DYNAMIC ANALYSIS OF PLANE NON-LINEAR EARTH STRUCTURES

by

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SYNOPSIS

A method is presented for computing both vertical and horizontal residual displacements in plane earth structures comprised of non-linear materials. The method uses a finite element program which incorporates non-linear stress-strain relations and both viscous and hysteretic damping. The responses of a slope and a dam to an earthquake are presented to illustrate the effects of hysteretic damping and non-linear stress-strain behaviour.

INTRODUCTION

Newmark (1) and Seed (2) have both criticized the concept of "factor of safety" as a means of assessing the probable performance of an earth dam during an earthquake. A factor of safety less than unity in a static analysis is not acceptable as it implies catastrophic displacements. However factors of safety of less than unity are acceptable in dynamic analyses; since the earthquake forces act for a short time and alternate in direction only small displacements may occur and these may be quite acceptable. For earth structures in general the magnitude of residual displacements is a more logical and better criterion of performance than the factor of safety.

Finn and Byrne (3) proposed a method for computing the residual displacements based on elastic-plastic shear slices which gave useful results for long slopes but was of limited practicality in analyzing dams because vertical displacements were ignored, and therefore the method could not predict losses in free-board due to an earthquake. Dibaj (4) and Farhoomand (5) developed more general methods for the computation of the response of non-linear earth structures.

In this paper a method is presented for computing both vertical and horizontal residual displacements. The method uses a finite element program which incorporates non-linear stress-strain relations and both viscous and hysteretic damping. The responses of a slope and of a dam to the first ten seconds of the N-S component of the El Centro earthquake of 1940 are presented, and the influence of hysteretic damping and of non-linear stress-strain behaviour is studied.

Although such an analysis has been theoretically practical for some time, great difficulty has been experienced in developing a computer pro-

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gram efficient enough to perform such an analysis at a reasonable cost. The computer program described in the paper is quite efficient in this respect. At present it is limited to 600 degrees of freedom which would appear to be adequate for most problems encountered in earthquake engineering practice involving soil structures.

THEORETICAL BASIS OF NON-LINEAR ANALYSIS

System Studied. The plane earth structure is represented by an assemblage of finite elements connected at the nodes. The displacement vector of these nodes $\{x\}$ is taken to represent the displacements of the actual structure. The equations of motion of the system may be written in the form

$$\{Q(\ddot{x},x)\} = \{F(t)\} \tag{1}$$

where Q represents the inertial and elastic forces of the system and F(t) represents the forces applied to the system by the environment. The mass matrix $[M]$ and the tangent stiffness matrix $[S]$ are then defined by

$$M_{ij} = \frac{\partial Q_i}{\partial \ddot{x}_j} \tag{2}$$

and

$$S(\{x\})_{ij} = \frac{\partial Q_i}{\partial x_j} \tag{3}$$

The mass matrix is constant and the non-linearity in the analysis arises from the tangent stiffness matrix. For simplicity of formulation here, the effects of viscous damping are ignored in this paper. Viscous damping can be readily incorporated and, in general, it may be desirable to provide such damping at low strains for materials with a finite proportional limit such as elastic-plastic materials.

Boundary Conditions. The system is divided into two regions; region 1 is subjected to known generalized forces $\{F_1\}$ and undergoes unknown generalized displacements $\{x_1\}$, and region 2 is subjected to known generalized displacements $\{x_2\}$. The governing differential equations are

$$\{dQ\} \equiv [M]\{d\ddot{x}\} + [S]\{dx\} = \{dF\} \tag{4}$$

Partitioning Equation (4) and expanding the part applicable to region 1 gives

$$[M_{11}]\{d\ddot{x}_1\} + [S_{11}]\{dx_1\} = \{dF_1\} - [M_{12}]\{d\ddot{x}_2\} - [S_{12}]\{dx_2\} \tag{5}$$

Integration of Equations of Motion

Let $\{\Delta x\} = \{x(t_i+\Delta t)\} - \{x(t_i)\} = \{x_{i+1}\} - \{x_i\}$

where Δt is the integration time step, and define $\{\Delta\ddot{x}\}$ and $\{\Delta F\}$ similarly.

The secant values of the elastic modulii at each point in an element for the i'th time step are defined such that the element stiffness matrix

$[\bar{S}]$ built using them is given by

$$[\bar{S}]\ \{\Delta x\} = \int_{t_i}^{t_i+\Delta t} S(\{x\}) \ \frac{d\{x\}}{dt} \ dt \tag{6}$$

Integrating Equation (5) over the interval $(t_i, t_i+\Delta t)$ gives

$$[M_{11}]\ \{\Delta\ddot{x}_1\} + [\bar{S}_{11}]\ \{\Delta x_1\} = \{\Delta F_1\} - [M_{12}]\ \{\Delta\ddot{x}_2\} - [\bar{S}_{12}]\ \{\Delta x_2\} \tag{7}$$

Equation (7) is a difference equation in time, and could be readily solved by an algorithm such as Newmark's Beta Method (7) if $[\bar{S}]$ were known. An iterative method to find $[\bar{S}]$ is as follows:

1. Estimate $\{x(t_i+\Delta t)\} = \{x(t_i)\} + \{\dot{x}(t_i)\}\Delta t + \{\ddot{x}(t_i)\} \frac{\Delta t^2}{2}$
2. Use the estimated $\{x_{i+1}\}$ to calculate the secant values of the elastic modulii for the step, and use these values to calculate an approximation to $[S]$, which we call $[S^*]$.
3. Using $[S^*]$ solve Equation 7 by numerical integration and find a better approximation to $\{x_{i+1}\}$.

Calculation of $[S^*]$. Knowing $\{x_i\}$ and having an estimate of $\{x_{i+1}\}$ the incremental strains at each point in an element can be calculated and the exact stress-strain relationship used to find the stress change over the interval $(t_i, t_i+\Delta t)$. Knowing the stress and strain changes, the secant values of the elastic modulii valid over the interval can be found and thus the element secant stiffness matrix for the interval can be calculated.

When constant stress elements are not used (as in this program), the above procedure is carried out at each point within an element, and the resultant spatially-varying modulii used to form $[S^*]$.

Integration Operator. Unlike the modal analysis which can be performed on linear systems, non-linear analysis retains the high-frequency components of the system. Accordingly, an unconditionally stable integration operator is essential. In addition, an operator which has artificial damping may create large non-conservative damping forces of unknown magnitude which could render the solution meaningless.

The behaviour of several unconditionally stable integration operators was studied by investigating the behaviour of a single degree of freedom system and comparing the computed results with the exact solution. The best results were given by Newmark's β-method (6) with the parameters β=.25, α=0.5. This operator has no artificial damping and gives good results for the forced vibration of very high frequency systems.

This result was in agreement with the investigations of Nickell (7) wherein the stability and damping characteristics of several operators were investigated. Nickell showed that Newmark's method with $\beta=0.25$ and $\alpha=0.5$ was unconditionally stable and had no artificial damping.

The Computer Program NDYNAMIC. NDYNAMIC was designed to be as versatile as possible while still being reasonably efficient. Force boundary conditions are specified by arbitrary patterns of horizontal and vertical loads, each of these two load-patterns being applied with an arbitrary time-intensity history. This permits the use of relative displacement analysis for systems with rigid moving boundaries (i.e. region 2 moves as a rigid body). Displacement boundary conditions are specified by a horizontal and a vertical accelerogram. These accelerograms moving through region 2 with specified direction and velocity, create a moving displacement field.

It was decided to assume that the soil would be isotropic, and therefore the volumetric and deviatoric behaviour of an element could be separated.

$$[S^*] = K^*[\frac{d(S)}{dK}] + G^*[\frac{d(S)}{dG}] \qquad (8)$$

where $[\frac{d(S)}{dK}]$ represents the stiffness matrix for an element with unit bulk modulus K and zero shear modulus, and $[\frac{d(S)}{dG}]$ represents the matrix for zero bulk modulus and unit shear modulus G. These constant matrices were evaluated at the beginning of the program, and permitted the rapid evaluation of [S*] when K* and G* were calculated.

The choice of the plane strain elements which would have non-linear stress-strain properties was quite complicated. It was not considered practical to use elements which would have modulii varying within each element, as the calculation of the [S*] matrix would have been exceptionally difficult and time-consuming. Accordingly, constant stress triangular elements seemed appropriate. However, these elements proved incapable of modelling the kind of non-linear response observed in soil structures. Consider Figure 1, an illustration of a dam modelled by triangular finite elements. Suppose the shaded elements are yielded; i.e. their shear modulus is zero but their bulk modulus is still greater than zero. A slip band should form along the path of the yielded elements, and sliding should occur. However, if the elements are of the constant stress type, no slip can occur. Consider the shaded elements which have a side on the rigid boundary below the dam. The only deformation pattern these elements can undergo without volumetric strain is for their upper nodes to move horizontally. Moreover, if there is to be no volumetric strain in the elements between them, their upper nodes must move equally. But such a motion will do work on the non-yielded elements to the left. Thus no slip can occur. In fact, even if <u>all</u> the elements were yielded no slip circle can form. The only slip possible is for each layer of elements to undergo uniform horizontal shear.

Accordingly, it was decided to employ 6-node triangles, which have a linear stress field and can model yielding quite effectively. However, the calculation of the stiffness matrix for such an element is very difficult as the modulii will vary over the element. The stiffness matrix, rather than being calculated exactly, is instead approximated at each step by using average values of the modulii for the entire element. These average values are determined from the incremental stresses and strains at the centroid of the element.

Rather than restrict the versatility of the program by building in stress-strain laws, a user-provided subroutine is used to define the stress-strain behaviour of the elements. Thus any behaviour, such as elastic-plastic, bilinear, Ramberg-Osgood, etc. can be defined by writing an appropriate subroutine. Because an unconditionally stable integration operator is used, it is feasible for the stress-strain law to entail zero or negative stiffnesses, a necessity for strain-softening materials.

Geometric matrices to model the non-linear effects of large displacements were not incorporated in the program. The program also includes the capacity to handle stiffness- and/or mass-proportional viscous damping. This adds considerable complexity to the analysis described in Section 2, and was not used in the examples studied herein.

Use of dynamically acquired core-storage, and of very efficient subroutines for matrix operations prevented the program from being exorbitantly expensive to run, but it is nevertheless an exceptionally complex analysis, and a great deal of computer time on a large computer is required for a single run.

EXAMPLES

For the purpose of illustrating the kinds of results that may be obtained using the non-linear analysis described above two examples are presented. The seismic responses of an infinite slope and an earth dam comprised of elastic-plastic material are compiled. In each case the bulk modulus is assumed constant. The stress-strain properties are shown in Figure 2.

In both examples the initial static stresses were found from a total-stress analysis by assuming that static loads were slowly applied to the system, which had bulk and shear modulii of 1,066,667 psf and 400,000 psf respectively (Poisson's ratio equal to 0.3), and a unit weight of 120 pcf.

The dynamic stress-strain-strength parameters were chosen to be typical of a medium strength compacted silty clay. The shear strength S_u is a function of the initial effective stresses:

$$S_u = 2160 + 0.268 \frac{\bar{\sigma}_{x_I} + \bar{\sigma}_{y_I}}{2} \text{ psf}$$

where $\bar{\sigma}_{x_I}$ and $\bar{\sigma}_{y_I}$ are the initial effective stresses in the x and y directions. The effective stresses are defined as the total stresses minus any pore-water pressure. The shear modulus $G_o = 200\ S_u$ psf and, assuming a value of Poisson's ratio of 0.49, the bulk modulus $K = 49.66\ G_o$ psf.

Static safety factors were computed by slowly applying weight-proportional loads to the system, starting from the initial stress state, and using the dynamic stress-strain-strength parameters.

Case 1: Infinite Slope. An infinite soil-layer 50 ft. thick resting on a rigid base is shown in Figure 3. Such slopes have been analyzed by Finn and Byrne (3) using a shear-beam technique with an elastic-plastic material.

The calculated shear strengths, based on the initial stresses, varied linearly from 2160 psf at the surface to 3230 psf at the base of the layer. Using the finite element mesh shown in Figure 3, the static factor of safety was computed as 1.16, which compared well to the exact value of 1.155.

The displacement of the surface of the slope relative to the base, in a direction parallel to the slope is shown in Figure 4 for three different analyses. The first was simply a linear analysis using the appropriate values of the shear modulus. The second was also a linear analysis with an attempt to model the effect of yielding by dividing the shear modulii by 3 and using 15% modal critical damping. The third analysis used the non-linear stress-strain behaviour, and produced results which were qualitatively dissimilar from the linear analyses. In each case, the base accelerations were the first five seconds of the N-S and vertical components of the El Centro earthquake of May 18, 1940.

The static safety factor for the slope was least at the base, and yielding occurred there. With the exception of 0.015 seconds during which the fourth element above the base yielded, only the lowest two elements yielded. As was shown by Finn and Byrne, the location of the yielding zones depends on the strength distribution within the soil profile.

Case 2: Earth-Fill Dam. A dam, 100 ft. high with a 25 ft. wide crest and 2.5:1 side slopes, is shown in Figure 5. The reservoir surface is assumed to be 10 ft. below the crest. Water forces are assumed to be hydrostatic, and are included in the static stress analysis. To find the initial effective stresses, pore-water pressures were computed by flow net analysis.

The factor of safety for vertical loads was computed as 2.6 using program NDYNAMIC with 40 finite elements, and as 3.0 using slip-circle analysis. The factor of safety for horizontal loads (maximum seismic coefficient) was computed as 0.42 using NDYNAMIC, and 0.55 using an assumed failure plane.

The dam was subjected to the first five seconds of the N-S and vertical components of the El Centro accelerogram. The acceleration amplitude was doubled to ensure severe yielding in the dam.

The first major yielding occurred at about 1.8 seconds, and large areas of the dam were yielded many times thereafter. The failure pattern, marked by the slumping off of the two elements in the upper right portion of the dam, became evident at 3.2 seconds, and was very noticeable by 3.6 seconds. Additionally, large plastic strains began to accumulate in the base of the dam at about the same time, but these strains did not grow as quickly as those of the two yielded elements at the crest. After 5 seconds, the strains along the base were greater than 100%, and the top had slumped some 50 ft. These results, of course, are not quantitatively reliable due to the small-strain small-rotation nature of the analysis.

Figures 6 and 7 show the deformed position of the dam after 1.7 and 3.7 seconds. In each case the vertical scale is twice the horizontal scale, and in Figure 6 the displacements have been magnified by a factor of 100.

It should be pointed out that in neither of the cases reported herein was an effort made to model the behaviour of a real system in any other than a qualitative manner.

CONCLUSION

A method has been presented which allows a more realistic appraisal of the probable performance of earth slopes and dams during an earthquake than appears to be currently available. The most important indicator of performance is the residual displacement field. In the case of dams for example, such displacements may lead to a loss in free-board. The method described in the paper allows the computation of the residual displacement field when realistic properties are assumed for the soil in the structures such as non-linear material properties, hysteretic and viscous damping.

ACKNOWLEDGEMENTS

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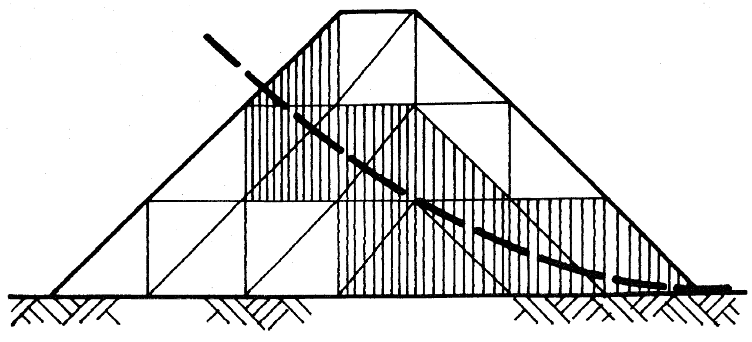

FIGURE 1. FAILURE PLANE IN DAM MODELLED BY FINITE ELEMENTS.

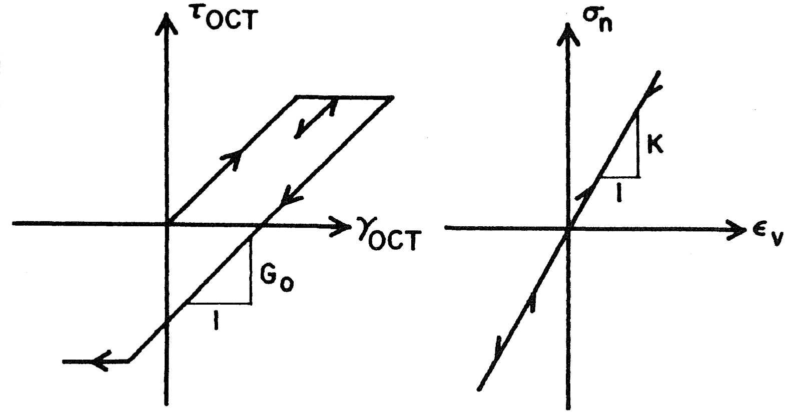

FIGURE 2. STRESS-STRAIN CURVES.

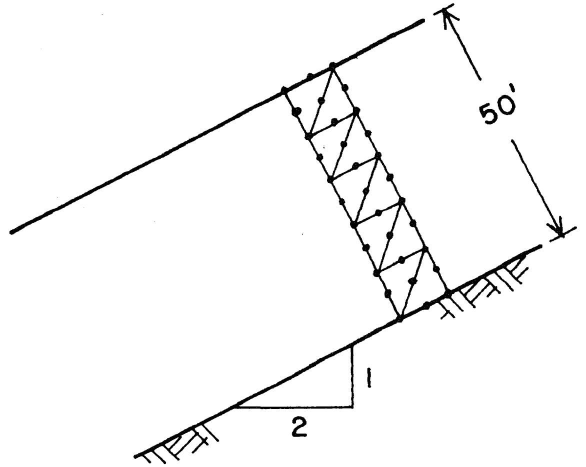

FIGURE 3. INFINITE SLOPE AND FINITE ELEMENT REPRESENTATION.

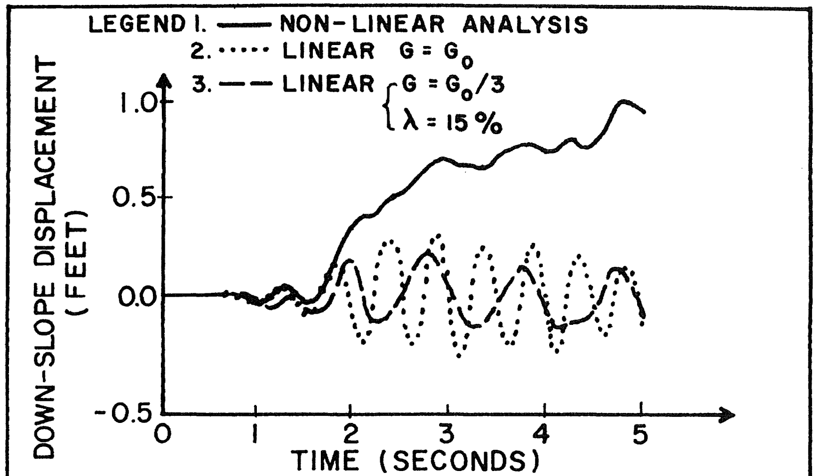

FIGURE 4. RESPONSE OF INFINITE SLOPE TO EL CENTRO ACCELEROGRAM.

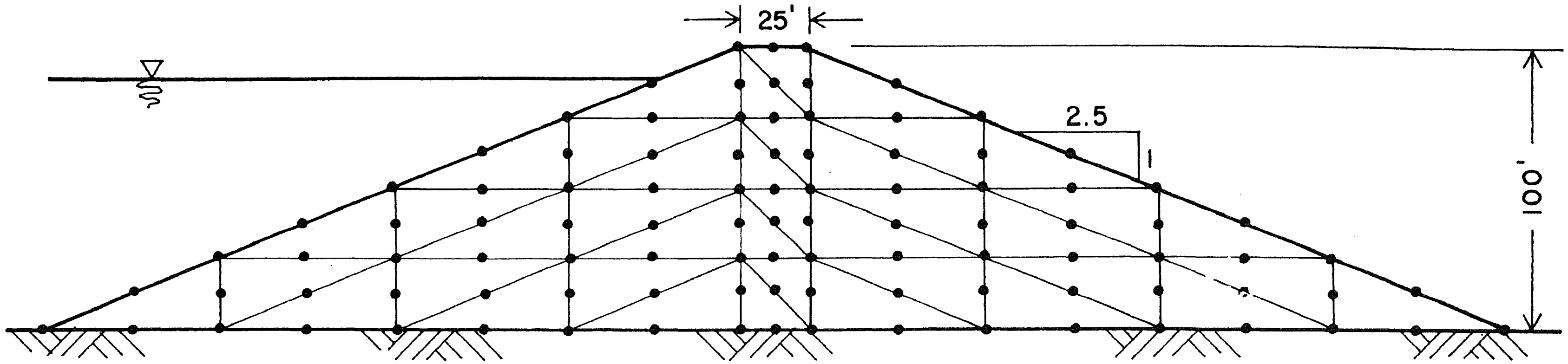

FIGURE 5. DAM SHOWING FINITE ELEMENT REPRESENTATION.

NOTE: VERTICAL SCALE = 2X HORIZONTAL SCALE.
: DISPLACEMENTS MAGNIFIED X 100.
: NO ELEMENTS YIELDED.

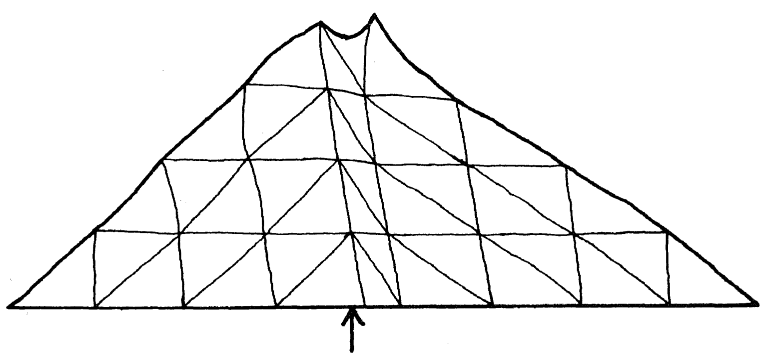

FIGURE 6. DISPLACED POSITION OF DAM AFTER 1.7 SECONDS.

NOTE: VERTICAL SCALE = 2X HORIZONTAL SCALE.
: YIELDED ELEMENTS ARE SHADED.

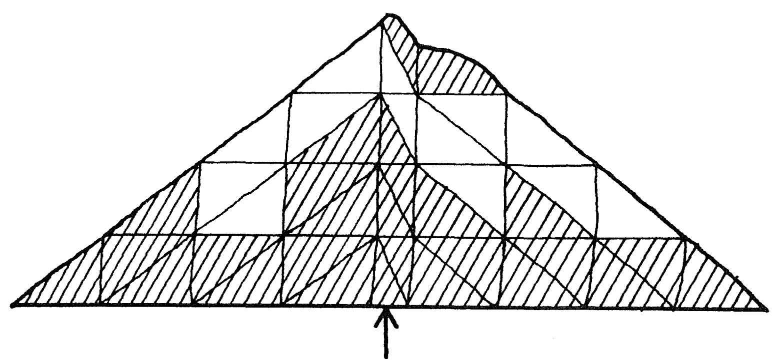

FIGURE 7. DISPLACED POSITION OF DAM AFTER 3.7 SECONDS.