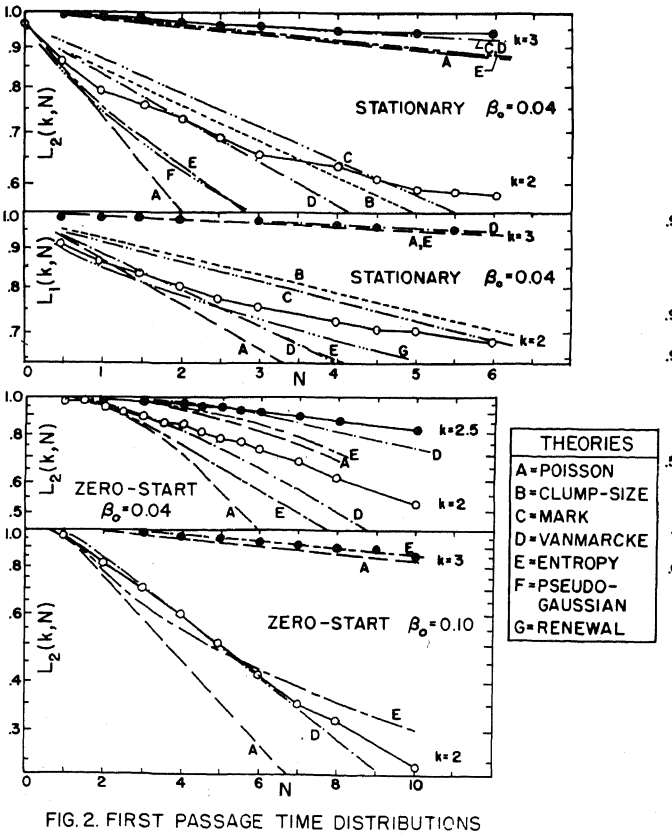

FIG. 2. FIRST PASSAGE TIME DISTRIBUTIONS

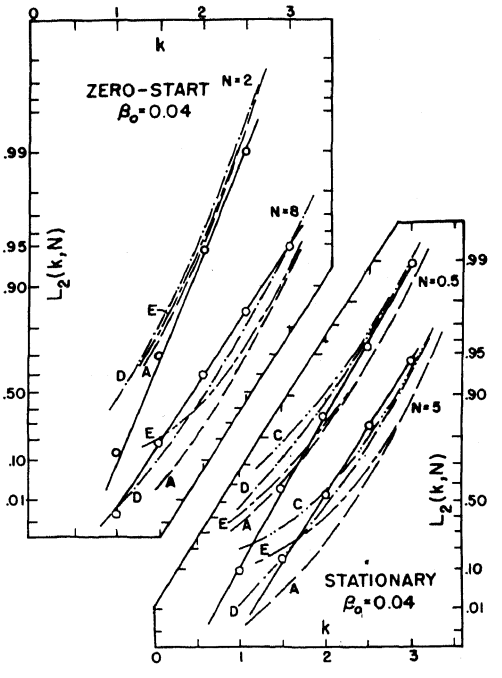

FIG. 4. EXTREME VALUE DISTRIBUTIONS