THE THREE DIMENSIONAL STRESS, STRAIN AND DAMPING RESPONSE OF A CYCLICALLY STRESSED SAND-CLAY

by

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SYNOPSIS

This paper reports an investigation into the forms that consummate constitutive relationships might assume for a soil subjected to low frequency cyclic triaxial compressive stress. The investigation combined a study of stress-strain relationships with an examination of energy storage and dissipation characteristics.

By conducting a series of tests in which the spherical and deviatoric components of stress were simultaneously varied it was found possible to derive comprehensive stress-strain-energy relationships using computer orientated statistical techniques. These relationships were derived in terms of stress and strain invariants and may represent an advance on earlier constitutive relationships.

INTRODUCTION

A knowledge of the forms that constitutive relationships may assume for dynamically loaded soils is an essential prerequisite to the analysis of foundations subjected to earthquakes and other dynamic loads. Ideally, a constitutive relationship for a soil should not merely embrace the effects of changes in the stress magnitude, σ, but should also include the geometry of the problem, x, the stress history, s, and the effects of stateparameters such as the voids ratio, e, and the degree of saturation, S_r. Moreover, it is often necessary to consider the effects of such environmental parameters as the temperature, T. Where dynamic loading conditions are encountered, the effects of time, t, and frequency, ω, must also be included. Thus, in general, a constitutive relationship might be written as

$$[\varepsilon] = f([\sigma], [x], [e], [S_r], s, T, t, \omega, \ldots) \qquad (1)$$

The effects of each of the factors listed in eqn.1 on the response of soils to cyclic loading have been reviewed elsewhere (Shackel, 1973a). This review includes a critical assessment of existing stress-strain relationships for cyclically stressed soils.

In the past it has been customary to simplify the problem of defining constitutive relationships by considering only a specified combination of state and environmental conditions and just one particular instant in the stress history experienced by the soil. This approach is, however, inadequate for defining the response of soils to earthquake or dynamic loadings. Here it is usually necessary to define the soil behaviour throughout some complex stress history during which the state and environmental parameters may sometimes alter.

Currently only two approaches to defining constitutive relationships are in common use. These are the phenomenological approach employing some idealised response model and the simulation service approach in

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which an attempt is made to simulate, in the laboratory, the stress conditions observed in the prototype soil mass. Of these two approaches the simulation services technique has an advantage over the phenomenological approach in that it requires fewer prior assumptions concerning the nature of the soil response. For this reason the simulation service approach was adopted in the work reported here.

Of the various tests that may be employed in the simulation service approach to defining constitutive relationships, the triaxial compression test is still the most versatile technique available. Unfortunately this test cannot simulate the full range of stresses (or strains) observed in prototype soil masses. Consequently it is not possible to derive completely generalised stress-strain relationships from triaxial testing techniques. However, provided that certain simplifying assumptions can be made, e.g., that the soil behaves isotropically, then the limitations of the test method can be largely overcome by expressing the observed stress-strain relationships in terms of invariants of the stress and strain tensors. Adopting this approach, it is then sometimes possible to validly apply a constitutive relationship derived for one particular stress or strain condition to the analysis of some other different, and perhaps more complex, stress and strain system.

CONSTITUTIVE RELATIONSHIPS FOR SOILS

As noted above, there are some advantages to be gained by expressing constitutive relationships for soils in terms of stress and strain invariants. However, it should be recognised that it is usually inadvisable to attempt to formulate stress-strain relationships in terms of just single, arbitrarily selected, stress and strain components. This can best be illustrated by reference to Fig.1. This figure shows data obtained by the author for the cyclically stressed soil used in the experimental work and is plotted for the 1st and 10,000th stress cycles. It will be seen that the relationships between the first and second stress invariants and the octahedral normal and shear strains cannot be represented by unique curves but depend on such factors as the principal stress ratio, σ_1 / σ_3 and the number of stress applications. Thus in order to obtain satisfactory stress-strain relationships it is necessary to derive the relationships in terms of <u>all</u> the relevant stress-strain and and stress history parameters.

Newmark (1960) has suggested that it may be possible to define a relationship between stress and strain, even in the non-linear range, by expressions of the form

$$\varepsilon_{oct} = f_1 (\sigma_{oct}) + f_2 (\tau_{oct}) + f_3 (\varphi) \qquad (2)$$

$$\tfrac{1}{2}\,\gamma_{oct} = f_4 (\sigma_{oct}) + f_5 (\tau_{oct}) + f_6 (\varphi) \qquad (3)$$

and

$$\theta = f_7 (\sigma_{oct}) + f_8 (\tau_{oct}) + f_9 (\varphi) \qquad (4)$$

where f_1 to f_9 are arbitrary functions, σ_{oct} and τ_{oct} are the octahedral normal and shear stresses and ε_{oct} and γ_{oct} are the corresponding octahedral normal and shear strains. Since the octahedral stresses and strains are functions of the first and second invariants of the stress and strain tensors, they are themselves invariant. The parameters, φ and θ , are functions of the third stress and strain invariants respectively.

For the purposes of this paper, the author has assumed that eqns. 2, 3 and 4 may be simplified to the following

$$\varepsilon_{oct} = f_1 (\sigma_{oct}) + f_2 (\tau_{oct}) \qquad (5)$$

$$\gamma_{oct} = f_3 (\sigma_{oct}) + f_4 (\tau_{oct}) \qquad (6)$$

The problem of defining constitutive relations is then resolved into determining suitable forms of the functions f_1 to f_4. This is now considered for the particular problem of a soil subjected to cyclic compressive stress. The method described is, however, applicable to a much wider range of stress conditions.

The effects of subjecting a soil to cyclic loading along a stress path for which σ_1/σ_3 remains constant are shown in Fig.2. Here the stress path is represented by OABC and, if the test is repeated at a number of different stress amplitudes, the peak stresses are represented by A, B, and C. For each particular stress amplitude, the soil will accumulate strain energy during each loading cycle such as OA_1, DA_2 etc. This input strain energy, W, is given by

$$W = 3\left(\int_{\gamma_i}^{\gamma_p} \tau_{oct}\, d\gamma_{oct} + \int_{\varepsilon_i}^{\varepsilon_p} \sigma_{oct}\, d\varepsilon_{oct} \right) \tag{7}$$

where the subscripts, i and p, refer to the initial and peak stress conditions. Similarly during unloading, part of this strain energy will be recovered. This recoverable strain energy, R, is given by

$$R = 3\left(\int_{\gamma_f}^{\gamma_p} \tau_{oct}\, d\gamma_{oct} + \int_{\varepsilon_f}^{\varepsilon_p} \sigma_{oct}\, d\varepsilon_{oct} \right) \tag{8}$$

Here the subscript, f, refers to the conditions at the conclusion of the unloading cycle. For most soils, the energy recovered will be less than the energy loaded into the soil and some energy dissipation or damping will occur. Using the nomenclature proposed by Lazan (1968), the Specific Damping Energy, D, is given by

$$D \doteq W - R \tag{9}$$

In order to define the functions f_1 to f_4 in eqns. 5 and 6 it is necessary to perform a series of cyclic loading tests both over a range of peak stress conditions, as shown in figure 2, and also over a range of principal stress ratios, σ_1/σ_3 (or, alternatively, of confining stresses, σ_3). It will then be found that, for some designated number of stress cycles and at some particular instant in the stress history, such as that corresponding to peak stress, the strains represented, for example, by points A_1, B_1, and C_1 in figure 2, will each lie on a surface inclined in three dimensional ε_{oct} (or γ_{oct}), τ_{oct}, σ_{oct} space. Such a surface will, however, only be unique for the particular instant in the stress history under study.

In general the strain-stress surfaces will be curved. However, by either restricting the range of stresses studied, or by replotting the stresses and strains on logarithmic or other suitable scales, the surface may often be represented, to a good approximation, by a flat plane. Then, by using the well established techniques of multiple linear regression, a suitable model can be drafted to represent this plane.

One such simple regression model is shown in figure 3, linking some arbitrary strain component, ε, to the applied stresses. This is represented by the plane EFGH in the figure and can be expressed as

$$\varepsilon = a\,\tau_{oct} - b\,\sigma_{oct} + c \tag{10}$$

where a, b, and c are empirical constants.

The selection of a suitable regression model must either be based on previously published relationships or must be chosen quite arbitrarily. However, once a model has been selected, then the linearity and the fit of the model to the experimental data can be tested using statistical procedures. The selection of the most suitable models for defining constitutive relationships must be carried out on a trial-and-error basis and there is no guarantee that any of the models examined will be fundamentally correct. Nevertheless, a model which is a good fit to the data can usually be employed with confidence to obtain engineering predictions of the strains resulting from any combination of stresses lying within the domain of the experimental observations.

As well as relating stress and strain, regression methods can also be used to derive relationships between the input or damping energies and the applied stresses. Thus, for example, the plane ABCD in figure 3 might represent a suitable regression model linking the Specific Damping Energy, D, to the octahedral stresses.

THE EXPERIMENTAL WORK

The purpose of the experimental work was to determine whether it might be possible to characterise the response of a cyclically stressed soil in terms of a number of simple linear and non-linear regression models, relating the strains and energies to the applied stresses, and to examine the effects of a well defined stress history on the constants needed to define the regression models. The influence of structural changes in the soil are not considered here but have been described earlier (Shackel, 1972a). The work reported here complements previously published stress, strain and damping studies (Shackel, 1971; 1972b; 1973b; 1973c).

Soil Characteristics and Testing Procedures

An artificial soil was used in the experimental work which combined both frictional and cohesive characteristics in roughly equal proportions. This comprised a mixture of 60% sand and 40% Kaolin by weight and had a liquid limit of 63% and a plastic limit of 35%. Specimens of the soil were prepared by floating mould compaction (Shackel, 1970b) to a dry density of 115.8 lb/ft^3 at a degree of saturation of 80%.

After curing for three days at 20°C, the specimens were subjected to 10,000 cycles of loading in a special triaxial cell incorporating frictionless end platens and comprehensive electronic instrumentation. This equipment has been described in detail elsewhere (Shackel, 1970a). The tests were essentially unconsolidated and undrained although the pore air pressures were allowed to equalise with atmospheric pressure. The test temperature was 20°C and the frequency of loading was 2cpm. Each test involved 10,000 stress applications and ran continuously for about 84 hours. Measurements of the axial and radial stresses and strains were recorded automatically, for complete cycles of loading and unloading , at nineteen predetermined stages during each test.

In order to examine the stress-strain-energy behaviour of the soil an incomplete, fully randomised factorial experiment without replication was implemented. This comprised 17 tests. The factors studied were the magnitude of the peak octahedral shear stress, τ_{oct} , and the octahedral stress ratio, τ_{oct}/σ_{oct} . Four levels of τ_{oct} ranging from 5.66 to 22.6 lb/in^2 and five levels of τ_{oct}/σ_{oct} ranging from 0.61 to 1.07 were studied. Thus each test involved a different peak value of the octahedral normal stress, σ_{oct} . In all the tests, the stress paths followed during loading and unloading were chosen to maintain the ratio τ_{oct}/σ_{oct} constant, i.e. the paths were similar to path OABC in Fig. 2.

Analytical Procedures

The data from the cyclic loading tests were reduced by means of specially written computer programs. These evaluated the various stress and strain parameters as well as the integrals needed to define the input and damping energies. The reduced data were then grouped according to the numbers of stress applications at which observations had been recorded. Each of these data groups served as an input to a multiple regression computer program. This program was used to determine the constants needed to define the various regression models selected for examination. The program also calculated the coefficients of multiple determination, r^2 ; a measure of the regression linearity.

RESULTS

It was found that, for the stress paths and stress levels used in the experimental work, the radial strains were extremely small and that consequently the soil was always closeto the K_o condition; the condition of earth pressure-at-rest, Consequently the contributions of the radial strain to the storage and dissipation of strain energy were neglible and the value of W and D reported here are based on axial strain measurements only.

The first models selected for study were simple linear models of the form

$$W = a_1 \sigma_{oct} + b_1 \tau_{oct} + c_1 \tag{11}$$

$$D = a_2 \sigma_{oct} + b_2 \tau_{oct} + c_2 \tag{12}$$

$$\varepsilon_{oct} = a_3 \sigma_{oct} + b_3 \tau_{oct} + c_3 \tag{13}$$

$$\gamma_{oct} = a_4 \sigma_{oct} + b_4 \tau_{oct} + c_4 \tag{14}$$

These models were studied both with the constants, C, included and with the constants suppressed. In general, for the model relating the input energy, W, to the stresses, the coefficients, a_1, b_1 and c_1 did not show any obvious trend with increasing numbers of stress cycles. However, the coefficients needed to define the models represented by eqns. 12, 13 and 14 all showed some dependence upon the number of load applications. This is shown in figure 4. From the figure, it may be seen that, in general, suppressing the constants, C, had only a minor influence on the coefficients of τ_{oct} and σ_{oct}. It may also be seen that, after the first load cycles, the coefficients needed to define the damping-stress model given as eqn. 12 did not appreciably alter. By contrast the coefficients needed to define the stress-strain models represented by eqns. 13 and 14 gradually increased as the tests proceeded.

The linear models represented by eqns. 11 and 14 were compared with non-limear models described elsewhere (Shackel, 1973b; 1973c). These non-linear models took the general form

$$D \text{ or } \varepsilon_{oct} \text{ or } \gamma_{oct} = K \cdot \tau_{oct}^{u} \cdot \sigma_{oct}^{-v} \tag{15}$$

where K, u and V were empirical constants, It was found that, in the case of stress-strain relationships, the linear models generally gave distinctly better values of r^2 than the non-linear models, irrespective of whether or not the constants, C, in the linear models were suppressed. However, in the case of the damping-stress relationship, the non-linear model was marginally superior to the linear model. For all the models, the r^2 values usually ranged between approximately 0.8 and 0.99.

It was found that, for a designated number of load applications, there was a unique relationship between the input energy, W, or the specific damping energy, D, and the two octahedral strains, ε_{oct} and γ_{oct}. This is shown diagramatically in figure 5 as ABCD. The relationship between the octahedral shear and normal strains, shown as AGFE in figure 5, could, as an excellent approximation, be written as

$$\gamma_{oct} = C \cdot \varepsilon_{oct} \qquad (16)$$

and values of the constant C are shown in Fig. 6 as a function of the number of load applications.

It was then determined that the input and damping energies, W and D, could be related to the octahedral shear strains, γ_{oct}, by the relationships

$$W = K_W \, \gamma_{oct}^{n_1} \qquad (17)$$

and

$$D = K_d \, \gamma_{oct}^{n_2} \qquad (18)$$

The empirical constants, K_W and K_d, and the exponents, n_1 and n_2, are shown in Fig. 7 as functions of the number of load applications.

An important inference can be drawn from a consideration of the various damping-stress, strain-stress and damping-strain relationships given as eqns. 12, 14 and 18. This is that neither the linear nor the non-linear damping-stress or strain-stress models discussed in this paper can be regarded as being complete. This is most easily illustrated for the case of the linear models. Similar comments, however, also apply to the non-linear models of the form given in eqn. 15. If the octahedral stresses are varied along a path, such as NP in fig. 3, so as to maintain the damping, D, constant, then, unless the ratio of the coefficients a_2/b_2 is equal to the ratio a_4/b_4 (eqns. 12 and 14). the octahedral shear strain, γ_{oct}, will vary as the stresses change i.e.,with reference to Fig. 3, JK will not be parallel to LM. This is not compatible with eqn 18.

It is therefore probable that any full constitutive relationship for the soil should at least include stress, strain and strain energy parameters. Two such relationships were arbitrarily selected for study. These could be written as

$$\gamma_{oct} = a\,\sigma_{oct} + b\,\tau_{oct} + cD + d \qquad (19)$$

and

$$\gamma_{oct} = K\, \tau_{oct}^{u} \cdot D^{v} \cdot \sigma_{oct}^{-w} \qquad (20)$$

where a, b, c, d, u, V and W were empirical constants. Regression analyses revealed that the linear model represented by eqn. 19 gave a marginally better fit to the data than the non-linear model given as eqn. 20. The coefficients neededto define the linear strain-stress-damping model are given in Fig.8 as functions of the number of load applications. Values of the coefficient of multiple determination, r^2, generally ranged between 0.83 and 0.97.

CONCLUSIONS

The work reported in this paper demonstrated that, by using regression techniques, it was possible to derive both linear and non-linear stress-strain, damping-stress and damping-strain relationships in more rigorous forms than most relationships reported hithertoo. The various models were derived in terms of invariant parameters. Consequently the models are not merely limited to describing the triaxial stress conditions under which they were derived , but, subject to certain assumptions (e.g. isotropy), may be applied to the analysis of more complex stress situations.

For the relatively simple models selected for study here, it was shown that it was necessary to incorporate parameters reflecting energy changes into the stress-strain relationships. However, it is not suggested that the stress-strain-damping model described here is, itself, complete.

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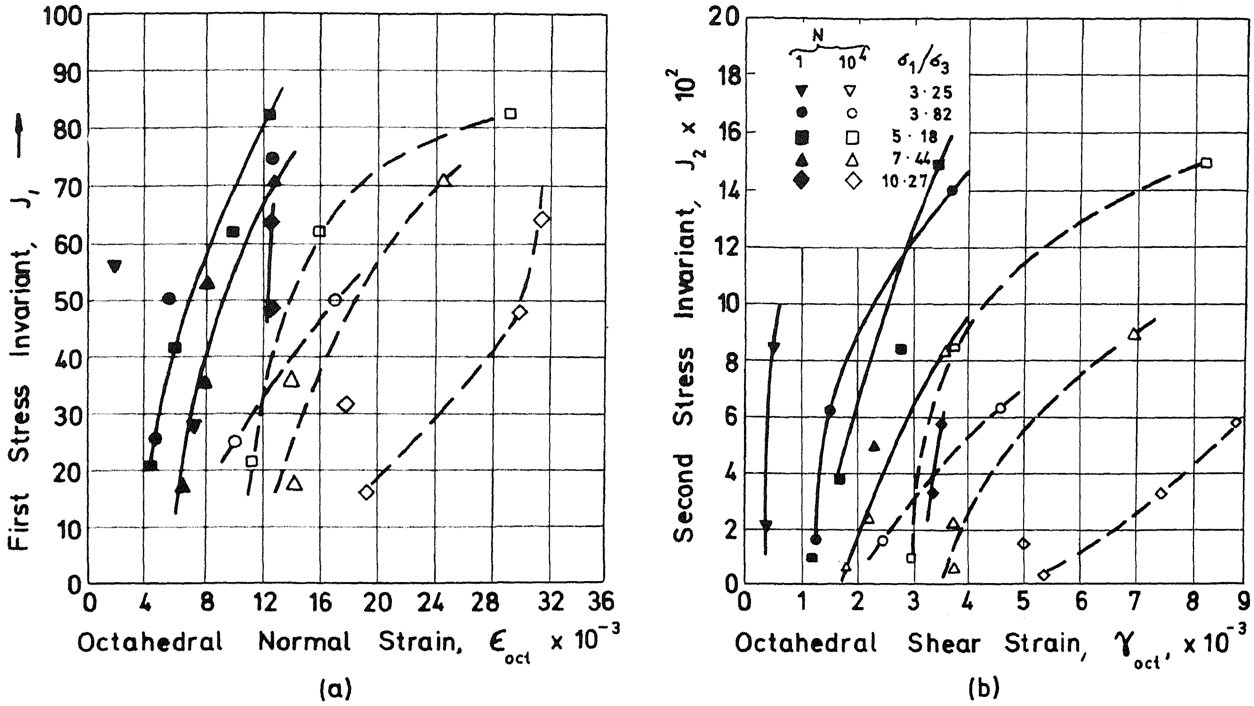

FIG. 1. THE OCTAHEDRAL NORMAL AND SHEAR STRAINS AS FUNCTIONS OF THE FIRST AND SECOND INVARIANTS

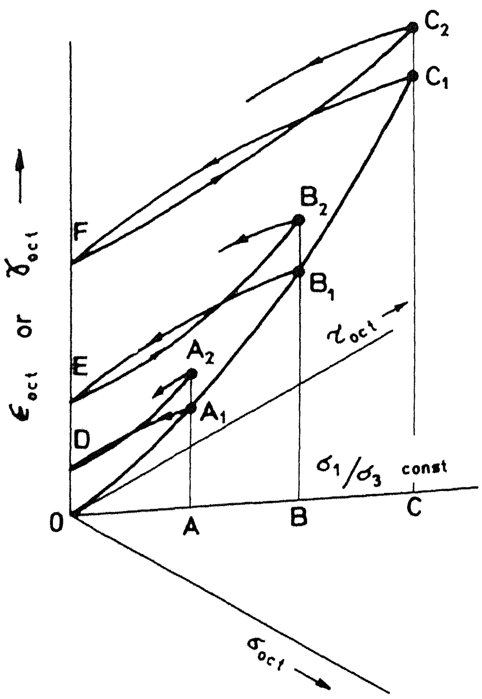

FIG. 2. EFFECT OF CYCLING THE APPLIED STRESSES

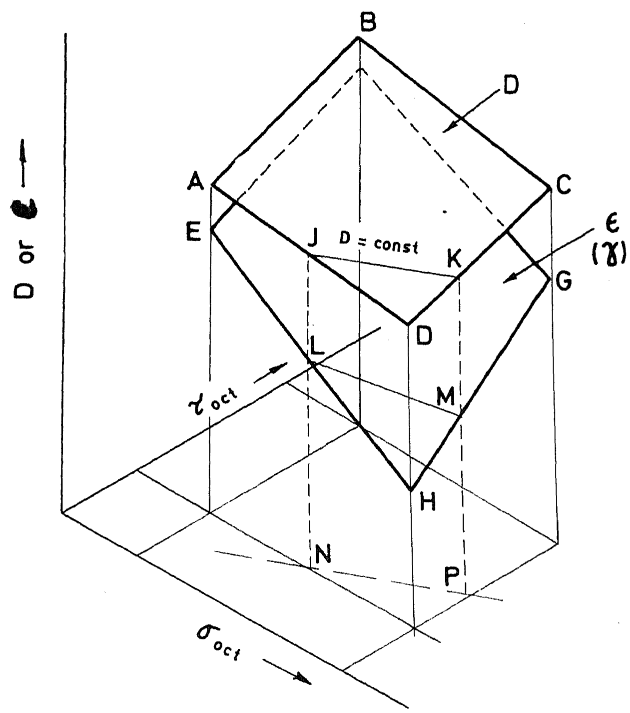

FIG. 3. THE REGRESSION MODELS FOR DAMPING AND STRAIN

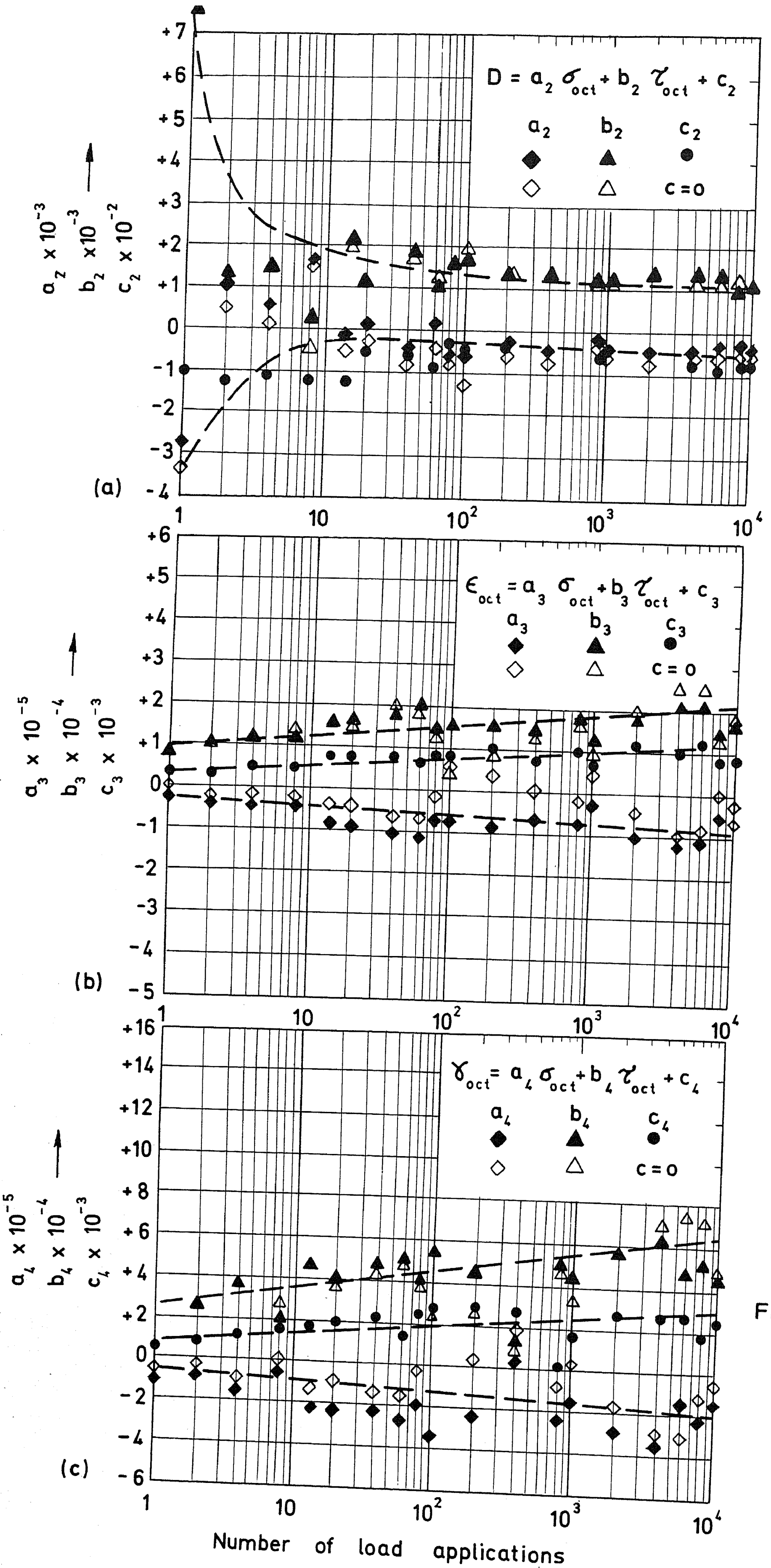

FIG. 4. THE LINEAR DAMPING AND STRESS STRAIN MODELS

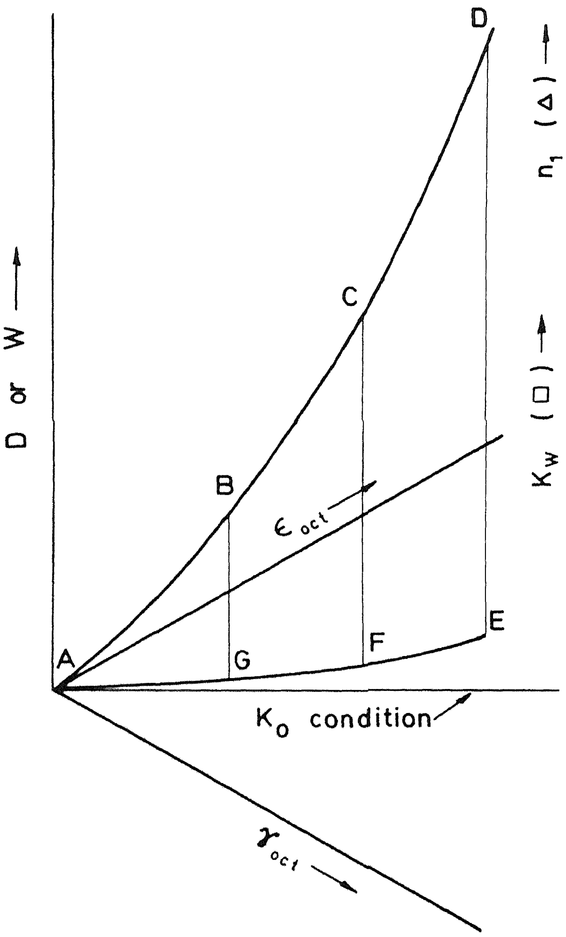

FIG. 5. DAMPING - STRAIN RELATIONSHIPS

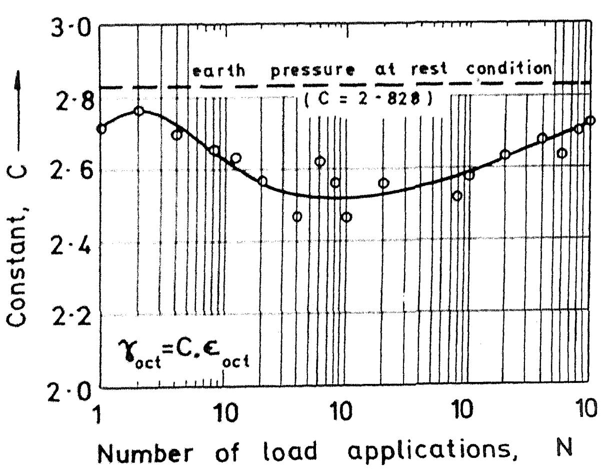

FIG. 6. THE RELATIONSHIP BETWEEN THE OCTAHEDRAL STRAINS

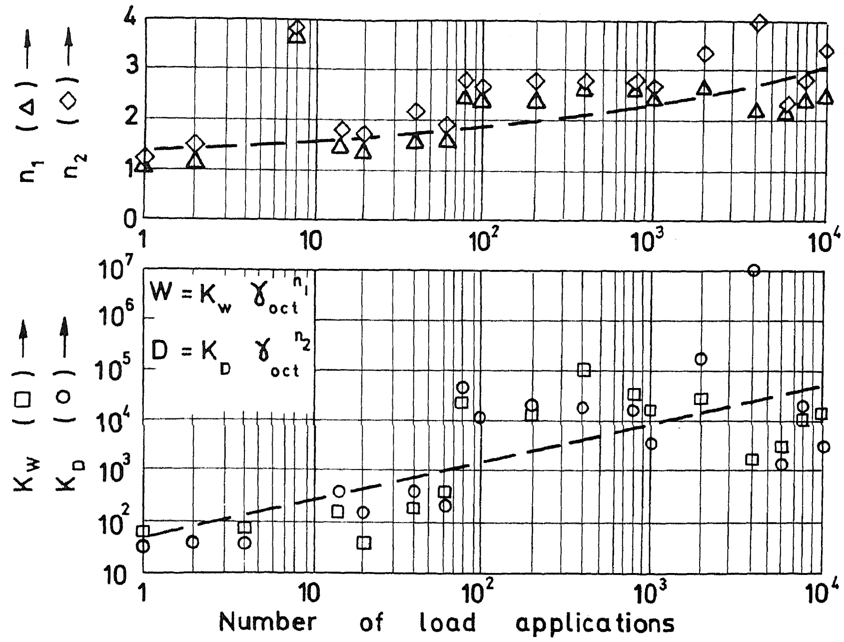

FIG. 7. DAMPING - STRAIN MODELS

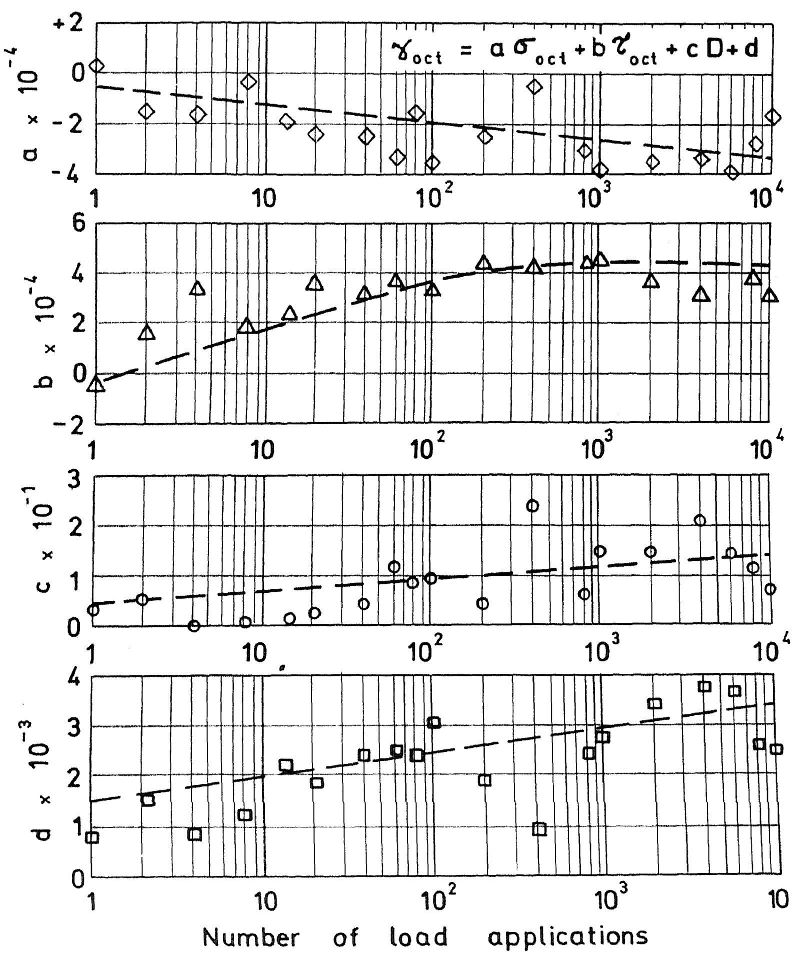

FIG. 8. THE LINEAR STRESS - STRAIN - DAMPING REGRESSION MODEL