A METHOD OF ANALYSIS FOR THE EVALUATION OF FOUNDATION-STRUCTURE INTERACTION

by

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ABSTRACT

The finite element method coupled with a stable step-by-step integration procedure is developed in order to evaluate the elastic earthquake response of complex structures and their foundations. Linear strain quadrilateral elements are introduced which are significantly more accurate than the previously used triangular elements. Computer techniques are discussed which form the basis for the development of a large capacity program for the efficient earthquake analysis of large foundation systems. The analysis of a large earth dam is presented in order to illustrate the application of the procedure.

INTRODUCTION

Earthquake forces are transmitted to a structure through its foundation. It is apparent that the magnitude of these forces depend on the properties of both the structure and the foundation. However, most earthquake analyses which have been performed to date have not included this interaction correctly.

Many earthquake studies have been conducted using a rigid foundation as shown in figure 1a. Two approximations are made in this idealization: the stiffness of the foundation is assumed to be rigid as compared to the stiffness of the structure, and the input acceleration is assumed not to be altered by the presence of the structure. It should be pointed out that for many structures the errors introduced by this idealization are small.

Another approximate approach to include the effect of foundation properties on the earthquake response of the structure is in the introduction of foundation springs and masses as shown in figure 1b. This, of course, is an oversimplification of the dynamic behavior of the foundation and in some cases may act as a filter for certain earthquake frequencies.

The separation of the foundation and structure into two separate analyses is also a common approach: this idealization is shown in the analysis of the foundation based on pure shear behavior of the material. This acceleration is then used in the analysis of the structure. Again the interaction is not considered.

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A true interaction model is shown in figure 1d; both the structure and its foundation are included and the earthquake acceleration is applied at a remote distance from the structure. Therefore, the true foundation-structure interaction is included automatically in the idealization. Such an approach is possible if the foundation is represented by a system of finite elements.

The finite element method has been used previously in the earthquake analysis of structures. However, limitation on the number of elements has prevented the consideration of large foundation areas. The purpose of this paper is to present a higher order finite element approach and a stable step-by-step integration procedure which may be used as an efficient tool in the analysis of complex foundation-structure interaction problems.

THE FINITE ELEMENT IDEALIZATION

The finite element method is a recently developed technique that has been extremely successful in the static and dynamic analysis of continuous structures.[1,2,3,4] The advantages of the finite element method, as compared to other numerical approaches, are many. The method is completely general with respect to geometry and material properties. Since each element in the system may have different properties, complex bodies composed of many different layered, anisotropic materials are easily represented. Displacement or stress boundary conditions may be specified at any point in the finite element system. Mathematically, it can be shown that the method converges to the exact solution as the number of elements is increased; therefore, any desired degree of accuracy may be obtained. In addition, for both static and dynamic analyses, the finite element approach generates equilibrium equations which produce a symmetric positive-definite matrix that may be placed in a band form and solved with a minimum of computer storage and time.

In the finite element idealization of solids the continuous structure is replaced by a finite number of elements, or regions, which have common joints, or nodal points. For the purpose of describing the behavior of the finite element system anaapproximate displacement field is assumed within each element. In the case of two-dimensional solids. expressions for both the x and y displacement fields are required. These fields within each element are expressed in terms of a discrete number of unknown displacements associated with the connecting nodal points. Therefore, for the dynamic response a lumped parameter idealization of the actual structure is possible, in which the mass properties of the system are separated from the elastic properties of the system. The advantage of this discrete mathematical formulation is that the force equilibrium of the system may be expressed by a set of ordinary differential equations rather than the partial differential equation required to describe the actual continuous structure.

The Linear Strain Quadrilateral

All previous dynamic two-dimensional finite element work has been based on the constant strain triangular element. In this investigation, a higher order quadrilateral element is introduced which results in a significant increase in accuracy with less computational effort than in the triangular element idealization. The linear strain quadrilateral is composed of two four nodal point triangles and is shown in figure 2.

The Area Coordinate System

Within a triangular element the displacements may be expressed as a function of space and the values of the displacements at the four nodes. A convenient form in which these displacements may be expressed directly is in the area coordinate system shown in figure 3. The dimensionless area coordinate is in the ratio of the subarea to the total area of the element. Or,

$$\eta_i = \frac{A_i}{A} \tag{1}$$

where

$$A = A_1 + A_2 + A_3$$

Therefore, a point within the triangle may be defined in terms of the global system x and y or in terms of the local area coordinates A_1, A_2 and A_3 or in terms of the dimensionless coordinates η_1, η_2, and η_3.

The Displacement Field Approximation

The advantage of the area coordinate system is that the compatible displacement functions can be written directly in a simple form. Hence, the x and y components of the displacement field are

$$u = \eta_1 u_1 + \eta_2(1-2\eta_3)\, u_2 + \eta_3(1-2\eta_2)\, u_3 + 4\eta_2\eta_3\, u_4 \tag{2a}$$

$$v = \eta_1 v_1 + \eta_2(1-2\eta_3)\, v_2 + \eta_3(1-2\eta_2)\, v_3 + 4\eta_2\eta_3\, v_4 \tag{2b}$$

Relationship Between Coordinate Systems

A typical subarea region of the triangular element is shown in figure 4. The area of the region is given by

$$A_k = \left(\frac{y+y_j}{2}\right)(x_j-x) + \left(\frac{y+y_i}{2}\right)(x-x_i) - \left(\frac{y_j+y_i}{2}\right)(x_j-x_i) \tag{3}$$

Or in dimensionless form

$$\eta_k = \frac{1}{2A}\left[(y+y_j)(x_j-x) + (y+y_i)(x-x_i) - (y_j+y_i)(x_j-x_i)\right] \tag{4}$$

The derivatives with respect to the global coordinates are in general

$$\frac{\partial\eta_k}{\partial x} = \frac{y_i-y_j}{2A} \text{ and } \frac{\partial\eta_k}{\partial y} = \frac{-x_j-x_i}{2A} \tag{5}$$

These evaluated for the three coordinates are

$$\frac{\partial \eta_1}{\partial x} = \frac{b_1}{2A} \qquad \frac{\partial \eta_1}{\partial y} = \frac{a_1}{2A}$$

$$\frac{\partial \eta_2}{\partial x} = \frac{b_2}{2A} \qquad \frac{\partial \eta_2}{\partial y} = \frac{a_2}{2A} \tag{6}$$

$$\frac{\partial \eta_3}{\partial x} = \frac{b_3}{2A} \qquad \frac{\partial \eta_3}{\partial y} = \frac{a_3}{2A}$$

where

$$a_1 = x_3 - x_2$$

$$a_2 = x_1 - x_3$$

$$a_3 = x_2 - x_1 \tag{7}$$

$$b_1 = y_2 - y_3$$

$$b_2 = y_3 - y_1$$

$$b_2 = y_3 - y_1$$

Expressions for Element Strains

The element strains may be derived from the displacement field by the use of the chain rule. Or

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta_1} \cdot \frac{\partial \eta_1}{\partial x} + \frac{\partial u}{\partial \eta_2} \cdot \frac{\partial \eta_2}{\partial x} + \frac{\partial u}{\partial \eta_3} \cdot \frac{\partial \eta_3}{\partial x}$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial \eta_1} \cdot \frac{\partial \eta_1}{\partial y} + \frac{\partial v}{\partial \eta_2} \cdot \frac{\partial \eta_2}{\partial y} + \frac{\partial v}{\partial \eta_3} \cdot \frac{\partial \eta_3}{\partial y}$$

$$\gamma = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{8}$$

$$= \frac{\partial u}{\partial \eta_1} \cdot \frac{\partial \eta_1}{\partial y} + \frac{\partial u}{\partial \eta_2} \cdot \frac{\partial \eta_2}{\partial y} + \frac{\partial u}{\partial \eta_3} \cdot \frac{\partial \eta_3}{\partial y}$$

$$+ \frac{\partial v}{\partial \eta_1} \cdot \frac{\partial \eta_1}{\partial x} + \frac{\partial v}{\partial \eta_2} \cdot \frac{\partial \eta_2}{\partial x} + \frac{\partial v}{\partial \eta_3} \cdot \frac{\partial \eta_3}{\partial x}$$

Therefore, the three global components of strain are given in terms of the local dimensionless coordinate system as

$$\epsilon_x = \frac{1}{2A}\Big[b_1 u_1 + (b_2 - 2b_3\eta - 2b_2\eta_3)\, u_2 + (b_3 - 2b_2\eta_3 - 2b_3\eta_2)\, u_3 + (4b_3\eta_2 + 4b_2\eta_3)\, u_4\Big]$$

$$\epsilon_y = \frac{1}{2A}\Big[a_1 v_1 + (a_2 - 2a_3\eta_2 - 2a_2\eta_3)\, v_2 + (a_3 - 2a_2\eta_3 - 2a_3\eta_2)\, v_3 + (4a_3\eta_2 + 4a_2\eta_3)\, v_4\Big] \tag{9}$$

$$\gamma = \frac{1}{2A}\Big[a_1 u_1 + (a_2 - 2a_3\eta_2 - 2a_2\eta_3)\, u_2 + (a_3 - 2a_2\eta_3 - 2a_3\eta_2)\, u_3 + (4a_3\eta_2 + 4a_2\eta_3)\, u_4 + b_1 v_1 + (b_2 - 2b_3\eta_2 - 2b_2\eta_3)\, v_2 + (b_3 - 2b_2\eta_3 - 2b_3\eta_2)\, v_3 + (4b_3\eta_2 + 4b_2\eta_3)\, v_4\Big]$$

Expression for Strain Energy

The total strain energy stored in the element of constant thickness t is

$$\Phi = \frac{t}{2}\int [\epsilon]^T [c][\epsilon] \cdot dA \tag{10}$$

In this case, $[c]$ is a 3 x 3 matrix of material properties and $[\epsilon]$ is a 3 x 1 matrix of the three components of strain which are linear functions of space. A closed form expression for the evaluation of a product of two linear functions has been developed and is given as

$$\int f \cdot g \cdot dA = \frac{A}{12} \langle f_1 f_2 f_3 \rangle \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \tag{11}$$

where the subscripts 1, 2, and 3 indicate the values of the functions evaluated at the corners of the triangular element.

The application of this integration formula to Eq. 10 results in the following expression for the strain energy in terms of the strains at the corner nodal points:

$$\Phi = \frac{At}{24} \begin{bmatrix} \bar{\epsilon}_x^T & \bar{\epsilon}_y^T & \bar{\gamma}^T \end{bmatrix} \begin{bmatrix} C_{11}Q & C_{12}Q & C_{13}Q \\ C_{21}Q & C_{22}Q & C_{23}Q \\ C_{31}Q & C_{32}Q & C_{33}Q \end{bmatrix} \begin{bmatrix} \bar{\epsilon}_x \\ \bar{\epsilon}_y \\ \bar{\gamma} \end{bmatrix} \qquad (12)$$

where the corner strain submatrices are defined as

$$\bar{\epsilon}_x = \begin{Bmatrix} \epsilon_{x1} \\ \epsilon_{x2} \\ \epsilon_{x3} \end{Bmatrix} \; ; \; \bar{\epsilon}_{y2} = \begin{Bmatrix} \epsilon_{y1} \\ \epsilon_{y2} \\ \epsilon_{y3} \end{Bmatrix} \; ; \; \bar{\gamma} = \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{Bmatrix} \qquad (13a,b,c)$$

and

$$Q = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad (14)$$

The evaluation of Eq. 9 results in the following expression for corner strains in terms of global displacements at the four element nodal points:

$$\begin{bmatrix} \bar{\epsilon}_x \\ \bar{\epsilon}_y \\ \bar{\gamma} \end{bmatrix} = \begin{bmatrix} U & \cdot \\ \cdot & V \\ V & U \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad (15)$$

where the submatrices are defined as

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \qquad (16a,b)$$

$$U = \frac{1}{2A} \begin{bmatrix} b_1 & b_2 & b_3 & \\ b_1 & b_2-2b_3 & -b_3 & 4b_3 \\ b_1 & -b_2 & b_b-2b_2 & 4b_2 \end{bmatrix} \qquad (16c)$$

$$V = \frac{1}{2A} \begin{bmatrix} a_1 & a_2 & a_3 & \cdot \\ a_1 & a_2-2a_3 & -a_3 & 4a_3 \\ a_1 & -a_3 & a_3-2a_2 & 4a_2 \end{bmatrix} \tag{16d}$$

The substitution of Eq. 15 into Eq. 12 yields the following equation for the strain energy of the element:

$$\Phi = \frac{1}{2} [u^T \; v^T] \; \begin{bmatrix} k \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{17}$$

With the strain energy written in this form it is apparent that [k] is the element stiffness matrix and is given by

$$[k] = \frac{At}{12} \begin{bmatrix} U^T & \cdot & V^T \\ \cdot & V^T & U^T \end{bmatrix} \begin{bmatrix} C_{11}Q & C_{12}Q & C_{13}Q \\ C_{21}Q & C_{22}Q & C_{23}Q \\ C_{31}Q & C_{32}Q & C_{33}Q \end{bmatrix} \begin{bmatrix} U & \cdot \\ \cdot & V \\ V & U \end{bmatrix} \tag{18}$$

Within the computer program given in this report, the submatrices U and V are formed and then the element stiffness matrix is formed directly in a series of operations which minimize programming effort and optimizes computer execution time.

Quadrilateral Stiffness Matrix

After the formation of the 8 x 8 stiffness matrices for the two triangular elements (application of Eq. 18) they are combined to form a 10 x 10 stiffness matrix by the direct stiffness technique. The unknown displacements associated with the center point are then eliminated by a "static condensation" procedure [5], and the 8 x 8 stiffness matrix for the quadrilateral element is developed. At the same time an approach stress-displacement transformation matrix is developed to be used later to calculate the stresses after the displacements are determined.

The application of the direct stiffness approach to the system of quadrilateral elements results in the stiffness matrix for the complete structure. Other types of elements, such as beam and truss elements, may be added to the system by the same approach.

DYNAMIC EQUILIBRIUM EQUATIONS

The force equilibrium of a system of structural elements is expressed by the following matrix equation:

$$\underline{M}\, \ddot{\underline{u}}_t + \underline{c}\, \dot{\underline{u}}_t + \underline{K}\, \underline{u}_t = \underline{P}_t \tag{19}$$

where $\underline{u}$, $\dot{\underline{u}}$, and $\ddot{\underline{u}}$ are vectors of nodal point displacements, velocities, and accelerations at time "t". In case of earthquake forces the load vector $\underline{P}_t$ is a function of the mass matrix and the earthquake acceleration at time "t". The formation of the stiffness matrix $\underline{K}$ for a finite element system is discussed in the previous section.

A formal mathematical development of the mass matrix $\underline{M}$ is possible. Such an approach would result in a mass matrix with the same coupling properties as the stiffness matrix. However, if the physical lumped mass approximation is made the mass matrix will be diagonal. The lumped mass approximation results in a small reduction in accuracy and a considerable saving in computer storage and time. In this investigation one-fourth the mass of each quadrilateral is assumed to be concentrated at each of the four nodal points.

For most structures the exact form of the damping matrix $\underline{c}$ is unknown. In the solution procedure the damping matrix may be completely arbitrary; however, there is little experimental justification for selecting specific damping coefficients. A form of viscous damping, which is sufficiently general for most structures, is given by the following matrix equation:

$$\underline{c} = \alpha \underline{M} + \beta \underline{K} \tag{20}$$

By assuming the damping matrix is proportional to the mass and stiffness matrices the effects of viscous damping is included without requiring additional storage within the computer program.

STEP-BY-STEP INTEGRATION OF EQUILIBRIUM EQUATIONS

The dynamic equilibrium of the finite element system is given by Eq. 19. The solution of this set of second order differential equations is accomplished by a step-by-step procedure.[6] The only approximation which is made is that the acceleration of each point in the system varies linearly within a small time interval, Δt. This assumption leads to a parabolic variation of velocity and a cubic variation of displacement within the time interval, $t - \Delta t$ and t

A direct integration over the interval gives the following equations for acceleration and velocity at the end of the time interval:

$$\ddot{\underline{u}}_t = \frac{6}{\Delta t^2} u_t - \frac{6}{\Delta t^2} \underline{u}_{t-\Delta t} - \frac{6}{\Delta t} \dot{u}_{t-\Delta t} - 2 \ddot{\underline{u}}_{t-\Delta t} \tag{21}$$

$$\dot{\underline{u}}_t = \frac{3}{\Delta t} \underline{u}_t - \frac{3}{\Delta t} \underline{u}_{t-\Delta t} - 2 \dot{\underline{u}}_{t-\Delta t} - \frac{\Delta t}{2} \ddot{\underline{u}}_{t-\Delta t} \tag{22}$$

The substitution of Eqs. (20), (21) and (22) into the equilibrium relationship, Eq. (19), results in a set of linear equations in terms of the unknown vector $\underline{u}_t$. A solution of this set of equations yields the displacements of the system at time t. The acceleration and velocities may then be found from Eqs. (21) and (22). This procedure may then be repeated for subsequent time steps.

STABILITY OF THE STEP-BY-STEP METHOD

The previously described step-by-step integration technique is accurate if the time step is small compared to the shortest period of the finite element system. If the time step is long compared to the shortest period, the method will become unstable and fail to produce realistic results. Newmark[7] has studied this instability and has suggested a constant acceleration method. Newmark's procedure was found to be stable when applied to finite element systems; however, spurious finite oscillations associated with the high frequencies of the system were still present in the results. Several other stable step-by-step methods were investigated with respect to finite element systems; the method found to be completely stable was a modification of the previously described linear acceleration method.

The instability in the linear acceleration method is first initiated by an oscillation of the displacements about the true solution. In the early stages of instability it is apparent that the displacements at the center of the time interval are a good approximation of the true solution. Therefore, if this mid-point solution is utilized, the tendency for oscillations to develop is eliminated.

In order to modify the previous step-by-step equations to reflect this approach a time increment of $2\Delta t$ is introduced and the acceleration, $\ddot{u}_{t+\Delta t}$, at the end of the time interval is calculated. The midpoint acceleration is calculated as

$$\ddot{\underline{u}}_t = \frac{1}{2}\left[\ddot{\underline{u}}_{t-\Delta t} + \underline{u}_{t+\Delta t}\right] \qquad (23)$$

The velocities and displacements at time "t" are calculated from

$$\dot{\underline{u}}_t = \dot{\underline{u}}_{t-\Delta t} + \frac{\Delta t}{2}\ \ddot{\underline{u}}_{t-\Delta t} + \frac{\Delta t}{2}\ \ddot{\underline{u}}_t \qquad (24)$$

$$\underline{u}_t = u_{t-\Delta t} + \Delta t\ \dot{\underline{u}}_{t-\Delta t} + \frac{\Delta t^2}{3}\ \ddot{u}_{t-\Delta t} + \frac{\Delta t^2}{6}\ \ddot{u}_t \qquad (25)$$

This modification eliminates all stability problems from the linear acceleration method. However, the new procedure tends to introduce damping in the higher frequencies of the system. Fortunately, this partial truncation of the higher modes is justified in an earthquake analysis since these frequencies are not accurately recorded in the input acceleration data.

This new stable step-by-step method, which is presented in a form which minimizes computer storage and execution time, is summarized below:

1. Initialization

   a. Form Stiffness Matrix $\underline{K}$ and Diagonal Mass Matrix $\underline{M}$

   b. Calculate the following constants:

$$\tau = 2\,\Delta t \qquad a_5 = \frac{3\beta}{\tau} a_4 - \frac{3}{\tau^2}$$

$$a_0 = \frac{6 + 3\alpha\tau}{\tau^2 + 3\beta\tau} \qquad a_6 = 2\,\beta a_4 - \frac{3}{\tau}$$

$$a_1 = \frac{6}{\tau^2} + \frac{3}{\tau}(\alpha - \beta\, a_0) \qquad a_7 = \frac{\beta - \tau\, a_4 - 1}{2}$$

$$a_2 = \frac{6}{\tau} + 2(\alpha - \beta a_0) \qquad a_8 = \frac{\Delta t}{2}$$

$$a_3 = 2 + \frac{\tau}{2}(\alpha - \beta a_0) \qquad a_9 = \frac{\Delta t^2}{3}$$

$$a_4 = \frac{3}{3\beta\tau + \tau^2} \qquad a_{10} = \frac{\Delta t^2}{6}$$

c. Form the "effective" Stiffness Matrix

$$\underline{\overline{K}} = \underline{K} + a_0\,\underline{M}$$

d. Triangularize $\underline{\overline{K}}$. This is normally banded and its triangularized form is also banded.

2. For each time increment

a. Form "effective" load

$$\underline{\overline{P}}_t = \underline{P}_{t+\Delta t} + \underline{M}\left[a_1\,\underline{u}_{t-\Delta t} + a_2\,\dot{\underline{u}}_{t-\Delta t} + a_3\,\ddot{\underline{u}}_{t-\Delta t}\right]$$

b. Solve for "effective" displacement vector $\underline{\overline{u}}_t$

$$\underline{\overline{K}}\,\underline{\overline{u}}_t = \underline{P}_t$$

This solution of a set of linear equations at each time step involves a minimum of computational effort since $\underline{\overline{K}}$ has been previously triangularized.

c. Calculate accelerations, velocities and displacements at time t .

$$\ddot{\underline{u}}_t = a_4\,\underline{\overline{u}}_t + a_5\,\underline{u}_{t-\Delta t} + a_6\,\dot{\underline{u}}_{t-\Delta t} + a_7\,\ddot{\underline{u}}_{t-\Delta t}$$

$$\dot{\underline{u}}_t = \dot{\underline{u}}_{t-\Delta t} + a_8\left[\ddot{\underline{u}}_{t-\Delta t} + \ddot{\underline{u}}_t\right]$$

$$\underline{u}_t = \underline{u}_{t-\Delta t} + \Delta t\,\dot{\underline{u}}_{t-\Delta t} + a_9\,\ddot{\underline{u}}_{t-\Delta t} + a_{10}\,\ddot{\underline{u}}_t$$

d. Calculate element stresses if desired

e. Repeat for next time increment

EXAMPLE

An earth dam was selected to illustrate the application of the method in the evaluation of the foundation-structure interaction of a large structure. The dimensions and properties of the structure and foundation and the finite idealization of the complete system are shown in figure 5. The input base acceleration was the N-S component of the 1940 El Centro Earthquake. Five percent damping was assumed. The maximum values of accelerations at the base of the dam are plotted in figure 6. In this particular example, the difference between the response at the base of the dam (24%) and the free-field surface displacements (30%) is significant. Because of the large numbers of parameters involved (shape and properties of dam, depth and properties of foundation, and characteristics of the earthquake) a complete parameter study of a particular type of structure is practically impossible.

The computer time required for this analysis which included the first six seconds of the earthquake was seven minutes on the CDC 6400. Therefore, to include the effects of foundation interaction for a structure of this type is certainly practical from an economic consideration.

FINAL REMARKS

A procedure is presented for the dynamic analysis of two-dimensional stress structures of arbitrary shape and material properties. The method is especially suited for the evaluation of foundation-structure interaction effects during earthquakes.

The stable step-by-step integration of the dynamic equilibrium equations provides an effective computer technique for finite element systems. Some of the advantages of the method compared to the mode superposition approach are:

1. The large eigenvalue problem is eliminated.

2. In most cases, the step-by-step method is faster.

3. A large capacity is possible since the basic computer storage required during the step-by-step solution is a triangularized form of the banded stiffness matrix (i.e., for a computer with 32 K storage a mesh site of 10 x 50 would be possible).

4. The approach is easily extended to include the effects of non-linear material properties. However, this would result in a large increase in computer time, since the stiffness matrix would be formed and triangularized many times during the step-by-step solution procedure.

Another application of the techniques presented in this paper is in the study of reservoir-dam interaction during earthquakes. The idealization of water as a structural finite element with zero shear modulus does not cause difficulties. This work is reported in more detail in a separate publication [8]