DAM-FOUNDATION INTERACTION DURING EARTHQUAKES

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SYNOPSIS

An analytical procedure to determine the dynamic characteristics of a dam resting on an elastic foundation is presented. The dam cross-section is represented as an assemblage of finite elements and the foundation is treated as a semi-infinite elastic medium. The analysis takes account of the dynamic interaction between the dam and foundation and determines the natural frequencies and mode shapes of the dam and the loss of energy by wave propagation into the foundation. Numerical results are presented and discussed for a typical earth dam for a wide range of foundation properties.

INTRODUCTION

During the past decade, one or more destructive earthquakes have occurred somewhere in the world nearly every year. Each of these earthquakes has caused, on the average, one-half billion dollars in property damage in addition to loss of life. The design of structures to safely withstand strong earthquakes is therefore extremely important. Earthquake resistant design of dams is a particularly important problem in view of the catastrophic consequences of dam failure. The state of knowledge of effects of earthquakes on dams is not as advanced as our knowledge of the performance of buildings during earthquakes. The problem of earthquake analysis of dams is inherently more complex but it must also be recognized that it has received relatively less attention.

It is only during recent years that a satisfactory technique has been developed to analyze the stresses caused in earth dams and concrete gravity dams during earthquakes[1]. The analysis technique is based on the finite element concept[2] and at present stage of development is applicable to two-dimensional continua with arbitrary geometry and material non-homogeneity. Thus the earthquake behavior of a dam can be analyzed by this technique if the problem can be idealized as a plane stress or plane strain system. Using the finite element method the effects of earthquakes on earth dams have been studied extensively[3].

The studies mentioned above have demonstrated that the two-dimensional finite element method provides a powerful tool for the earthquake analysis of dams. At present, the method is restricted to linearly elastic materials. Considerable amount of additional research needs to be done before it will be possible to satisfactorily determine the response of dams to earthquakes. The analytical procedures need to be extended to include effects of inelastic deformations. The interaction between the reservoir and the dam[4], and the manner in which water affects

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the response of the dam, represents a field of study about which our knowledge is meagre. In addition, little is known about the effects of the flexibility of the foundation soil or rock on the earthquake response of the dam. It was to shed some light on this problem that the present study was undertaken.

The effects of an elastic foundation may be included in the finite element analysis[2,5] by extending the finite element mesh into a zone of the foundation. A finite element idealization of the dam-foundation system, by necessity, would include only a finite zone of the foundation. If the foundation zone included is large enough such an idealization can probably represent satisfactorily the inertia and stiffness effects of even an infinite foundation, such as a half space. However, the idealization will be incapable of including the apparent loss of energy by wave propagation into the half space. This loss of energy would occur even if the foundation medium possessed no material damping and is caused entirely by the infinite geometry of the system. It will in the following be referred to as "radiation damping". The finite element approach, however, offers other advantages: irregular geometry of the dam-foundation system, material non-homogeneity that will be encountered at the junction of the dam and foundation and a non-homogeneous and anisotropic foundation material can all be realistically treated in a straight forward manner.

The purpose of this investigation is to present an entirely different approach in which the foundation is treated as an elastic half space i.e. a semi-infinite, elastic medium. Radiation damping is now represented satisfactorily because the infinite extent of the foundation is recognized. However, it becomes desirable to assume that the foundation medium is homogeneous and isotropic so that the dynamic elasticity problem for the half space can be conveniently solved.

EQUATIONS OF MOTION

Dam-Foundation System

The dam is assumed to be prismatic and infinitely long; the foundation to be a half space bounded by the plane $y = 0$. The dam may be composed of non-homogeneous anisotropic linearly elastic materials, whereas the half space is assumed to be homogeneous, isotropic and linearly elastic. The dynamic excitation of the system is assumed to be uniform along the length of the structure. Thus, the dam-foundation system constitutes a plane strain problem. The coordinate system for this two-dimensional problem is shown in Fig. 1.

Equations of Motion for the Dam

The dam cross-section is represented by an assemblage of triangular finite elements[2] interconnected at their nodal points. Compatibility between the edges of adjacent elements is maintained by requiring that the displacements within each element be linear functions of the x- and y- coordinates. This assumption leads to constant strains in the element. More refined elements, assuming higher order displacement functions have been introduced[6]. However, the constant strain triangular element is considered to be appropriate for the present investigation. The stiffness matrix $[K_d]$ and the diagonal mass matrix $[M_d]$ for the dam cross-section can then be assembled by standard techniques[1, 2]. The equations of motion for the undamped system may be written as

$$[M_d]\{\ddot{r}_d\} + [K_d]\{r_d\} = \{R_d(t)\} \tag{1}$$

Eq. 1 represents a system of 2N differential equations where N is the number of nodal points in the finite element idealization, i.e. each nodal displacement may include both x and y components in general. $\{r_d\}$ is the vector of nodal point displacements, $\{\ddot{r}_d\}$ is the corresponding acceleration vector and $\{R_d(t)\}$ is the vector of applied dynamic loads.

Steady State Harmonic Vibration of the Foundation

The equations of motion for the foundation are derived with respect to a set of nodal points selected at its surface, y = 0, as shown in Fig. 2. A "dynamic influence coefficient", S_{ij}, is defined as the complex frequency response of force at nodal point "i" associated with a displacement excitation at nodal point "j" i.e. when the excitation is a displacement $e^{i\omega t}$ at the nodal point "j" then the force response at nodal point "i" is $S_{ij}e^{i\omega t}$. A typical system of these dynamic influence coefficients is shown in Fig. 2. The nodal points have been selected at equal spacing for convenience. However, this is not essential to the procedure that follows.

The surface displacement between adjacent nodal points is assumed to vary linearly in order to ensure compatible deformation between the surface of the foundation and the base of the finite element system representing the dam. Thus, a unit harmonic vertical displacement at point "0" results in the following surface displacements:

$$u(x,0) = 0 \;;\; v(x,0) = h(x)e^{i\omega t} \tag{2}$$

where $h(x) = 1 - \frac{x}{a}$, $0 < x < a$; $1 + \frac{x}{a}$, $-a < x < 0$; 0 , $x < -a$ and $x > a$. (3)

Similarly, a unit harmonic horizontal displacement at point "0" results in the following surface displacements

$$u(x,0) = h(x)e^{i\omega t} \;;\; v(x,0) = 0 \tag{4}$$

The equations of motion for a two-dimensional homogeneous elastic solid may be expressed in terms of two wave equations

$$\frac{\partial^2\phi}{\partial t^2} = C_1^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)$$
$$\frac{\partial^2\psi}{\partial t^2} = C_2^2\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \tag{5}$$

where $C_1 = \{(\lambda + 2\mu)/\rho\}^{\frac{1}{2}}$ and $C_2 = (\mu/\rho)^{\frac{1}{2}}$ are respectively the velocities of irrotational and equivoluminal waves in the elastic medium. In these equations ρ is the density, λ and μ are Lame's constants which are related to the modulus of elasticity E and Poisson's ratio ν:

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} , \quad \mu = \frac{E}{2(1 + \nu)}$$

The x and y components of displacement, u and v are given by

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} , \quad v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \tag{6}$$

The normal stresses σ_x and σ_y and the shearing stress τ_{xy} are related to the displacements by the equations

$$\begin{aligned} \sigma_x &= \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu \frac{\partial u}{\partial x} \\ \sigma_y &= \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu \frac{\partial v}{\partial y} \\ \tau_{xy} &= \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \end{aligned} \tag{7}$$

The steady state response of the foundation subjected to the harmonic excitations defined by Eqs. 2 and 4 may be expressed as

$$\begin{aligned} \phi(x,y,t) &= H_\phi(x,y,\omega)e^{i\omega t} \\ \psi(x,y,t) &= H_\psi(x,y,\omega)e^{i\omega t} \end{aligned} \tag{8}$$

where H_ϕ and H_ψ are the complex frequency response functions for ϕ and ψ. Substituting from Eq. 8, Eq. 5 becomes

$$\begin{aligned} \frac{\partial^2 H_\phi}{\partial x^2} + \frac{\partial^2 H_\phi}{\partial y^2} + n^2k^2H_\phi &= 0 \\ \frac{\partial^2 H_\psi}{\partial x^2} + \frac{\partial^2 H_\psi}{\partial y^2} + k^2H_\psi &= 0 \end{aligned} \tag{9}$$

in which $k^2 = \omega^2/C_2^2$ and $n^2 = (1 - 2\nu)/(2 - 2\nu)$. The stress responses may be expressed as

$$\sigma(x,y,t) = H_\sigma(x,y,\omega)e^{i\omega t} \tag{10}$$

Eq. 10 applies to each of the stress components σ_x, σ_y and τ_{xy} and corresponding complex frequency response functions are $H\sigma_x$, $H\sigma_y$ and $H\tau_{xy}$. The complex frequency responses for stresses may be expressed in terms of those for ϕ and ψ from Eqs. 6, 7, 8 and 10:

$$\begin{aligned} H_{\sigma_x} &= \mu \left(-k^2H_\phi - 2 \frac{\partial^2 H_\phi}{\partial y^2} + 2 \frac{\partial^2 H\psi}{\partial x \partial y}\right) \\ H_{\sigma_y} &= \mu \left(-k^2H_\phi - 2 \frac{\partial^2 H_\phi}{\partial x^2} - 2 \frac{\partial^2 H_\psi}{\partial x \partial y}\right) \\ H_{\tau_{xy}} &= \mu \left(-k^2H_\psi + 2 \frac{\partial^2 H_\phi}{\partial x \partial y} - 2 \frac{\partial^2 H_\psi}{\partial x^2}\right) \end{aligned} \tag{11}$$

Eqs. 9 subject to the boundary conditions of Eq. 2 are solved by the

Fourier transform technique[7,8], for the functions $H_\phi(x,y,\omega)$ and $H_\psi(x,y,\omega)$. The functions $H_{\sigma_x}(x,y,\omega)$, $H_{\sigma_y}(x,y,\omega)$ and $H_{\tau_{xy}}(x,y,\omega)$ are then determined from Eqs. 11. These functions are expressed in terms of non-dimensional quantities: frequency ratio $a_o = \omega a/C_2$ and $z = x/a$. In particular, with the aid of tables of integral transforms[9] the functions at the surface are given by[10]

$$H_{\sigma_y}(z,0,\omega) = \frac{2\mu a_o^2}{\pi a} \int_0^\infty \frac{(\zeta^2 - a_o^2)^{\frac{1}{2}} (1 - \cos\zeta) \cos(z\zeta)}{\zeta^2\{\zeta^2 - (\zeta^2 - n^2 a_o^2)^{\frac{1}{2}} (\zeta^2 - a_o^2)^{\frac{1}{2}}\}} d\zeta$$

$$H_{\tau_{xy}}(z,0,\omega) = \begin{cases} -\dfrac{2\mu}{a} + \dfrac{2\mu a_o^2}{\pi a} \cdot I, & 0 < z < 1 \\ \dfrac{2\mu a_o^2}{\pi a} \cdot I, & 1 < z < \infty \end{cases} \tag{12}$$

where

$$I = \int_0^\infty \frac{(1 - \cos\zeta) \sin(z\zeta)}{\zeta\{\zeta^2 - (\zeta^2 - n^2 a_o^2)^{\frac{1}{2}} (\zeta^2 - a_o^2)^{\frac{1}{2}}\}} d\zeta .$$

The limit of the complex frequency responses for stresses as ω(or k) $\to 0$ check with the solution of the corresponding static problem[11].

The dynamic influence coefficients, which are the nodal forces associated with the surface stresses (Eq. 12), may be found by applying the principle of virtual displacement. To determine the vertical nodal force at point "j" resulting from the prescribed vertical surface displacement at point "0" (Eq. 2), it is necessary to apply a unit vertical virtual displacement $\bar{r}_j^y = 1$ as shown in Fig. 3. The resulting vertical virtual displacements are given by

$$\eta(z) = \begin{cases} z - (j - 1), & (j - 1) < z < j \\ (j + 1) - z, & j < z < (j + 1) \\ 0, & z < (j - 1),\ z > (j + 1) \end{cases} \tag{13}$$

The principle of virtual displacements then leads to

$$S_{jo}^{yy} = a \int_{(j-1)}^{(j+1)} H_{\sigma_y}(z,0,\omega)\, \eta(z)\, dz \tag{14}$$

Similarly, the horizontal nodal force S_{jo}^{xy} at point "j" resulting from the prescribed vertical surface displacement at point "0" is given by

$$S_{jo}^{xy} = a \int_{(j-1)}^{(j+1)} H_{\tau_{xy}}(z,0,\omega)\, \eta(z)\, dz \tag{15}$$

Substituting from Eqs. 12 and 13 into Eqs. 14 and 15, interchanging the order of integration and integrating first with respect to z, the coefficients S_{jo}^{yy} and S_{jo}^{xy} can be written as indefinite integrals in ζ. The integrands may be imaginary, complex or real depending on the value of ζ and the integrals are split accordingly[8,10].

The dynamic influence coefficients S_{jo}^{yx} and S_{jo}^{xx}, associated with a unit horizontal displacement at point "0" (Eq. 4) can be derived in a similar fashion. All these dynamic influence coefficients depend on the excitation frequency ω and are defined only for harmonic excitations. For a specific value of E, ν, ρ, a and ω these coefficients are determined by numerically evaluating the integrals in ζ.

Equations of Motion for the Foundation

A 2 x 2 matrix, S_{jo}, may be defined as follows:

$$S_{jo} = \begin{bmatrix} S_{jo}^{xx} & S_{jo}^{xy} \\ S_{jo}^{yx} & S_{jo}^{yy} \end{bmatrix} \tag{16}$$

The equations for steady state harmonic vibration of the foundation may be expressed as

$$[S(\omega)]\{r\} = \{R\} \tag{17}$$

where {R} is the vector of harmonic forces applied at the nodal points at the surface of the half plane and {r} is the vector of steady state harmonic displacements of the nodal points. The frequency dependent foundation matrix $[S(\omega)]$ may be constructed by assembling the 2 x 2 matrices S_{jo}. Eq. 17 for a system of nodal points such as those shown in Fig. 4 may be written as

$$\begin{bmatrix} S_{00} & S_{10} & S_{20} & \cdots & S_{m+1,0} & 0 & \cdots & 0 & 0 \\ S_{10}^{T} & S_{00} & S_{10} & \cdots & S_{m0} & S_{m+1,0} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ S_{m+1,0}^{T} & S_{m0}^{T} & S_{m-1,0}^{T} & \cdots & S_{00} & S_{10} & \cdots & S_{m+1,0} & 0 \\ 0 & S_{m+1,0}^{T} & S_{m0}^{T} & \cdots & S_{10}^{T} & S_{00} & \cdots & S_{m0} & S_{m+1,0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & S_{m+1,0}^{T} & S_{m0}^{T} & \cdots & S_{00} & S_{00} \\ 0 & 0 & 0 & \cdots & & S_{m+1,0}^{T} & \cdots & S_{10}^{T} & S_{10}^{T} \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{m+1} \\ r_{m+2} \\ \vdots \\ r_{2m+N_f-1} \\ r_{2m+N_f} \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ \vdots \\ R_{m+1} \\ R_{m+2} \\ \vdots \\ R_{2m+N_f-1} \\ R_{2m+N_f} \end{Bmatrix} \tag{18}$$

in which r_j and R_j are two-dimensional vectors of displacements and forces at nodal point "j",

$$\{r_j\} = \begin{Bmatrix} r_j^x \\ r_j^y \end{Bmatrix} \; ; \; \{R_j\} = \begin{Bmatrix} R_j^x \\ R_j^y \end{Bmatrix}$$

where the superscripts x and y denote the x- and y- components respectively.

The foundation matrix $[S(\omega)]$ in Eq. 18 is banded and the band width represents the number of significant nodal forces generated by an applied nodal displacement. The number of nodal points considered on the foundation surface must be considerably more than those directly under the dam in order to represent the foundation deformation satisfactorily (Fig. 4). The foundation surface is assumed to be restrained outside of the region represented by these nodal points, as shown in Fig. 4. Eq. 18 may be written in partitioned form as

$$\begin{bmatrix} b & c & d \\ c^T & e & c_1 \\ d^T & c_1^T & b \end{bmatrix} \begin{Bmatrix} r_L \\ r_f \\ r_R \end{Bmatrix} = \begin{Bmatrix} R_L \\ R_f \\ R_R \end{Bmatrix} \tag{19}$$

where $\{r_L\}$, $\{r_f\}$ and $\{r_R\}$ are the displacement vectors for the left, base and right zones respectively. $\{R_L\}$, $\{R_f\}$ and $\{R_R\}$ are the corresponding load vectors. Since no external loads will be applied outside the base of the dam, $\{R_L\} = \{R_R\} = \{0\}$ and Eq. 19 may be reduced to

$$[G(\omega)]\{r_f\} = \{R_f\} \tag{20}$$

where

$$[G(\omega)] = [e] - [c^T \; c_1] \begin{bmatrix} b & d \\ d^T & b \end{bmatrix}^{-1} \begin{bmatrix} c \\ c_1^T \end{bmatrix} \tag{21}$$

and will be referred to as the "reduced foundation matrix". The number of nodal points to be released on either side of the base of the dam depends on the degree of accuracy required in the analysis. The elements of $[G(\omega)]$ are in general complex and may be expressed as

$$[G(\omega)] = [K_f(\omega)] + i\,\omega\,[C_f(\omega)] \tag{22}$$

and since $\{\dot{r}_f\} = i\,\omega\,\{r_f\}$ for steady state harmonic vibration, Eq. 20 may be written as

$$[C_f(\omega)]\{\dot{r}_f\} + [K_f(\omega)]\{r_f\} = \{R_f\} \tag{23}$$

Eq. 23 governs the steady state harmonic vibration of the foundation at an excitation frequency ω. The matrix $[C_f(\omega)]$ may be regarded as a damping matrix associated with the loss of energy due to propagation of waves into the infinite foundation. The matrix $[K_f(\omega)]$ may be regarded as a dynamic stiffness matrix for the foundation.

Equations for Steady-State Harmonic Vibration of the Dam-Foundation System

The equations of motion for the finite element system representing the dam (Eq. 1) may be expressed as

$$\begin{bmatrix} M_d^{aa} & 0 \\ 0 & M_d \end{bmatrix} \begin{Bmatrix} \ddot{r}_d^a \\ \ddot{r}_d \end{Bmatrix} + \begin{bmatrix} K_d^{aa} & K_d^a \\ (K_d^a)^T & K_d \end{bmatrix} \begin{Bmatrix} r_d^a \\ r_d \end{Bmatrix} = \begin{Bmatrix} R_d^a \\ R_d \end{Bmatrix} \tag{24}$$

in which the matrices have been partitioned in order to separate the nodal points above the foundation (denoted by superscript a) from those on the foundation. The steady-state harmonic vibration of the foundation is governed by Eq. 23. Displacements of the foundation nodal points in the base zone must be the same as those of the nodal points at the base of the dam in order to satisfy compatibility, i.e.,

$$\{r_f\} = \{r_d\} \equiv \{r_c\} \qquad (25)$$

Eqs. 23 and 24 may now be combined as

$$\begin{bmatrix} M_d^{aa} & 0 \\ 0 & M_d \end{bmatrix} \begin{Bmatrix} \ddot{r}_d^a \\ \ddot{r}_c \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C_f(\omega) \end{bmatrix} \begin{Bmatrix} \dot{r}_d^a \\ \dot{r}_c \end{Bmatrix} + \begin{bmatrix} K_d^{aa} & K_d^a \\ (K_d^a)^T & K_c(\omega) \end{bmatrix} \begin{Bmatrix} r_d^a \\ r_c \end{Bmatrix} = \begin{Bmatrix} R_d^a \\ R_c \end{Bmatrix} \qquad (26)$$

or, $[M]\{\ddot{r}\} + [C(\omega)]\{\dot{r}\} + [K(\omega)]\{r\} = \{R\}$ (27)

where, in Eq. 26,

$$[K_c(\omega)] = [K_d] + [K_f(\omega)] \; ; \; \{R_c\} = \{R_d\} + \{R_f\} \qquad (28)$$

Eq. 27 governs the steady-state harmonic vibration of the coupled dam-foundation system. It should be noted that the coefficient matrices of Eq. 27 depend on the excitation frequency ω.

FREQUENCIES, MODE SHAPES AND MODAL DAMPING

The general equations of motion for a structure may be expressed as

$$[M]\{\ddot{r}(t)\} + [C]\{\dot{r}(t)\} + [K]\{r(t)\} = \{R(t)\} \qquad (29)$$

[M], [C] and [K] are the mass, damping and stiffness matrices for the structure. Usually, all elements of these matrices are constant. The frequencies and mode shapes may then be determined by solving the standard eigenvalue problem, provided [C] satisfies certain restrictions[12]. It is apparent from Eq. 27 that the coefficient matrices in the equations for steady state harmonic vibration of the dam-foundation system depend on the excitation frequency ω. If one were to write the corresponding equations of motion for arbitrary dynamic excitation, the coefficient matrices would be time-dependent and the standard eigenvalue approach would not apply. However, the natural frequencies and mode shapes of such a system may be determined by evaluating the steady-state response to harmonic excitations over a wide range of frequencies and analyzing the resonance phenomenon. This procedure is not particularly suitable for analysis of the dam-foundation system because the natural frequencies may not be well separated. An alternative approach, based on the theory of resonance testing[13], is therefore used in this study.

Natural Frequencies

Under steady state harmonic vibration $\{\dot{r}\} = i\,\omega\,\{r\}$ and $\{\ddot{r}\} = -\omega^2\{r\}$.

Eq. 27 therefore becomes

$$(-\omega^2[M] + i\ \omega\ [C(\omega)] + [K(\omega)])\{r\} = \{R\} \tag{30}$$

Solving for the response $\{r\}$

$$\{r\} = [X(\omega)]\{R\}\ ;\ [X(\omega)] = (-\omega^2[M] + i\ \omega\ [C(\omega)] + [K(\omega)])^{-1} \tag{31}$$

$[X(\omega)]$ is often called the "admittance matrix"; it is, in general, a complex matrix and may be expressed as

$$[X(\omega)] = [Y(\omega)] + i\ [Z(\omega)] \tag{32}$$

The natural frequencies of the system are the zeros of $|Y(\omega)|$ [13] where the determinant of $[Y(\omega)]$ is denoted by $|Y(\omega)|$. Hence, the natural frequencies of the dam-foundation system may be determined by forming the admittance matrix, evaluating the determinant $|Y(\omega)|$ of its real part as a function of ω and determining the zeros of this function.

Natural Mode Shapes

Let $\{R_m\}$ be a real vector such that the response of the system to force vector $\{R_m\}e^{i\omega_m t}$ is entirely in the m^{th} natural mode (natural frequency = ω_m). This response may be expressed as $\{\phi\}e^{i\omega_m t}$ and from Eqs. 31 and 32

$$\{\phi\} = ([Y(\omega_m)] + i\ [Z(\omega_m)])\{R_m\} \tag{33}$$

Then $\{\phi\}$ is the m^{th} natural mode if and only if $\{\phi\}$ and $\{R_m\}$ are 90° out of phase[13], i.e.,

$$[Y(\omega_m)]\{R_m\} = \{0\} \tag{34}$$

The coefficient matrix of Eq. 34 is singular and the set of homogeneous equations may be solved for $\{R_m\}$. The shape of the m^{th} natural mode of vibration $\{\phi_m\}$ is given by Eq. 33 which by Eq. 34 reduces to

$$\{\phi_m\} = [Z(\omega_m)]\{R_m\} \tag{35}$$

In numerical computations, the determinant of $[Y(\omega_m)]$ will not be exactly zero due to round-off errors. However, $\{R_m\}$ can be approximated by the eigenvector corresponding to the eigenvalue of $[Y(\omega_m)]$ with the smallest modulus[13].

Modal Damping

Assuming that the "radiation damping" for the dam-foundation system can be represented in terms of an equivalent viscous damping, the generalized damping in the various natural modes may be determined by standard techniques[14]. These are applicable to systems with light to moderate damping (less than 20% of critical damping) and with well separated natural frequencies. The response of the dam-foundation system to the force vector $\{R_m\}e^{i\omega_m t}$ may be equated to the response of a generalized one degree of freedom system for the m^{th} natural mode.