COMPUTATION OF INDIVIDUAL FOURIER SPECTRUM ORDINATES

by

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SYNOPSIS

A method is presented to calculate selected ordinates of Fourier amplitude and phase spectra. The approach is based on the relation between the Fourier spectra of an accelerogram and the final displacement and velocity of an undamped oscillator subjected to the same accelerogram. Because only the final values of response are required, superposition is used to form equivalent excitations with durations of only one-quarter of the period of the corresponding undamped oscillator.

INTRODUCTION

Fourier spectrum techniques have become common in earthquake engineering in recent years because of the efficiency of the Fast Fourier Transform (FFT) and the general use of Fourier methods in other technical fields. Typically, the FFT operates on 2^N equally spaced points in the time domain to produce 2^{N-1} Fourier amplitude spectrum ordinates, equally spaced in the frequency domain from 0 to $1/2\Delta T$ cps, Δt being the interval of digitization. To complement the capabilities of the FFT, there is occasionally a need to calculate a few spectrum points in narrow frequency bands or to analyze, over selected frequency bands, records of longer duration than can be accommodated conveniently by standard FFT programs. The present calculation method, based on the relation between Fourier spectra and the response of an undamped linear oscillator, permits accurate and rapid calculation of selected Fourier spectrum ordinates.

FOURIER SPECTRA AND EARTHQUAKE RESPONSE

Ordinates of Fourier amplitude spectra are closely related to the response of an undamped single degree of freedom oscillator (1, 2, 3, 4). The analysis can also be extended to relate the ordinates of phase spectra to the undamped response (5). If $|F(\omega)|$ and $\psi(\omega)$ are the Fourier amplitude and phase spectrum ordinates as a function of ω for an accelerogram $\ddot{z}(t)$ of duration T, and if $x(t)$ and $\dot{x}(t)$ are the displacement and velocity of an undamped oscillator with natural frequency ω subjected to the same accelerogram, then it can be shown (5) that

$$|F(\omega)| = [\dot{x}^2(T) + \omega^2 x^2(T)]^{\frac{1}{2}} \quad ; \quad \tan\psi = \frac{\omega x(T)\ \cos\ \omega T - \dot{x}(T)\ \sin\omega T}{\dot{x}(T)\ \cos\ \omega T + \omega x(T)\ \sin\omega T} \qquad (1)$$

These results can be related to an associated free vibration problem as shown in Fig. 1. First as indicated in 1a and 1b, a segment of zero excitation and response is added at the beginning to make the total duration an integer multiple of the natural period, $T_o = 2\pi/\omega$. Then, as shown

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in 1c, an undamped oscillator released with $x(T)$ and $\dot{x}(T)$ will have these same values at the end of the excitation. If the free vibration is treated as a response and Eq. (1) applied, it will be seen that the associated free vibration has the same Fourier ordinates as the accelerogram. The conditions at the original origin, $t = 0$, can be analyzed to show that $|F(\omega)|^2$ is twice the total energy per unit mass of the associated free vibration, and $\tan^2\psi$ is the quotient of the initial potential and kinetic energies of the associated free vibration.

CALCULATION OF RESPONSE AND FOURIER ORDINATES

The calculation is performed by computing the velocity and displacement, at the end of the earthquake, of an undamped oscillator subjected to $\ddot{z}(t)$. Equations (1) are then used to find the Fourier spectrum ordinates. To set up the calculations, the length of record can be taken as a multiple of the natural period (by assuming an initial segment of zeroes) without loss of generality. The computation method is based on superposition; Fig. 2 shows that two impulses at corresponding times with respect to the period can be superposed as far as the response at $t = T$ is concerned. This feature allows the excitation to be cut into segments of duration T_o and superposed to form an equivalent excitation, $\ddot{z}_o(t)$, with duration of only one natural period.

The response of the undamped oscillator to $\ddot{z}_o(t)$ is basically four integrations of the accelerogram with one-quarter of a sine wave. By appropriate definition of quarter-cycle time scales and associated accelerograms, the calculation of $x(T)$ can be reduced to the calculation of the final velocity of an undamped oscillator to an equivalent quarter-cycle accelerogram. Similarly, $\dot{x}(T)$ is the final displacement of an undamped oscillator subjected to a different quarter-cycle accelerogram. These relations can be written as

$$x(T) = -\frac{1}{\omega}\dot{x}^*(T_o/4, \ddot{z}_5) \quad \dot{x}(T) = \omega x^*(T_o/4, \ddot{z}_6) \tag{2}$$

in which $\ddot{z}_5(t)$ and $\ddot{z}_6(t)$ are reduced, quarter-cycle accelerograms and x^* and $\dot{x}^*$ are the final displacement and velocity, respectively, of an undamped oscillator subjected to the reduced accelerograms in the usual manner. Hence, the calculation of $x(T)$ and $\dot{x}(T)$ is reduced to two calculations of the final response of an undamped oscillator to accelerograms of duration $T_o/4$. Any method can be used to calculate the response.

A subroutine has been developed to perform the calculations outlined above (5), using the technique developed by Nigam and Jennings (6). Examples are shown in Fig. 3 ($T_o = 1$ sec) and Fig. 4 ($T_o = 0.1$ sec). In each case the original accelerogram is artificial earthquake C-1 (7) for which $\Delta t = 0.025$. The equivalent one-period excitations (3b and 4b) are shown along with the reduced quarter-cycle accelerograms $\ddot{z}_5$ and $\ddot{z}_6$ for use in Eq. (2) (3c, 3d, 4c, 4b).

DISCUSSION AND CONCLUSIONS

It has been found that the method can calculate about 16 Fourier spectrum ordinates for a 1024-pt record in the same time the FFT calculates all 512, indicating the relative computing time involved. The absolute time is 22 ms/pt on the IBM 370/155. Of course, the present method can be used to calculate ordinates at any selected period and for records of any length, and is not intended to duplicate the computation of the entire spectrum as given by the FFT. The two methods are related, however, (5), and this relation can be used to guide the use of the FFT in earthquake engineering.

In mathematical terms, the essential feature of the method is the periodicity of the kernel in the Fourier transform, which allows superposition of the function to be transformed. This fact is well known in the field of numerical analysis, but these mathematical studies have not yet found their way into earthquake engineering.

One possible application appears to be in the analysis of data from ambient vibration tests, where the method might be used to investigate resonance peaks in detail, after the general nature of the Fourier spectra had been established by use of the FFT on a relatively short record. This application illustrates the two primary advantages of the computing method: the capability of selecting a small number of arbitrary frequencies for analysis, and the ability to handle long records with little difficulty.

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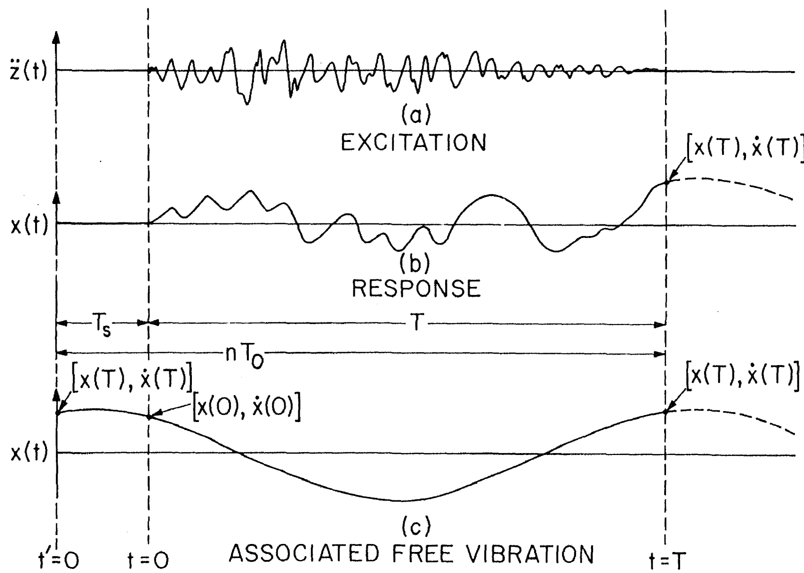

Figure 1. Excitation, response and associated free vibration.

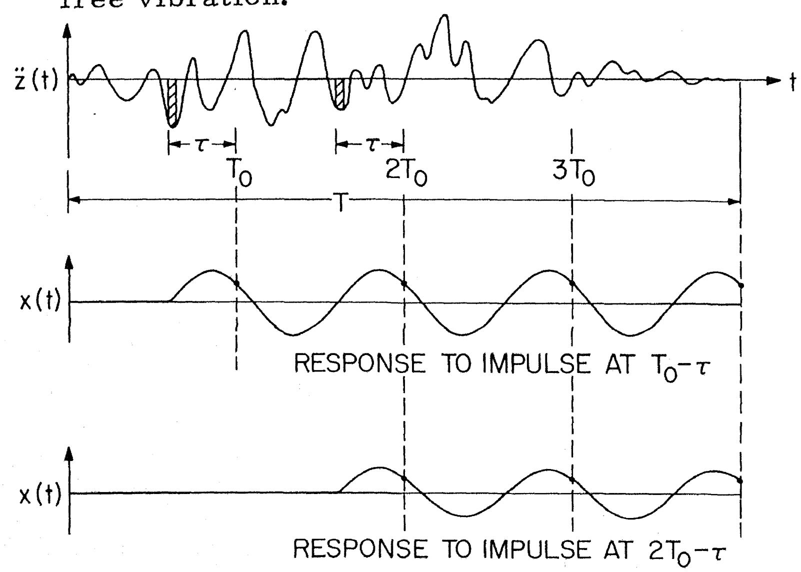

Figure 2. Superposition of impulse responses.

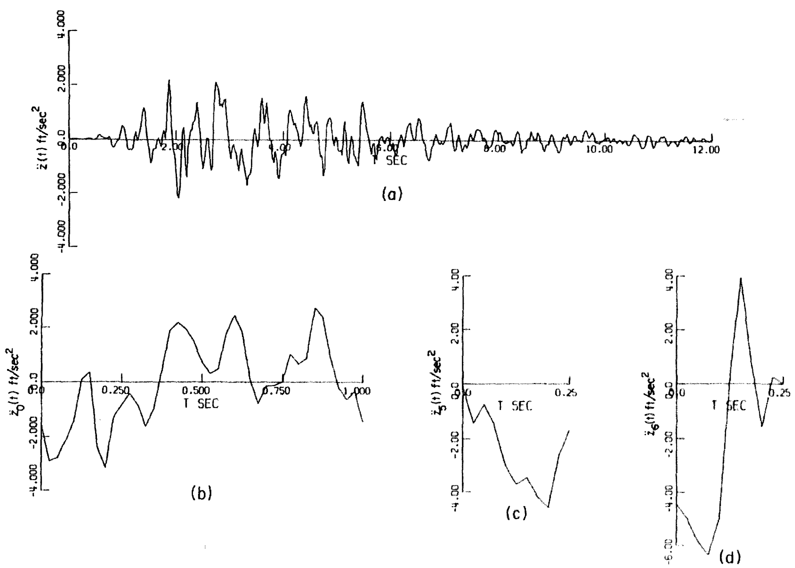

Figure 3. Application of the method to artificial earthquake C-1 for T_o = 1 sec.

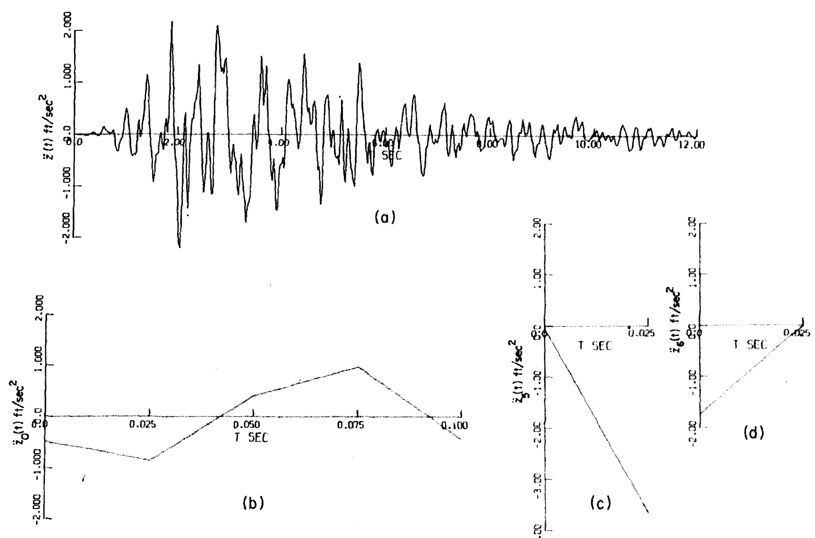

Figure 4. Application of the method to artificial earthquake C-1 for T_o = 0.1 sec.