A FINITE ELEMENT-ANALOGUE METHOD FOR DETERMINING THE DYNAMIC CHARACTERISTICS OF AN ARCH DAM-RESERVOIR SYSTEM

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Synopsis

This paper contains a study of the coupled natural frequencies and mode shapes of an Arch Dam-reservoir system. Both geometrically linear and nonlinear responses of the Dam have been considered. Results obtained by the Finite Element method show that if geometrical nonlinearity of the Dam is admitted then, under certain conditions which may arise in practice, the dynamic characteristics of the system will be significantly different from those obtained with the linear assumption. Results also show that a simple Electric Analogy method can be used with advantage to determine the low natural frequencies of the system.

Introduction

The dynamic behaviour of a structure, which is totally or partially submerged in a fluid medium, is characterized by the interaction that occurs between the vibrating structure and the fluid surrounding it. Such problems are recognized as essentially Elasto-Hydrodynamic in which the structural deformations and the Hydrodynamic pressures are modified by each other. It is convenient, when dealing with 'coupled' problems of this type, to think of the structure-fluid 'system' as a single dynamical unit responding to a given excitation with its own natural frequencies and mode shapes which are functions of, but distinct from those of the structure and the fluid body separately. It can be shown (1,2,3) that the degree of interaction and hence the coupled characteristics are determined by the stiffness of the structure compared to that of the fluid body: if the structure is flexible, then the coupled response of the system can be reasonably accurately calculated by the well known 'added mass' (4,5,6) method. This, however, is not valid for relatively stiff structures for which more rigorous methods will have to be employed. In practice the fundamental and the next few higher system frequencies and their associated mode shapes are only required for evaluating the system response to an excitation of known spectra.

Although a mathematical model of the structure-fluid system is easily constructed, its rigorous solution by conventional means however presents insurmountable difficulties, particularly as practical

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problems of this type usually have complex geometrical configurations. The Finite Element method, on the other hand, can be used for analysis very effectively and with comparative ease.

In this study an investigation is made into the coupled natural frequencies of an Arch Dam-reservoir system: the main objectives here are to determine (a) the effect, if any, of the geometrical nonlinearity of the Dam on the system frequencies and, (b) the conditions under which a simple Electric Analogy method, which is valid for incompressible fluid response, can be used to represent the Hydrodynamic effects.

Equations of motion

Fig.1a shows the geometry of the reservoir-Dam system under consideration: the Dam is cylindrical in shape and is located in a U-shaped valley, its thickness varying linearly along the y axis as shown in Fig.1b.

Ignoring viscosity and if in addition the so called 'convective accelerations' within the water are negligibly small then, for relatively small amplitudes of motion it can be demonstrated (7,8) that the Hydrodynamic pressures, $p(x,y,z,t)$, generated within the water will be governed by the Classical 'wave' equation

$$\nabla^2 p = (1/c^2)\,\ddot{p} \tag{1}$$

in which ∇^2 denotes the Laplace operator in a three dimensional rectangular Cartesian framework and c the acoustic velocity in water; dots denote differentiation with respect to time. Introducing into Eq.(1) the representation

$$p(x,y,z,t) = \mathrm{Re}(Pe^{i\omega t}), \tag{2}$$

where $i=(-1)^{\frac{1}{2}}$ and ω the circular frequency of oscillation, we can then show that the quasi-Harmonic Hydrodynamic response will be governed by the equation

$$\nabla^2 P + (\omega/c)^2 P = 0 \tag{3}$$

As the water is assumed frictionless, the following condition will hold (7) at the moving interface (to be called simply 'interface' henceforth) between the Dam and the water in contact with it:

$$\frac{\partial P}{\partial n} + \rho a_n = 0 \tag{4}$$

in which ρ denotes mass density of water and a_n the amplitude of total acceleration along the outward normal n, drawn to this interface.

Confining ourselves to ground acceleration along z only, let a_g denote the harmonic amplitude of ground acceleration along this direction and let a_{gn} denote the component of a_g resolved along n. It is a simple matter then to show (1,2) that

$$a_n = (a_{gn} - \omega^2 w_n) \qquad (5)$$

where w_n denotes the harmonic deflection response of the Dam along n. Furthermore, Eqs.(4) and (5) show that

$$\frac{\partial P}{\partial n} = 0 \qquad (6)$$

on rigid boundaries on which $a_{gn}=0$. The only other boundary condition concerns the free water surface; it is reasonable to assume

$$P = 0 \qquad (7)$$

at this surface (9,10).

Eq.(3) and the above boundary conditions uniquely specify the Hydrodynamic aspect of the problem. Subdividing the reservoir into a system of discrete elements, the finite element version of Eq.(3) can be written as

$$[G - \omega^2 m]\{P\} = \left\{\begin{array}{c} f' \\ \hline 0 \end{array}\right\}^{I} \qquad (8)$$

in which [G] and [m] are respectively the overall 'stiffness' and 'inertia' matrices (11) of the discretized reservoir. The subvector

$$\{f'\} = \rho[H]\{\omega^2 w - a_g\}, \qquad (9)$$

in which [H] is a diagonal weighting matrix of elemental areas of the 'interface', refers to this 'interface' only (It is understood throughout this text that {f'} as indeed all forces, displacements etc. have been suitably transformed so that they all refer to a common 'global' system of rectangular coordinates).

As for the structural aspect, the Hydrodynamic pressure resul-

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I Throughout this paper [ ] and { } will denote square (or rectangular) matrices and vectors respectively.

tants will obviously act as external nodal forces on the upstream face of the Dam. Listing these forces as $\{F\}$, let matrix $[T]$ transform the Hydrodynamic pressures into $\{F\}$ as

$$\{F\} = [T]\{P\} \tag{10}$$

(As usual, the finite element transformation of pressures into nodal forces in this way will result in transverse forces as well as moments in $\{F\}$). Observing that $\{F\}$ opposes the Dam's motion and ignoring damping due to internal friction, we can show that the equation of motion of the discretized Dam, when subjected to ground acceleration a_g, is

$$[K - \omega^2 M]\begin{Bmatrix} w \\ U \end{Bmatrix} = -\{F\} - [M]\begin{Bmatrix} a_g \\ 0 \end{Bmatrix} = -[T]\{P\} - [M]\begin{Bmatrix} a_g \\ 0 \end{Bmatrix} \tag{11}$$

in which $[K]$ and $[M]$ are respectively the overall stiffness and inertia matrices of the Dam. The subvector $\{w\}$ lists the transverse nodal deflections of the Dam and $\{U\}$ its other nodal displacements.

Eqs.(8) and (11) are obviously the sought coupled equations of motion of the reservoir-Dam system. For convenience, let us now partition $[K]$ as

$$[K] = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{12}^t & K_{22} \end{array}\right] \tag{12}$$

in which the submatrices $[K_{11}]$ and $[K_{22}]$ correspond with $\{w\}$ and $\{U\}$ respectively. Also, let

$$[M] = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{12}^t & M_{22} \end{array}\right] \tag{13}$$

Then, Eqs.(8) and (11) can be combined and written as

$$\left[\begin{array}{c|c|c} K_{11} & K_{12} & T \\ \hline K_{12}^t & K_{22} & 0 \\ \hline 0 & 0 & G \end{array}\right]\begin{Bmatrix} w \\ U \\ P \end{Bmatrix} - \omega^2\left[\begin{array}{c|c|c} M_{11} & M_{12} & 0 \\ \hline M_{12}^t & M_{22} & 0 \\ \hline \rho H & 0 & m \end{array}\right]\begin{Bmatrix} w \\ U \\ P \end{Bmatrix} = -\begin{Bmatrix} [M_{11}]\{a_g\} \\ [M_{12}^t]\{a_g\} \\ \rho[H]\{a_g\} \end{Bmatrix} \tag{14}$$

Clearly, the response of the system to a given ground acceleration and at a given forcing frequency will be computed by processing Eq.(14). When the determinant of this equation is set equal to zero, an eigenvalue equation of the general form

$$[A]\{x\} = \omega^2[B]\{x\} \tag{15}$$

emerges. Obviously, the roots and vectors of this equation are respectively the sought coupled natural frequencies and mode shapes of the reservoir-Dam system.

In the above formulation the Dam was assumed to respond in a geometrically linear fashion. However, if its geometrical nonlinearity is significant, then the interaction between the in-plane and transverse stiffnesses will have to be considered. This situation may well arise when a relatively thin arch Dam-reservoir complex is subjected to large Earthquake accelerations. From the finite element point of view an accurate analysis of such a problem would require the application of the well known 'incremental stiffness' procedure. A somewhat less sophisticated but nevertheless straightforward method can be used, on the other hand, to predict approximately the effect of geometrical nonlinearity on the system frequencies (work on an accurate algorithm is currently in progress). In this, the overall geometrical stiffness matrix (11,12) of the Dam, computed at a given forcing frequency on the basis of stresses corresponding to the displacement response obtained from Eq.(14), is added to [K] of Eq.(11); replacing the [K] of Eq.(14) by this modified [K], the system response at that frequency is recalculated from this equation. Repeating this process for a range of forcing frequencies, the natural system frequencies are found from the peaks of the resulting normalized frequency-response plot.

Problem solution

In view of its relative thinness, the Dam was subdivided into a system of rectangular 'plate' elements, interconnected by a total of 17 nodes (symmetrical half only). At each node 5 'local' degrees of freedom - 3 associated with transverse bending and 2 in-plane, were assumed and the local element characteristics were transformed into those referring to a global system of coordinates. The reservoir (symmetrical half only) was represented by a system of tetrahedral and rectangular 'prism' field elements (11), interconnected by a total of 163 nodes. The geometrically linear natural frequencies and mode shapes of the system were determined by processing Eq.(14) in terms of Eq.(15).

For comparison, a well tried Electric Analogy method was also used to determine the incompressible 'Hydrodynamic mass matrix' (4,13) of the Dam, to be used to represent the effect of the reservoir on response. For this an analogue model of the reservoir-Dam system was prepared in perspex in which the nodes were represented by 'finite electrodes' (7). A brief outline of the analogy will be given in Appendix 1; further details can be found in references 4 and 7. In both finite element and analogue models the reservoir length was taken as 4 times the height of the Dam.

Results and observations

The computed values of the fundamental and the first harmonic frequencies of the Dam, the reservoir and the reservoir-Dam system are listed in Table 1. Clearly, the geometrically nonlinear frequencies computed with $a_g = g$, where g is gravity acceleration, are significantly lower than those obtained with the linear assumption. This is to be expected, however, considering the decrease in the total stiffness of the Dam brought about by the predominantly compressive stresses within it. Furthermore, the Electric Analogy method, when used in conjunction with the Finite Element model of the Dam, is found to give reasonably accurate estimates of the fundamental and the first harmonic frequencies of the reservoir-Dam system. The effect of geometrical nonlinearity on the frequencies of this Dam ceased to be significant for a_g less than 0.5g.

Because of their relative thinness, the dynamic response of an Arch Dam or a similar submerged structure would be modified by its geometrical nonlinearity when subjected to large ground accelerations. It is easily recognized, however, that geometrical nonlinearity ought not to be considered in isolation, but instead in a wider context which includes material nonlinearities as well as those arising from the convective accelerations within the fluid. This is obviously a very difficult proposition and considerable work is yet to be done in this area.

Owing to its finiteness and the conditions that must be prescribed at its far boundaries, the Finite Element model of the fluid body inevitably leads to a standing-wave type solution, characterized by the repeated reflections of stress waves within these boundaries. At first sight such a solution does not appear realistic for fluid bodies extending to very large distances, as it does not recognize the 'radiation damping' in which energy is carried away to infinity by the stress waves. However, the validity of a standing wave solution in problems of this type can be justified by the fact that in real situations the reservoir bed usually rises slowly in the upstream direction away from the Dam. Strictly speaking, a part of the stress wave incident on the water-soil interface will as a result be refracted into the soil and the remainder reflected back. In general, a standing wave solution of the type used here will therefore be incorrect only to the extent that it assumes total reflection of waves at this interface and also to the extent that some energy will still be lost by radiation damping.

Conclusions

The method outlined herein can be used very effectively to evaluate the dynamic response of a geometrically linear or nonlinear structure-fluid system. Results obtained for a specific problem show that in a relatively thin Arch Dam subjected to large ground accelerations, the effect of geometrical nonlinearity on response will be significant. Also, a simple Analogy method can be used to represent approximately the effect of the fluid body. In many complex problems this will be found to be very useful.

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Appendix 1 The Electric Analogy

The flow of electric current in an isotropic resistive conductor is governed by the equation

$$\nabla^2 V = 0 \tag{16}$$

where $V(x,y,z)$ denotes electric potential (voltage). Also, according to Ohm's law, the intensity of current, i_r, along any direction r is related to V through

$$i_r = -\frac{1}{k}\cdot\frac{\partial V}{\partial r} \tag{17}$$

in which k denotes the specific resistance of the conductor.

If the fluid is assumed to be incompressible, then Eq.(3) reduces to the Laplacian form

$$\nabla^2 P = 0 \tag{18}$$

The similarity between Eqs.(16) and (18) and that between Eqs. (4) and (17) forms the basis of the Analogy. It permits us to determine the incompressible pressures, $\{P'\}$, at the 'interface' by replacing the reservoir-Dam model by an analogue model in which the fluid is represented by a conductor, such as ordinary tap water, and the reservoir boundaries by a nonconducting material; the interface nodes are then represented by 'finite electrodes' (7). Having simulated the prescribed boundary conditions into the analogue, the influence coefficients of voltages at the interface can now be obtained by feeding unit current densities at each of these electrodes, in turn, and by measuring the resulting voltages at all the electrodes. Let these coefficients be organized into the matrix [C]. Then, by definition,

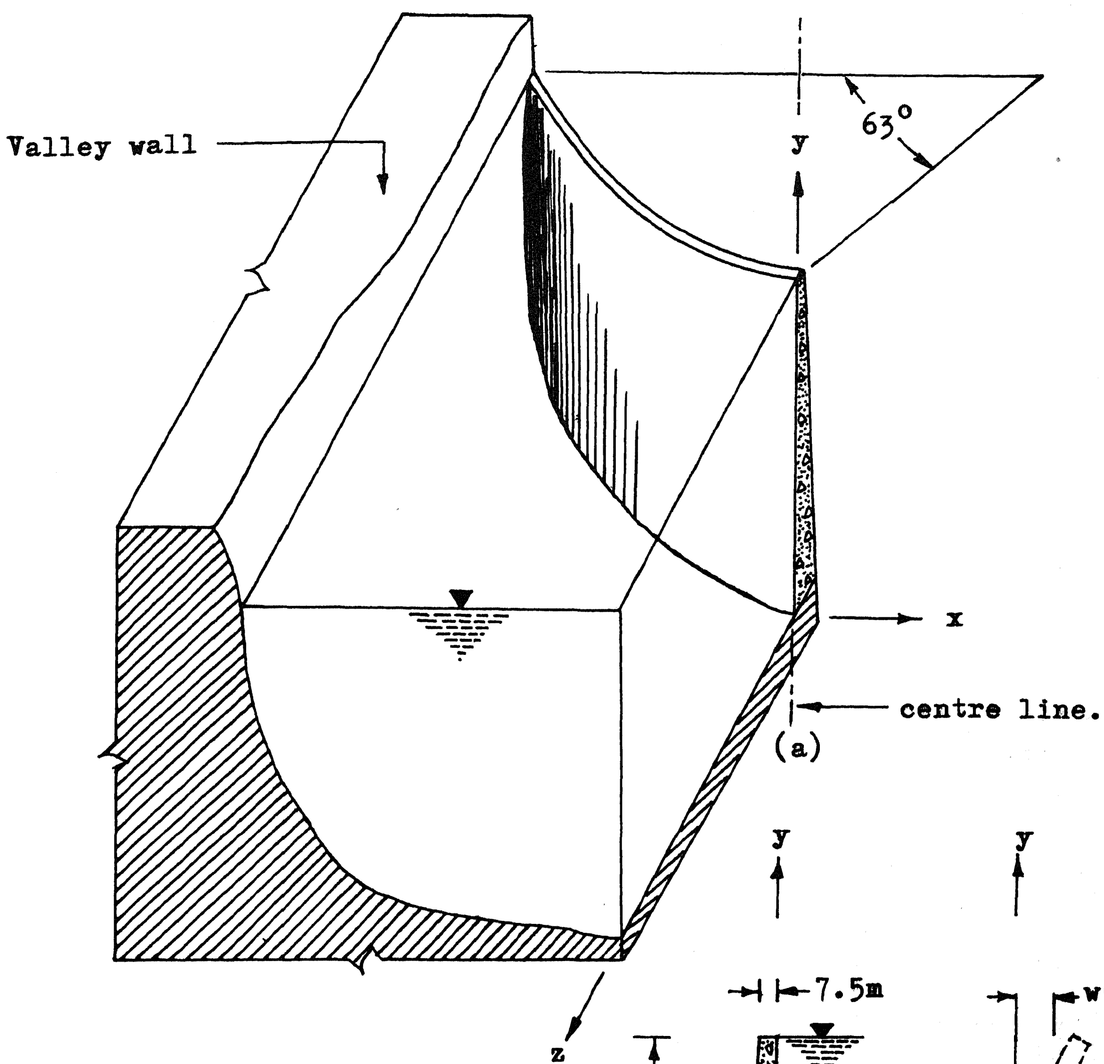
$$\{P'\} = q[C]\{f'\} \tag{19}$$

in which q is a scale factor between voltages and Hydrodynamic pressures. Defining [T'] as

$$\{F\} = [T']\{P'\} \tag{20}$$

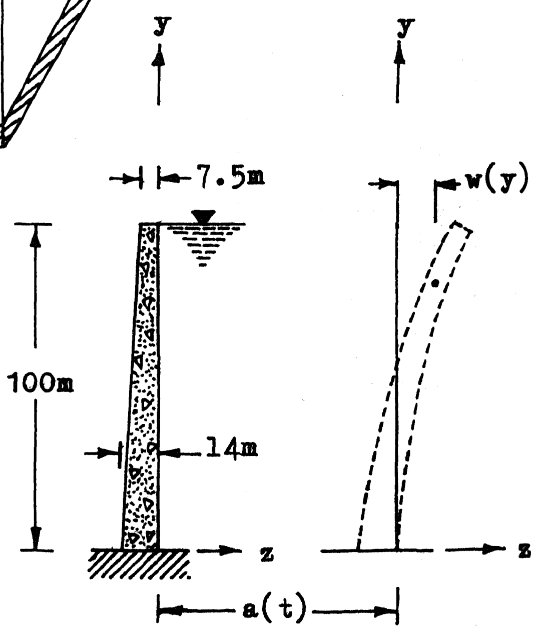
and substituting this into Eq.(11), the coupled system response can then be evaluated by processing the resulting equation and Eq.(19). These equations can be combined, as before, to form a single matrix equation similar to Eq.(14).

Since $\{P'\}$ represents incompressible pressures, this method is obviously valid only for relatively flexible structures, whose solution by the added mass method is adequate. Also, it would still be necessary to represent the structure by a numerical model.



Details of Dam

Type	Cylindrical
Radius	112m
Poisson's ratio	0.20
Young's modulus	31 GN/m^2
Mass density	2400 kg/m^3



$a(t) = a_g \sin(\omega t)$

$a_n(t) = a(t) + \ddot{w}$

(b)

Fig.1 (a) Geometry of reservoir-Dam.
(b) Displacement of Dam section at $x = z = 0$.

Table 1 Computed natural frequencies.

Case	Fundamental (Hz)	1st. Harmonic (Hz)
(geometrically linear)		
Uncoupled Dam, reservoir empty.	1.88	3.19
Uncoupled reservoir.	4.21	12.89
Coupled reservoir-Dam, Eqs.(14) and (15).	1.36	2.54
Coupled reservoir-Dam, by analogy.	1.47	2.73
(geometrically nonlinear, $a_g = g$)		
Uncoupled Dam, reservoir empty.	1.86	3.16
Uncoupled reservoir.	4.21	12.89
Coupled reservoir-Dam, Eqs.(14) and (15).	1.23	2.46
Coupled reservoir-Dam, by analogy.	1.39	2.58