EQUIVALENT LUMPED SYSTEM FOR STRUCTURE FOUNDED UPON STRATUM OF SOIL

Robert V. Whitman (I)

SYNOPSIS

The theory for the dynamic loading of an elastic stratum is first used to distinguish three cases - depth of soil>>width of building, depth of soil≈width of building, depth of soil<<width of building - and to suggest limiting values of the depth/width ratio. Then results from this theory are used to guide choice of equivalent spring constants and masses. Tentative suggestions are made concerning the choice of equivalent damping ratios.

1. INTRODUCTION

Simple arrangements of "lumped" springs and dashpots are often used to simulate the effect of flexibility in the underlying soil upon the response of a structure to earthquake ground motions. Sometimes the spring constant and damping factor are made variable with frequency (1,2); more commonly, constant parameters are employed (3). With this simple "lumped" representation of the soil, the effect of varying the assumptions concerning the behavior of the soil may be investigated readily so as to bracket the probable behavior. Modern computer-based techniques (such as the finite element method) make it possible to calculate with essentially "continuous" representations of the soil. However, because of their complexity, these more sophisticated techniques will find their major use in research, and the simple "lumped" representation will continue to have wide use for practical calculations.

Considerable attention has already been given to development of methods for evaluating spring constants and damping ratios suitable for specific problems (4,5). While the majority of the field evaluations of these methods has involved foundations for machines, some have involved large foundations for radar towers. When the soil beneath a structure is essentially homogeneous to a depth which is large compared to the width of the structure, equivalent spring constants, masses and damping ratios can be evaluated with some confidence. However, the case where the soil beneath a structure is of limited depth has not previously been clarified. This paper presents a preliminary examination of this important case, revealing the essential features of the problem.

I. Professor of Civil Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts, U.S.A.

2. GENERAL DISCUSSION

The soil conditions at the site of a building affect the earthquake response of the building in two ways (see Figure 1).

First, even if no building is present, the motion $x_2(t)$ at the top of the soil in general differs from the motion $x_1(t)$ of first ground. This aspect of the problem, which is generally called *soil amplification*, has been treated in detail by many engineers (6,7).

Second, the presence of the building causes the motions $x_3(t)$ at the base of the building to differ from the motions $x_2(t)$ of the unloaded soil. Moreover, because of rocking of the building on the soil, the motions $x_4(t)$ at the top of even a rigid building will differ still further from the motions $x_2(t)$. This second aspect of the effect of local soil conditions on response of buildings to earthquakes is referred to as *soil-structure interaction*. For typical multi-storied structures of flexible construction, soil-structure interaction may have little effect upon response (Ref. 3). However, for stiff buildings (containment structures for large nuclear reactors are an important example), soil-structure interaction generally does have a significant influence upon response.

Both of these two effects of local soil conditions are functions of the modulus and damping of the soil and of the thickness of the soil.

In the case where the depth of the soil is much greater than the width of the building, the effects are independent of each other. The representation shown in Figure 2 may be used to analyze soil-structure interaction; the motion x_2 which serves as input is determined by analysis of the effect of the unloaded soil. Theories for a mass resting upon an elastic half-space may be used to guide the choice of the spring constants and damping ratios (4,5).

However, when the depth of soil is similar to or less than the width, then the two effects are interdependent and any analysis must account simultaneously for both.

Section 3 illustrates this interdependence for a special case of a rigid structure undergoing pure horizontal motion. Section 4 outlines the procedure which may be used to determine suitable parameters for the equivalent lumped system.

3. EXACT SOLUTION FOR A SPECIAL CASE

The simplest special case is shown in Figure 3: a rigid structure which experiences no rocking rests upon a homogeneous elastic stratum which in turn rests upon a rigid but moving base. The essential features of this case may be understood by examining the response of the system to a periodic base motion $X_1\exp(i\Omega t)$ where Ω is circular frequency. The motions of the top of unloaded soil and of the structure are $X_2(\Omega)\exp[i(\Omega t+\alpha)]$ and $X_3(\Omega)\exp[i(\Omega t+\beta)]$, respectively, where α and β are phase angles.

The amplitude of the motion of the structure may be written as

$$X_3(\Omega) = A(\Omega)\ I(\Omega)\ X_1 \qquad (1)$$

where $A(\Omega) \equiv X_2/X_1$ is the *amplification spectrum*
$I(\Omega) \equiv X_3/X_2$ is the *soil-structure interaction spectrum*

$A(\Omega)$ may be evaluated by the methods discussed in Reference 6. For the special case considered here:

$$A(\Omega) = \sec a_1 \qquad (2)$$

where $a_1 = \Omega H/C_s$
$C_s = \sqrt{G/\rho}$ is the shear wave velocity of the stratum
G = shear modulus of the stratum
ρ = mass density of the stratum

$A(\Omega)$ becomes infinite at the resonant frequencies of the stratum in shear: i.e. for $(a_1)_n = (2n-1)\pi/2$ where n is an integer.

The first step in the valuation of $I(\Omega)$ is the theory for the response of a stratum to a periodic applied load (8). The force P between the structure and the stratum is a function of the relative displacement $y = x_3 - x_2$ ∴

$$y = \frac{P}{Gr}\ (f_1 + if_2) \qquad (3)$$

where r = half-width (or radius) of loaded area
f_1, f_2 = functions of a_o and H/r, Poisson's ratio of stratum, shape of loaded area
$a_o = \Omega r/C_s$

The functions of f_1 and f_2 are given in Figure 4 for the case $H/r = 4$, based upon the deflection at the centerline of a uniformly loaded square area (9). The function f_1 is infinite for $a_o = (a_1)_n/4$, i.e. at the resonant frequencies of the stratum in shear, and also for other a_o which correspond to three-dimensional standing waves. The function f_2 is zero for frequencies less than the fundamental frequency in shear, since at such frequencies surface waves propagating outward to infinity cannot exist and thus there is not radiation damping. For larger frequencies there is radiation damping.

For the special case of a rigid mass:

$$P = -M(\ddot{x}_2 + \ddot{y}) \qquad (4)$$

where M is the mass of the structure and the dots indicate differentiation with respect to time. Combining Equations 3 and 4 leads to:

$$I(\Omega) = 1/\sqrt{(1-a_o^2 bf_1)^2+(a_o^2 bf_2)^2} \qquad (5)$$

where $b = M/\rho r^3$. Figure 5 shows $I(\Omega)$ for b = 10 and for two different values of H/r. The following patterns may be noted:

(1). For $a_o=(a_1)_n r/H$, $I(\Omega) = 0$. The product $A(\Omega)I(\Omega)$ is indeterminate at this frequency. Hence at the resonant frequency of the stratum, the motions of a structure are not necessarily infinite.

(2). At some frequency slightly smaller than the fundamental frequency of the stratum, $(1-a_o^2bf_1) = 0$. At this frequency $f_2 = 0$, and hence $I(\Omega)$ is infinite, and so too is the product $A(\Omega)I(\Omega)$. The difference between this frequency and the fundamental frequency of the stratum reflects the effect of the structure upon the fundamental frequency of the system. This effect may be seen by comparing columns 3 and 4 in Table 1; column 3 gives a_o at the fundamental frequency as computed by the exact theory while column 4 gives r/H times the fundamental frequency of the stratum in shear. Even for a large mass (b=10), this effect is negligible for $H/r \geq 4$ (I).

(3). In the curve for H/r = 4, $I(\Omega)$ has a peak at approximately a_o=0.6. This peak is similar to that in the curve for the half-space. Thus in this case, the effects of amplification and soil-structure interaction are more-or-less independent; the first, infinite peak represents the resonance of the stratum, modified by the presence of a mass, while the second peak is that caused by soil-structure interaction. Figure 6 compares the exact solution for the product $A(\Omega)I(\Omega)$ with the product of $A(\Omega)$ computed for a stratum without mass (Equation 2) and $I(\Omega)$ for a half-space. It is seen that the exact and approximate curves are for all practical purposes the same.

I. Similarly, $(1-a_o^2bf_1)$ may also become zero for larger values of a_o (provided that b is not too large), so that there may be higher order resonances of the overall system. While the behavior of f_2 near these higher frequencies has not been calculated accurately as yet, presumably $f_2 \neq 0$ so that $I(\Omega)$ is finite at the system's higher order resonances.

Table 1

Values of a_o at Fundamental Frequency

(1) b	(2) H/r	(3) Exact	(4) Stratum no mass	(5) Stratum uniform mass	(6) 2-DOF lumped system
10	1	0.75	1.57	0.63	0.75
	2	0.63	0.79	0.39	0.58
	4	0.39	0.39	0.25	0.37
1	1	1.35	1.57	1.30	1.36
	2	0.78	0.79	0.70	0.74
	4	0.39	0.39	0.39	0.39

Thus, for cases of large H/r where $I(\Omega)$ has not been calculated (for example, H/r=6), $I(\Omega)$ for the half-space may be used to obtain a good approximation for $A(\Omega)I(\Omega)$.

(4). For a very thin stratum, the response of the stratum plus the structure should be similar to the response of a stratum carrying a distributed mass (with the same mass per unit area as for the structure) over its entire surface. The fundamental frequency for the latter situation (a one-dimensional problem) is given in column 5 of Table 1 (10,11). Comparing columns 3 and 5, it may be seen that for b=1 the agreement is good for H/r =1. For a larger mass (b=10) the agreement becomes good for $H/r \leq 0.5$.

Thus, examination of the exact solution has led to two important conclusions of a general nature: (a) for $H/r \geq 4$, amplification and soil structure interaction may be treated separately and soil-structure interaction may be analyzed using equations derived for a half-space; and (b) for $H/r \leq 0.5$, the problem is essentially one-dimensional. These conclusions are depicted in Figure 7.

4. EQUIVALENT SPRING CONSTANTS

Reference 12 has proposed the approximate 2 degree-of-freedom system shown in Figure 7b. The parameters of this model are selected as follows:

(1) The mass M is the mass resting on the surface of the stratum

(2) The spring constant k_s is the force-deflection ratio for the elastic cylinder which replaces the stratum. The radius of this cylinder is

$$r_c = r_e + H$$

where r_e is the radius of a circle having the same area as the square loaded area

$$r_e = 2r/\sqrt{\pi}$$

Thus

$$k_s = \pi \frac{G}{H} \left(\frac{2}{\sqrt{\pi}} r + H \right)^2 \tag{6}$$

(3) The mass M is chosen such that

$$\frac{1}{2\pi} \sqrt{\frac{k_s}{M_s}} = \frac{1}{4H} \sqrt{\frac{G}{\rho}} \tag{7}$$

That is, the natural frequency of the mass M_s with the spring k_s just equals the fundamental frequency of the stratum itself.

(4) The spring constant k_f, which represents the soil-structure interaction, is evaluated from

$$\frac{1}{k} = \frac{1}{k_f} + \frac{1}{k_s} \tag{8}$$

where k is the static force-deflection relationship for the loaded area upon the stratum.. One possible way to obtain values of k is by the equation:

$$k = Gr/f_1 \tag{9}$$

where f_1 is obtained from figure 4 at zero frequency. As H/r approaches zero, k_f becomes zero; that is, the spring constant k_s fully accounts for the deflections caused by a static load. As H/r becomes very large, k_f approaches the spring constant for a half-space.

With the parameters of the 2-DOF system thus devined, the fundamental frequency of the system may then be computed. Results are given in column 6 of Table 1. There is quite good agreement with the "exact" results in column 3. Predictions of this approximate model have also been compared with exact and experimental results for a circular loaded area, and excellent agreement was found (12).

For $H/r \leq 0.5$, the single degree-of-freedom system shown in Figure 7d may be used to represent the combined effects of amplification and soil-structure interaction. The spring constant k_s and mass M_s are determined, using the same rules as in (2) and (3) above, from a prism with an area just equal to the loaded area. Thus:

$$k_s = 4Gr^2/H \tag{10}$$

$$M_s = 16\rho Hr^2/\pi^2 \tag{11}$$

The total mass is $M + M_s$. This equivalent system gives the fundamental frequency of the system within 5%.

5. EQUIVALENT DAMPING RATIOS

For $H/r \leq 0.5$, since this situation is essentially one dimensional, a damping ratio corresponding to the internal damping of the soil should be used (5,11). This value of damping ratio must be adjusted to correspond to the level of strain occurring in the soil during the earthquake.

For $0.5 \leq H/r \leq 4$, the following tentative procedure is suggested, pending the results of further research. Assign to the spring k_s a damping ratio corresponding to the internal damping of the soil. Assign to the spring k_f a damping ratio equal to the internal damping plus the radiation damping (4,5) for the half-space with the same mass ratio b. An average overall damping ratio for the 2 degree-of-freedom system is then obtained by weighting the damping ratios in proportion to the energy stored in the respective springs.

6. FINAL COMMENTS

This paper has shown how the results from the theory for a dynamically loaded elastic stratum may be used to guide the choice of spring constants and masses for an equivalent lumped system. This use of the elastic theory for a stratum parallels the use made in earlier publications of the elastic theory for a half-space.

The next logical steps are to carry out the same type of study for the case of rocking, and to use results from two-dimensional visco-elastic theory to guide the choice of equivalent damping ratios.

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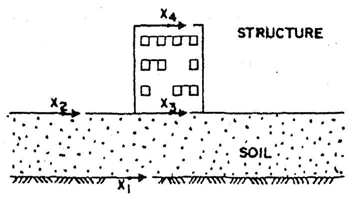


FIGURE 1

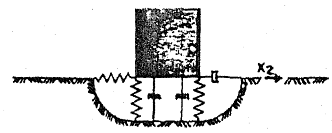


FIGURE 2

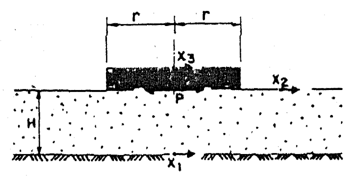


FIGURE 3

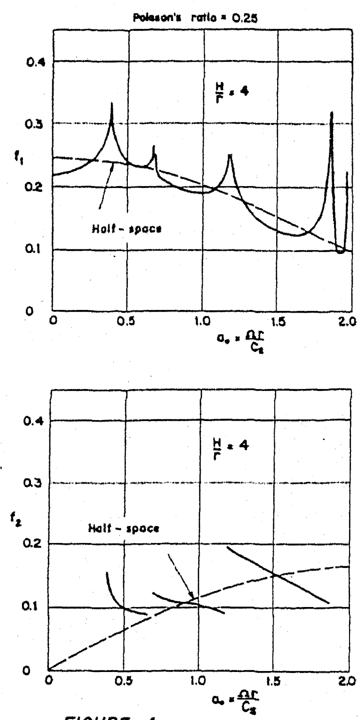
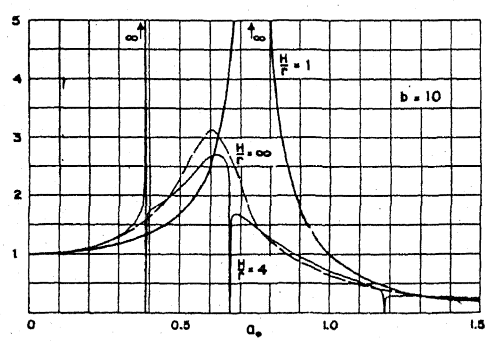


FIGURE 4



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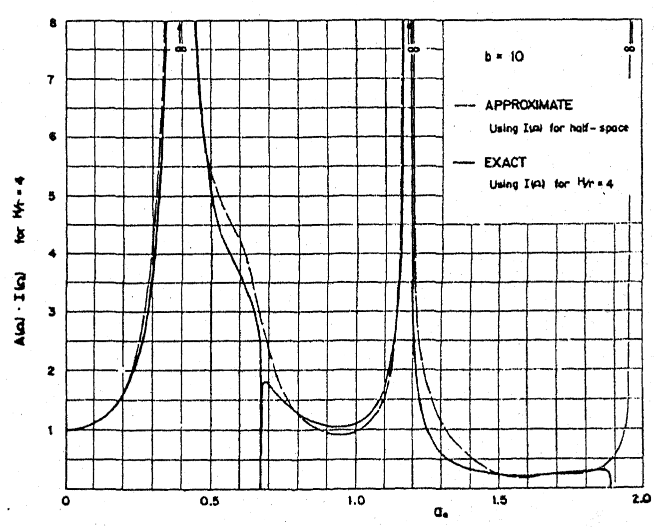

FIGURE 6

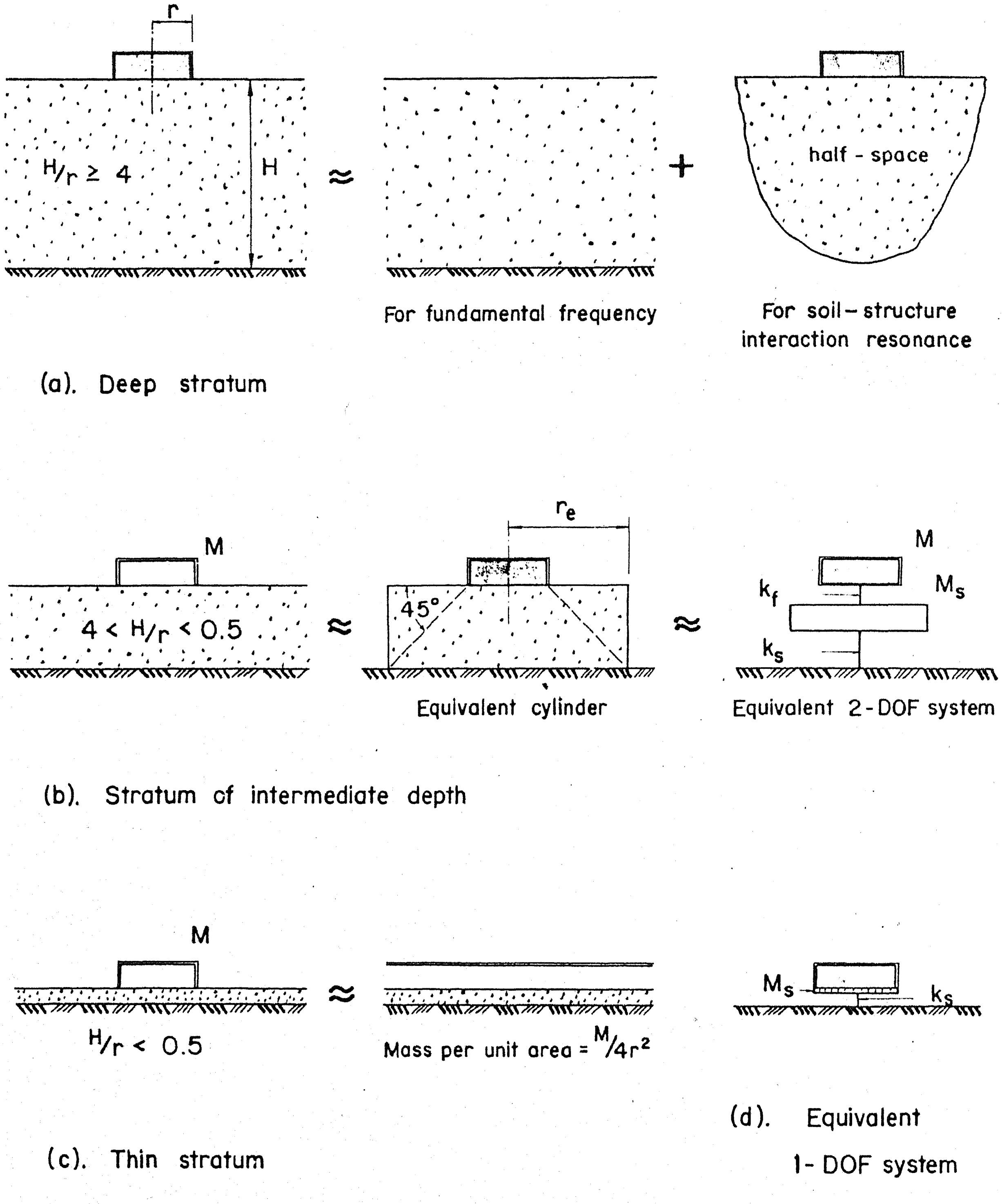
r
H/r ≥ 4
H
≈
+
half - space
For fundamental frequency
For soil – structure interaction resonance
(a). Deep stratum
M
4 < H/r < 0.5
≈
r_e
45°
Equivalent cylinder
≈
M
k_f
M_s
k_s
Equivalent 2-DOF system
(b). Stratum of intermediate depth
M
≈
H/r < 0.5
Mass per unit area = M/4r²
M_s
k_s
(c). Thin stratum
(d). Equivalent 1-DOF system

FIGURE 7