LINEAR AND NONLINEAR EARTHQUAKE ANALYSIS OF COMPLEX STRUCTURES

by

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SYNOPSIS

A survey of recent developments in the deterministic earthquake response analysis of large, complex structures is presented. The finite element method of discretization is assumed and emphasis is directed to the procedures used to solve the system equilibrium equations. Recently developed eigenproblem algorithms are presented, which solve economically systems of very large order and bandwidth. Various popular direct integration schemes are briefly compared and an improved version of the theta algorithm is proposed as a very effective scheme. For nonlinear analysis an incremental formulation of the equations of motion is presented and the experience gained with the stability and accuracy of the solution is discussed.

INTRODUCTION

The earthquake analysis of large and complex structures has become increasingly important during recent years. This is particularly exemplified by the requirement that nuclear reactor power generating plants be designed to resist relatively strong earthquake motions. The safety of a structure may be increased and the cost be reduced, if a refined linear and, in particular, nonlinear analysis can be carried out. However, a refined analysis may mean that for many time points a very large discrete system of the continuous structure need be analyzed.

The finite element analysis of structures has proven to be highly effective and a linear analysis of practically any structure can now be performed [1] [5] [7] [11]. With regard to nonlinear analysis, practical solutions to many different problems have been obtained. However, apart from the research required to define appropriately the element material properties, improvements in the numerical procedures for the solution of the finite element system equilibrium equations need be sought. The cost of an analysis is directly related to the efficiency of solving the system equilibrium equations, and can be prohibitively large, if inappropriate solution algorithms are used.

The purpose of this paper is to summarize the latest developments in the solution procedures used at the University of California, Berkeley. It is assumed that the actual structure is idealized as an assemblage of finite elements. The paper discusses briefly the formulation of the equations of motion, the solution of the eigenvalue problem and the direct

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integration of the equations of equilibrium for both linear and nonlinear systems.

FORMULATION OF EQUATIONS OF MOTION

The first step in a finite element earthquake analysis is the discretization process, in which the actual continuous structure is idealized as an assemblage of discrete finite elements interconnected at node points [1] [5] [7] [12]. The number of finite elements selected, in general, depends on the physical complexity of the structure and the required accuracy in the displacements and stresses. The analysis of the system of structural elements involves the calculation and solution of the equations of motion corresponding to the nodal degrees of freedom. Various procedures can be used to arrive at the equations of equilibrium, which may be written as

$$F_t^i + F_t^d + F_t^e = R_t \tag{1}$$

where the subscript t indicates that equilibrium at time t is considered, and F^i, F^d, F^e and R are the inertia, damping, elastic and externally applied forces, respectively.

When the system is linear, the force vectors can be expressed directly in terms of the physical properties of the finite elements, i.e.,

$$F_t^i = M\ddot{u} \quad ; \quad F_t^d = C\dot{u} \quad ; \quad F_t^e = Ku \tag{2}$$

where M, C and K are, respectively, the constant mass, damping and stiffness matrices of the element assemblage; u are the dynamic nodal point displacements, and a "dot" denotes time derivative. Hence Eq. (1) becomes

$$M\ddot{u} + C\dot{u} + Ku = R_t \tag{3}$$

The load vector R_t is a result of the ground displacements u_g at the supports of the structure,

$$R_t = [MK^{-1} K_g - M_g]\ \ddot{u}_g + [CK^{-1} K_g - C_g]\ \dot{u}_g \tag{4}$$

where M_g, C_g and K_g are rectangular mass, damping and stiffness matrices which represent the coupling between the structure nodes not connected to the ground and the support displacements [5]. Generally, the contribution due to damping in Eq. (4) is neglected; also, if a lumped mass idealization is used, M_g is a null matrix.

In the case of nonlinear behavior, an incremental formulation is developed by considering Eq. (1) also at time t plus an increment in time Δt [10], i.e.,

$$(F_t^i + \Delta F_t^i) + (F_t^d + \Delta F_t^d) + (F_t^e + \Delta F_t^e) = R_{t+\Delta t} \tag{5}$$

where the force vectors F_t^i, F_t^d and F_t^e are evaluated directly using the virtual work principle with the displacements, velocities and accelerations at time t; the changes of the force vectors over the time interval Δt are assumed to be given by

$$\Delta F_t^i = M_t \Delta\ddot{u}_t \quad , \quad \Delta F_t^d = C_t \Delta\dot{u}_t \quad , \quad \Delta F_t^e = K_t \Delta u_t \tag{6}$$

where M_t, C_t and K_t are the tangent mass, damping and stiffness matrices at time t; $\Delta\ddot{u}_t$, $\Delta\dot{u}_t$ and Δu_t are the changes in the accelerations, velocities and displacements during the time interval. Equation (5) can therefore be written as

$$M_t \Delta\ddot{u}_t + C_t \Delta\dot{u}_t + K_t \Delta u_t = R^*_{t+\Delta t} \tag{7}$$

where

$$R^*_{t+\Delta t} = R_{t+\Delta t} - F^i_t - F^d_t - F^e_t \tag{8}$$

The numerical integration scheme used relates $\Delta\ddot{u}_t$ and $\Delta\dot{u}_t$ to Δu_t. Using Eq. (7) the displacement increments can therefore be solved for, after which the required displacements, velocities and accelerations can be calculated.

The relations in Eq. (6) are only approximations; therefore, depending on the size of the time step and the nonlinearities present in the system the residual force vector,

$$R^r_{t+\Delta t} = R_{t+\Delta t} - F^i_{t+\Delta t} - F^d_{t+\Delta t} - F^e_{t+\Delta t} \tag{9}$$

which is evaluated directly using the virtual work principle, may not be negligable. It is then necessary to use an equilibrium iteration. Without equilibrium being satisfied in each time step (to some tolerance) solution accuracy cannot be expected [8].

SOLUTION OF THE EIGENVALUE PROBLEM

The essential idea for the solution of the generalized eigenvalue problem

$$K\Phi = \omega^2 M \Phi \tag{10}$$

where ω is an undamped circular frequency and Φ the corresponding vibration mode, is to find that subspace of the n dimensional operators K and M in which the essential response of the structure occurs [5] [6]. Experience has shown that in earthquake analysis the structural response is usually contained in the least dominant subspace $\mathcal{E}_\infty$ of the operators, spanned by $\Phi_1, \Phi_2, \ldots, \Phi_p$, which are the M-orthonormal eigenvectors corresponding to the smallest eigenvalues $\omega_1^2 \leq \omega_2^2 \leq \ldots \leq \omega_p^2$, where $p \ll n$. The required eigenproblem solution can therefore be written as

$$K\,\Phi = M\,\Phi\,\Omega^2 \tag{11}$$

where

$$\Phi = [\Phi_1, \ldots, \Phi_p] \quad ; \quad \Omega^2 = \text{diag}\,(\omega_i^2),\ i = 1, \ldots, p \tag{12}$$

Using $u = \Phi X$, where X is a vector of modal displacements, the equilibrium equations in the p-dimensional least dominant subspace are

$$\tilde{M}\ddot{X} + \tilde{C}\dot{X} + \tilde{K}X = \Phi^T R \tag{13}$$

where $\tilde{M}$, $\tilde{C}$ and $\tilde{K}$ are the projections of M, C and K onto $\mathcal{E}_\infty$, i.e. $\tilde{M} = \Phi^T M \Phi$, $\tilde{C} = \Phi^T C \Phi$, $\tilde{K} = \Phi^T K \Phi$. Although the linear problem is usually considered, it should be noted that such transformation may also be used for the solution of nonlinear problems, i.e. the repeated solution of Eq. (7). In this case, depending on the nonlinearities in the system, it may not be necessary to solve the eigenvalue problem in each time step, because the solution is accurate as long as the nonlinear response is contained in the subspace.

In linear analysis $\tilde{M}$ is the identity matrix, $\tilde{K} = \text{diag}(\omega_i^2)$ and in case of no damping coupling [6] $\tilde{C} = \text{diag}(2\omega_i \xi_i)$, where the ξ_i, $i = 1,\ldots,p$, are the modal damping ratios; otherwise $\tilde{C}$ is a full $p \times p$ matrix. In nonlinear analysis the matrices may initially be diagonal and then become full, until a new eigenvalue problem is solved. The solution of Eq. (13) can be obtained using a step-by-step integration algorithm, and is naturally most economical when no damping coupling is assumed. However, nearly always, the most expensive phase in the analysis is the solution of the generalized eigenvalue problem, Eq. (12). Two different algorithms have been found to be very effective; namely, a determinant search technique [2] and a subspace iteration method [3]. Both solution techniques solve the generalized eigenvalue problem directly without a transformation to the standard form.

The Determinant Search Solution

The determinant search method is most effective when K and M have small bandwidths. Basically, the solution algorithm combines triangular factorization and vector inverse iteration in an optimum manner to calculate the p lowest eigenvalues and corresponding eigenvectors in sequence starting with the least dominant eigenpair ω_1^2, Φ_1. An efficient accelerated secant iteration procedure, which operates on the characteristic polynomial

$$p(\omega^2) = \det(K - \omega^2 M) \tag{14}$$

is used to obtain a shift near the next unknown eigenvalue. The eigenvalue separation theorem (Sturm sequence property) is used in this iteration. Once a shift near the unknown eigenvalue has been obtained, inverse iteration is used to calculate the corresponding eigenvector. These calculations also give the eigenvalue to the required precision. Typical solution times are listed in Table 1.

The Subspace Iteration Solution

The subspace iteration method is most efficient, when the bandwidth of the system is large, and the matrices K and M are too large to be contained in the high speed storage of the computer. The idea in the

solution is to iterate directly for the p-dimensional required subspace, i.e., to calculate all required eigenvalues and eigenvectors simultaneously.

The iteration is performed with q linearly independent vectors, where $q > p$. In the k'th iteration, the vectors span the q-dimensional subspace $\mathcal{E}_k$ and "best" eigenvalue and eigenvector approximations are calculated; i.e. when the vectors span the p-dimensional least dominant subspace, the required eigenvalues and eigenvectors are obtained.

Let V_o store the starting vectors, then the k'th iteration is described as follows:

Solve for vectors $\bar{V}_k$ which span $\mathcal{E}_k$

$$K \bar{V}_k = M V_{k-1} \tag{15}$$

Calculate the projections of K and M onto $\mathcal{E}_k$

$$K_k = \bar{V}_k^T K \bar{V}_k \tag{16}$$

$$M_k = \bar{V}_k^T M \bar{V}_k \tag{17}$$

Solve for the eigensystem of K_k and M_k

$$K_k Q_k = M_k Q_k \Omega_k^2 \tag{18}$$

and calculate the k'th improved approximation to the eigenvectors

$$V_k = \bar{V}_k Q_k \tag{19}$$

Provided that the starting subspace is not orthogonal to any of the required eigenvectors, the iteration converges to the desired result, i.e. elements of $\Omega_k^2 \rightarrow \Omega^2$ and $V_k \rightarrow \Phi$ as $k \rightarrow \infty$. This iteration can be interpreted as a repeated application of the Ritz method, where the computed eigenvectors from one step are used as the trial basis vectors for the next iteration, until convergence to the required p eigenvalues and eigenvectors is obtained.

The number of vectors q used in the iteration is taken greater than the desired number of eigenvectors in order to accelerate the convergence of the process. The number of iterations required to achieve satisfactory convergence depends, of course, on the quality of the starting vectors V_o. Unless requested otherwise, an algorithm is used to generate q starting vectors, where $q = \min(2p, p+8)$, which has proven to be effective. At convergence a Sturm sequence check can be requested to verify the results. Table 2 gives a few typical solution times using the program generated starting vectors.

The subspace iteration technique is ideally suited for solution when a good estimate of the required eigensystem is known, such as in dynamic optimization or in the solution of a nonlinear problem, Eq. (7).

In these cases the previously calculated eigenvectors should be used as the starting vectors V_o in the iteration.

Step-by-Step Integration

The solution of the equations of motion represented in the original finite element coordinate basis or in the eigenvector basis, in which case they may be decoupled, can be obtained using step-by-step integration. For the solution of linear problems, the accuracy and stability of integration schemes can be readily studied [4]. In earthquake analysis, it is important to use an unconditionally stable method, because the time step size Δt can then be selected giving only regard to the required accuracy in the low mode response, i.e. the time step size need not be small enough to satisfy a stability criterion.

Various unconditionally stable integration procedures, for linear analysis, are currently in use, among which the Wilson θ-method has been found very effective. In this integration scheme a linear variation of acceleration over the time interval $\tau = \theta \Delta t$ is assumed, where for unconditional stability $\theta \geq 1.37$ [4]. Hence

$$\dot{u}_{t+\tau} = \dot{u}_t + \frac{\tau}{2}(\ddot{u}_{t+\tau} + \ddot{u}_t) \tag{20}$$

$$u_{t+\tau} = u_t + \tau \dot{u}_t + \frac{\tau^2}{6}(\ddot{u}_{t+\tau} + 2\ddot{u}_t) \tag{21}$$

which gives

$$\ddot{u}_{t+\tau} = \frac{6}{\tau^2} \Delta u_t - \frac{6}{\tau} \dot{u}_t - 2\ddot{u}_t \tag{22}$$

$$\dot{u}_{t+\tau} = \frac{3}{\tau} \Delta u_t - 2\dot{u}_t - \frac{\tau}{2} \ddot{u}_t \tag{23}$$

where now

$$\Delta u_t = u_{t+\tau} - u_t \tag{24}$$

Therefore, in the solution of the equilibrium equations a time step increment of τ is considered and the Δu_t in Eq.(24) are calculated. Using the linear acceleration assumption $u_{t+\Delta t}$, $\dot{u}_{t+\Delta t}$ and $\ddot{u}_{t+\Delta t}$ can then be obtained. The method is most accurate when θ is about 1.4. As an indication of the integration accuracy obtained using the Wilson θ-method, the Houbolt method and Newmark's integration scheme [4], Figs. 1 to 4 show the period elongations and amplitude decays resulting from the errors in the integration of the initial value problems indicated. It is noted that Wilson's method is much more accurate when $\theta = 1.4$ is used, instead of the earlier presented averaging scheme using $\theta = 2.0$ [9]. Also, quite equivalent to a mode superposition analysis, the Wilson and Houbolt method have the advantage of truncating the frequency domain, i.e., depending on the size of the time step the response in the high modes of the finite element system is not included in the analysis. This frequency truncation is appropriate because of the errors in the high modes, which are inherent in the finite element approximation.

These conclusions on the accuracy and stability have been arrived at assuming a linear system; however, the Wilson integration scheme has also been used effectively to obtain solutions to many practical nonlinear problems [10]. The "numerical damping", i.e. the amplitude decay in the high frequency components is beneficial in nonlinear analysis, in that the equilibrium iteration will converge faster. In the solution mass, damping and stiffness nonlinearities could be present, most frequently, however, only stiffness nonlinearities are considered. Table 3 gives a computer oriented formulation of the integration scheme for linear and stiffness nonlinear analysis.

CONCLUSIONS

Considerable improvements in the procedures for the solution of the finite element equations of motion have been achieved regarding the solution of large eigenvalue problems and the direct step-by-step integration. The required eigensystem of very large finite element systems can be solved economically, using the determinant research technique or the subspace iteration method. In step-by-step integration the Wilson method with $\theta = 1.4$ is much more accurate than the earlier reported averaging scheme. The method is used in nonlinear analysis, for which an efficient incremental formulation has been developed.

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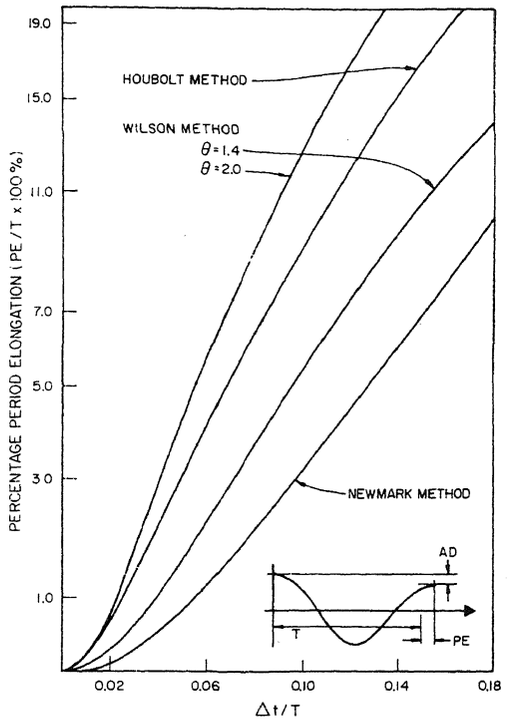

FIGURE 1 PERCENTAGE PERIOD ELONGATIONS

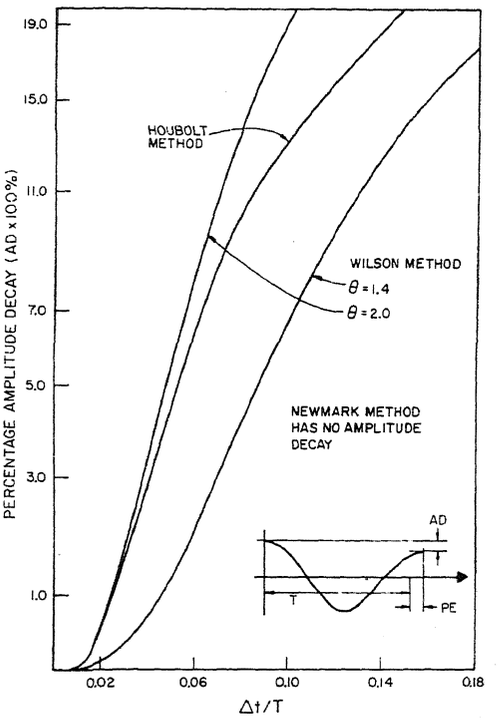

FIGURE 2 PERCENTAGE AMPLITUDE DECAYS

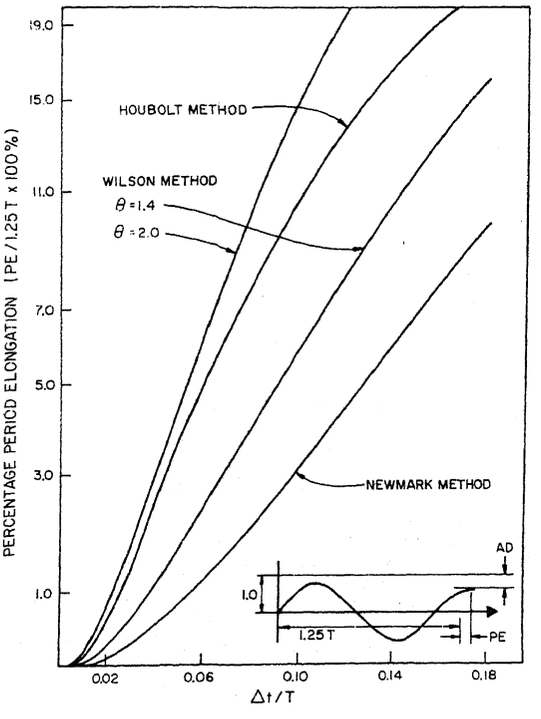

FIGURE 3 PERCENTAGE PERIOD ELONGATIONS

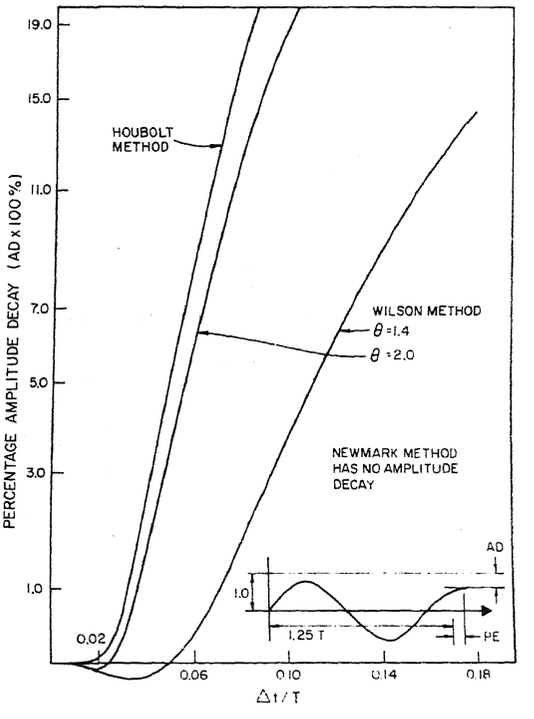

FIGURE 4 PERCENTAGE AMPLITUDE DECAYS

TABLE 1 CALCULATION OF FREQUENCIES AND MODE SHAPES USING THE DETERMINANT SEARCH METHOD

SYSTEM	SYSTEM ORDER n	MAXIMUM HALF BAND WIDTH	NUMBER OF REQ'D FREQN. AND MODE SHAPES p	COMPUTER USED	CENTRAL PROCESSOR SEC
PLANE FRAME	297	30	3	CDC 6400	40
PIPING SYSTEM	566	12	7	CDC 6600	11
BUILDING	340	32	7	CDC 6600	20
CONTAINER	265	65	40	CDC 7600	58

TABLE 2 CALCULATION OF FREQUENCIES AND MODE SHAPES USING THE SUBSPACE ITERATION METHOD

SYSTEM	SYSTEM ORDER n	MAXIMUM HALF BAND WIDTH	NUMBER OF REQ'D FREQN. AND MODE SHAPES p	COMPUTER USED	CENTRAL PROCESSOR SEC
PLANE FRAME	297	30	3	CDC 6400	25
PIPING SYSTEM	566	12	28	CDC 6600	142
BLDG. WITH FOUNDATION	1174	138	45	CDC 6600	890
3-DIM BLDG. FRAME	468	156	4	CDC 6400	160

TABLE 3 SUMMARY OF STEP-BY-STEP INTEGRATION

- - INITIAL CALCULATIONS - - -

1. Form linear stiffness matrix K, mass matrix M and damping matrix C; initialize u_o, $\dot{u}_o$, $\ddot{u}_o$

2. Calculate the following constants:

$\theta \geq 1.37$; $\tau = \theta \Delta t$; $tol \leq 0.01$; $nitem \geq 3$

$a_o = 6/\tau^2$ $a_1 = 3/\tau$ $a_2 = 2a_1$ $a_3 = \tau/2$ $a_4 = a_o/\theta$

$a_5 = -a_2/\theta$ $a_6 = 1 - 3/\theta$ $a_7 = \Delta t/2$ $a_8 = \Delta t^2/6$

3. Form effective linear stiffness matrix: $\tilde{K} = K + a_o M + a_1 C$

4. In linear analysis triangularize $\tilde{K}$

- - FOR EACH TIMESTEP - - -

A. IN LINEAR ANALYSIS

(i) Form effective loadvector:

$$\tilde{R}_t = R_t + \theta(R_{t+\Delta t} - R_t) + M(a_o u_t + a_2 \dot{u}_t + 2\ddot{u}_t) + C(a_1 u_t + 2\dot{u}_t + a_3 \ddot{u}_t)$$

(ii) Solve for displacement increments:

$$\tilde{K} u_{t+\tau} = \tilde{R}_t \quad ; \quad \Delta u_t = u_{t+\tau} - u_t$$

(iii) Go to C.

B. IN NONLINEAR ANALYSIS

(i) Update $\tilde{K}$ for nonlinear stiffness effects to obtain $\tilde{K}_t$ and triangularize

(ii) Form effective loadvector:

$$\tilde{R}_t = R_t + \theta(R_{t+\Delta t} - R_t) + M(a_2 \dot{u}_t + 2\ddot{u}_t) + C(2\dot{u}_t + a_3 \ddot{u}_t) - F_t^e$$

(iii) Solve for displacement increments:

$$\tilde{K}_t \Delta u_t = \tilde{R}_t$$

(iv) If required, iterate for dynamic equilibrium; then initialize $\Delta u_t^{(1)} = \Delta u_t$, $i = 0$

(a) $i = i + 1$

(b) Calculate i'th approximation to accelerations, velocities, and displacements:

$$\ddot{u}_{t+\tau}^{(i)} = a_o \Delta u_t^{(i)} - a_2 \dot{u}_t - 2\ddot{u}_t \quad ; \quad \dot{u}_{t+\tau}^{(i)} = a_1 \Delta u_t^{(i)} - 2\dot{u}_t - a_3 \ddot{u}_t \quad ;$$

$$u_{t+\tau}^{(i)} = u_t + \Delta u_t^{(i)}$$

(c) Calculate i'th out-of-balance loads: $R_{t+\tau}^{r(i)} = R_t + \theta(R_{t+\Delta t} - R_t) - M\ddot{u}_{t+\tau}^{(i)} - C\dot{u}_{t+\tau}^{(i)} - F_{t+\tau}^{e(i)}$

(d) Solve for i'th correction to displacement increments: $\tilde{K}_t \Delta\Delta u_t^{(i)} = R_{t+\tau}^{r(i)}$

(e) Calculate new displacement increments: $\Delta u_t^{(i+1)} = \Delta u_t^{(i)} + \Delta\Delta u_t^{(i)}$

(f) Iteration convergence if $\|\Delta\Delta u_t^{(i)}\|_2 / \|\Delta u_t^{(i+1)} + u_t\|_2 < tol$

If convergence: $\Delta u_t = \Delta u_t^{(i+1)}$ and go to C;

If no convergence and $i < nitem$:

go to (a); otherwise restart using a smaller time step size.

C. CALCULATE NEW ACCELERATIONS, VELOCITIES, AND DISPLACEMENTS

$$\ddot{u}_{t+\Delta t} = a_4 \Delta u_t + a_5 \dot{u}_t + a_6 \ddot{u}_t \quad ; \quad \dot{u}_{t+\Delta t} = \dot{u}_t + a_7(\ddot{u}_{t+\Delta t} + \ddot{u}_t) \quad ;$$

$$u_{t+\Delta t} = u_t + \Delta t \dot{u}_t + a_8(\ddot{u}_{t+\Delta t} + 2\ddot{u}_t)$$