ELASTIC AND INELASTIC RESPONSE OF FRAMED STRUCTURES DURING EARTHQUAKES

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SYNOPSIS

The ground motion during an earthquake is represented by three components. For framed structures, it is important to calculate the response to the three components acting simultaneously. For elastic structures such calculations are needed to estimate the true margin of safety against yielding. For inelastic structures, it is necessary to do so because the inelastic behavior at a section depends on interaction between forces acting at the section during yielding. Examples are presented to illustrate the influence of inelastic interaction on the energy input, maximum velocity, maximum displacement and drift. The significance of these results for the design of structures is discussed.

GLOSSARY

a_{y_1}, a_{y_2}	=	yield acceleration of the frame;
D	=	damping matrix;
h	=	height of the columns;
i	=	subscript;
$K_{\alpha\alpha}$, $K_{\alpha\beta}$	=	stiffness matrices;
k_1, k_2	=	stiffness coefficients;
M_1, M_2	=	bending moments in the columns;
M_{y_1}, M_{y_2}	=	yield moments in the columns;
m	=	mass on the frame;
$\overline{P}(\overline{x})$	=	restoring force function;
p_1, p_2	=	nondimensional lateral force;
Q_1, Q_2	=	lateral force;

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Q_{y_1}, Q_{y_2} = lateral yield force;

q_1, q_2 = lateral displacement;

q_{y_1}, q_{y_2} = lateral yield displacement;

T = duration of the response;

T_1, T_2 = period of the frame;

u_1, u_2 = nondimensional lateral displacement;

$\dot{W}$ = rate of internal work;

W_1^p, W_2^p = rate of plastic work;

$\bar{x}$ = displacement vector;

$\ddot{z}_1, \ddot{z}_2$ = base acceleration;

$\hat{\ddot{z}}_1, \hat{\ddot{z}}_2$ = r.m.s. of base acceleration;

α, β = subscripts;

γ_1, γ_2 = acceleration ratios;

ζ = ratio of natural frequencies;

ξ_1, ξ_2 = fraction of critical damping;

τ = transformed time;

Φ = scalar function defining yield surface; and

$\omega, \omega_1, \omega_2$ = natural frequency.

INTRODUCTION

The three components of ground acceleration recorded by an accelerograph during a strong-motion earthquake represent the history of ground motion at a point and provide information regarding the base motion of a structure resting on the ground. To obtain the dynamic response of a structure to an earthquake, it is necessary, therefore, to analyze its behavior under the simultaneous action of the three components. As a first step, consider the response of a structure under the action of the horizontal components only. If the structure is such that its displacement along the two components of ground motion is resisted independently by two different sets of structural elements such as mutually perpendicular shear walls, it is clear that its response can be obtained independently for each component. If, however, the motion of the structure is resisted by the same structural elements, such as columns in a typical building-frame, the behavior of such elements must be examined under the simultaneous action of the two components. For this purpose, it is, in general, necessary to treat a framed structure as a space-frame and to know the force-displacement relations for the individual elements under the most general loading conditions. For elastic behavior, the force-displacement relations for a typical one-dimensional element of framed structures are well known. For inelastic behavior, such relations depend upon the interaction between forces and moments acting at a section during yielding, and can be derived within the framework of the known rules of plasticity (1). Knowing the force-displacement relations for each element, the equations of motion of a framed structure can be derived and the response obtained by the integration of these equations.

In this paper, the elastic and inelastic response of framed structures is examined under the simultaneous action of the three components of ground acceleration. It is shown that the elastic and inelastic behavior at a section must be defined in relation to all the forces acting at the section. For elastic structures such an approach is needed to estimate the true margin of safety against yielding. For inelastic structures, it is necessary to do so because inelastic behavior at a section depends on interaction between forces acting at the section during yielding. A simple space-frame is used to illustrate the effects of such interactions on the response. The significance of these results for the design of structures is discussed.

EQUATIONS OF MOTION

A framed structure consists of an assemblage of one-dimensional elements interconnected at their ends. If the mass of the structure is assumed to be lumped at the nodes, the equations of motion can be written in the form

$$M\ddot{\bar{x}} + D\dot{\bar{x}} + \bar{P}(\bar{x}) = \bar{z}(t) \qquad (1)$$

in which M = the mass matrix, D = the damping matrix, $\bar{x}$ = the displacement vector, $\bar{P}(\bar{x})$ = the restoring force vector function, $\bar{z}(t)$ = the forcing function and $\dot{\bar{x}} = d\bar{x}/dt$.

The restoring force function $\bar{P}(\bar{x})$ represents the aggregate stiffness of the structure and is obtained by summing the stiffness contributions from the individual elements. Hence, if the force-displacement relations for each element and the geometry of the structure are known at any time, the restoring force function $\bar{P}(\bar{x})$ can be computed and the response obtained by the integration of Eq. 1.

BEHAVIOR AT A SECTION

To derive the force-displacement relations for an element made of elasto-plastic material, it is necessary to define the elastic and plastic behaviors at a section. Consider a section α of the typical element of a framed structure shown in Fig. 1. In general, it will be acted upon by an axial force, a torsional moment, two bending moments and two shear forces. Using the definition of the force and displacement in the generic sense, there exists thus a set of six forces and displacements at each section of the element. In any specific problem, the number of forces acting at a section may be less than six or one may choose to ignore the effect of some of these forces on the response. Consider, therefore, a n-dimensional ($n \leq 6$) force-space generated by such a force system. Let $\overline{Q}_\alpha$ and $\overline{q}_\alpha$ denote the force and displacement vector at the section α such that the internal work is of the form

$$W_\alpha = \langle \overline{Q}_\alpha, \overline{q}_\alpha \rangle \tag{2}$$

Then $\overline{Q}_\alpha$ and $\overline{q}_\alpha$ are the generalized force and displacement vectors respectively.

Assuming an elastic-perfectly-plastic relationship between stress and strains and neglecting work-hardening, yield behavior at a section is characterized by two surfaces in the force-space generated by the generalized forces:

1. Initial yield surface (Y'), which is defined as the locus of all points in the force-space such that yielding is imminent at one or more points of the section. Within Y', the relation between forces and displacements is linearly elastic (no geometric nonlinearities).

2. Limit yield surface (Y), which is defined as the locus of all points in the force-space such that yielding is imminent at all points of the section except possibly along a line.

The limit yield surface, Y, encloses the origin and the initial yield surface Y'. On Y the section is fully plastic and in the space between Y and Y', it is partly-elastic and partly-plastic. For a two-dimensional force-space, the initial and the limit yield surfaces are shown in Fig. 2. With reference to the behavior at a section described above, elastic and inelastic response of a framed structure can now be defined. The response is elastic if the force vector at every section of the structure lies within the initial yield surface during the entire response. If at any time the force vector at any section lies on or outside the initial yield surface, the response becomes inelastic. It is important to note that in order to define elastic and inelastic response of a structure it is necessary to know the complete time history of all the forces acting at each section of the structure.

ELASTIC RESPONSE

If the response of a structure is elastic, the force-displacement relations for each element can be written in the form

$$\bar{Q}_\alpha = K_{\alpha\alpha}\bar{q}_\alpha + K_{\alpha\beta}\bar{q}_\beta \tag{3}$$

in which $K_{\alpha\alpha}$, $K_{\alpha\beta}$ = the stiffness matrices of the element α-β.

The restoring force function, $\bar{P}(\bar{x})$ in Eq. 1, is obtained by combining the force-displacement relations of all the elements, and if the equations of motion include all possible degrees of freedom, the response of a structure can be obtained under the simultaneous action of three components of ground acceleration. If the configuration of the structure is such that the equations of motion are uncoupled along one or more directions, the response along these directions can be obtained independently. Whether the response of a structure is obtained in parts, or as a whole, it is clear that in order to design a structure with a prescribed margin of safety against yielding, a complete time history of the forces acting at each section must be known. Similar considerations apply to other design parameters such as maximum displacement. If the response of a structure is obtained by integration of the equations of motion for the three components of ground acceleration, such computations can be made without any difficulty.

Response Spectrum Curves - For the elastic design of structures, the response spectrum curves have been used extensively. Since these curves represent the maximum response of a simple oscillator to a single component of an earthquake, it is clear that all the details of the response, except the maximum values, are lost in the process of obtaining these curves. Also, a superposition of spectral values to obtain the total maximum value of a response parameter will, in general, overestimate the true value of the parameter. For the design of simple structures such as chimneys, water tanks and simple frames, it is possible, however, to modify the spectrum curves so that on superposition the true value of a response parameter is obtained (2). Such curves can be used directly for the design.

INELASTIC RESPONSE

It was shown above that a section is elastic if the force vector at the section lies inside the initial yield surface. If the loads are increased, the yielding begins and grows, the section becoming partly-elastic and partly-plastic and finally becomes fully plastic. The force vector now lies on the limit yield surface. To derive the force-displacement relations for the inelastic behavior of an element, it is necessary to disregard the transition from elastic to fully-plastic state (Y' to Y) and to assume that the section remains elastic up to Y and fully plastic on Y (hereafter called the yield surface). Under this simplifying assumption the yield behavior at a section, in terms of forces and displacements, becomes analogous to yield behavior at a point in a continuum, in terms of stresses and strains. The incremental force-displacement relations for such behavior can be derived within the framework of the known rules of plasticity and are given in Appendix I.

It is shown in Appendix I that for inelastic behavior, each element of a framed structure has four possible states of yielding with force-displacement relations and criteria for transition from one state to another defined by Eqs. AI.2 to AI.10. Since the state of yield of the elements changes with the response, it is necessary to use a step-by-step method for the integration of the equations of motion and to assume that the state of the elements does not change during an integration step. Starting with a known state of the elements, the inelastic response of framed structures can therefore be obtained by following the history of yielding of individual elements and reassembling the restoring force function, $\overline{P}(\overline{x})$ (Eq. 1), at the end of each integration step using the appropriate force-displacement relationships. The integration can be carried out by one of the several available methods of step-by-step integration. (3, 4, 5)

A degenerate case of the force-displacement relations represented by Eqs. AI.2 through AI.10 is obtained by assuming that yield behavior at a section is governed by a single force (1). These relations represent the one-dimensional elastic-perfectly-plastic force-displacement relationship used by several investigators (6, 7, 8, 9) for inelastic response of structures to earthquake type excitation. The investigations based on such a force-displacement relationship, therefore, neglect the inelastic interaction between forces during yielding. The effects of such interaction on the dynamic response of framed structures are discussed below with reference to a simple structure.

AN EXAMPLE

Consider the single-story space frame shown in Fig. 3. It consists of a rigid mass supported on four uniform and identical columns rigidly clamped at the top and bottom. The principal axes of the columns lie in the directions 1-1 and 2-2. The weight of the columns is assumed to be small compared to the supported weight mg and is neglected. Damping is assumed to be interfloor and viscous. The base is mounted so as to move freely in the horizontal plane. It is further assumed that the shear and mass-centers of the frame coincide and the influences of axial and shear forces on yielding are neglected. The base of the frame is subjected to a horizontal base motion $\overline{z}(t)$ and the effects of gravity and vertical component of ground motion during an earthquake are neglected. Under these simplifying assumptions, yielding occurs simultaneously at the top and bottom sections of all the four columns and the interacting forces at these sections are the bending moments M_1 and M_2 in the directions 1-1 and 2-2. The behavior of the frame is elastic when the end sections are elastic and the frame becomes fully plastic when yielding occurs at the end sections. Thus in this simple case, the states in which one end of the columns is elastic and the other plastic, do not exist. This leads to considerable simplification in the equations of motion of the frame.

Let the lateral restoring forces in the directions 1-1 and 2-2 be chosen as the generalized forces, and denoted by Q_1 and Q_2. The generalized displacements, q_1, and q_2, are then the lateral displacements in the directions 1-1 and 2-2. The equation of the yield surface can be expressed as

$$\Phi(Q_1, Q_2) = 1 \tag{4}$$

Using the force-displacement relations in Appendix I, the equations of motion of the frame can be derived (10). These are given below in the dimensionless form.

$$\ddot{u}_1 + 2\xi_1\dot{u}_1 + p_1 = -\gamma_1 g_1(\tau/\omega_1) \tag{5a}$$

$$\zeta^2\ddot{u}_2 + 2\xi_2\zeta\dot{u}_2 + p_2 = -\gamma_2 g_2(\tau/\omega_2\zeta) \tag{5b}$$

with

$$\begin{Bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}$$

$$\text{if } \Phi(p_1, p_2) < 1; \text{ or if } \Phi(p_1, p_2) = 1 \text{ and } \dot{W}^P < 0 \tag{5c}$$

and

$$\begin{Bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{Bmatrix} = \frac{1}{k_1\left(\frac{p_1}{Q_{y_1}}\right)^2 + k_2\left(\frac{p_2}{Q_{y_2}}\right)^2} \begin{bmatrix} k_2\left(\frac{p_2}{Q_{y_2}}\right)^2 & -\frac{k_2 p_1 p_1}{Q_{y_1}Q_{y_2}}\frac{q_{y_2}}{q_{y_1}} \\ -\frac{k_1 p_1 p_2}{Q_{y_1}Q_{y_2}}\frac{q_{y_1}}{q_{y_2}} & k_1\left(\frac{p_1}{Q_{y_1}}\right)^2 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}$$

$$\text{if } \Phi(p_1, p_2) = 1 \text{ and } \dot{W}^P \geq 0 \tag{5d}$$

where

$$\left.\begin{aligned} &\ddot{z}_1(t) = \hat{\ddot{z}}g_1(t) \quad , \quad \ddot{z}_2(t) = \hat{\ddot{z}}g_2(t) \\ &\hat{\ddot{z}} = (\hat{\ddot{z}}_1^2 + \hat{\ddot{z}}_2^2)^{\frac{1}{2}} \\ &\omega_i = \sqrt{\frac{k_i}{m}} \quad ; \quad u_i = \frac{q_i}{q_{y_i}} \\ &p_i = \frac{Q_i}{Qy_i} \end{aligned}\right\}$$

$$a_{y_i} = \frac{Q_{y_i}}{m} \quad ; \quad \gamma_i = \frac{\hat{\ddot{z}}}{a_{y_i}}$$
$$\tau = \omega_1 t \quad ; \quad \zeta = \frac{\omega_1}{\omega_2} \qquad (6)$$

in which $\hat{\ddot{z}}_1$, $\hat{\ddot{z}}_2$ = r.m.s. values of the two components of base acceleration. q_{y_i} = yield displacements, Q_{y_i} = lateral yield force and i = 1, 2.

To understand the effects of inelastic interaction, it is necessary to obtain the response of the frame when its behavior is elastic and also when the effects of interaction are disregarded and the frame is assumed to yield independently along the two principal directions. These responses are called elastic, denoted by E, and elasto-plastic, denoted by EP, respectively. The response of the frame incorporating the effects of inelastic interaction is represented by Eqs. 5 and denoted by EPI. The equations of motion for elastic behavior are represented by Eqs. 5a, 5b and 5c at all times. The equations of motion for elasto-plastic behavior are given by Eqs. 5a and 5b, with p_i, i = 1, 2, defined by

$$\dot{p}_i = \dot{u}_i \ \text{ if } \ |p_i| < 1 \ ; \ \text{or if } \ |p_i| = 1 \text{ and } \dot{W}_i^p < 0 \qquad (7a)$$

$$= 0 \ \text{ if } \ |p_i| = 1 \ \text{ and } \ \dot{W}_i^p \geq 0 \qquad (7b)$$

From Eqs. 5 and 7 it is seen that for elastic and elasto-plastic behavior without interaction, the equations of motion can be reduced to a system of two uncoupled second order differential equations which can be integrated independently. The response of the frame, in these cases, is, therefore, independent in the directions 1-1 and 2-2. For elasto-plastic behavior with interaction, the equations of motion are coupled and must be integrated simultaneously. In this case, therefore, the response in the direction 1-1 depends on the response in the direction 2-2 and vice versa. This interdependence characterizes a basic difference between elasto-plastic response with and without interaction.

To illustrate the effects of interaction, Eqs. 5 and 7 were integrated numerically using third order Runge-Kutta scheme of integration for $T_1 = 1.0$, $T_2 = 0.75$, $\gamma_1 = 0.5$, $\gamma_2 = 0.3$ and $\xi_1 = \xi_2 = \xi = 0.02$, using the horizontal components of Taft 1952 earthquake as the base excitation. Furthermore, it was assumed that the yield surface is given by

$$p_1^2 + p_2^2 = 1 \qquad (8)$$

Figures 4 and 5 show the displacement-time response of the frame for elastic and elasto-plastic behavior with and without interaction. The plastic drift of the frame during yielding is also indicated on these figures. Comparison of these figures shows the following features of the response:

1. The response for both elasto-plastic behavior and elasto-plastic behavior with interaction is considerably smaller and much more

uniform than the response for elastic behavior. This shows the effect of yielding in reducing the response of structures. Careful comparison of the responses with and without interaction shows that interaction leads to smaller and more uniform response.

2. The oscillatory part of the response for elasto-plastic and interactive elasto-plastic behavior is quite similar but the drift pattern is very different. For response with interaction, yielding occurs at lower force levels and therefore increments of plastic drift occur many more times than if there were no interaction. Hence the tendency to drift is expected to be greater.

Figure 6 shows the response of the frame in the force-space for the circular yield surface defined by Eq. 8. On the yield curve the frame is plastic and inside it is elastic. It is seen that the force-vector during yielding follows the assumed yield surface very closely. This is an indication of the accuracy of numerical integration.

<u>Response of the frame to the Taft Earthquake and an ensemble of pseudo-earthquakes.</u> -The response of the frame to the Taft earthquake and to an ensemble of four pairs of pseudo-earthquakes (10) is shown in Figs. 7, 8 and 9. These figures show the variation of energy input, maximum velocity and maximum radial displacement against the natural period of the frame. Since the equations of motion contain a large number of parameters, the presentation is restricted to the special case of symmetrical frames for the following values of the parameter, $T_1 = T_2 = 0.25$, 0.5, 0.75, 1.0 1.5, 2.0 and 2.5; $\xi_1 = \xi_2 = 0.02$; $\gamma_1 = \gamma_2 = 0.2$, 0.5 and 1.0 (for Taft Earthquake); $\gamma_1 = \gamma_2 = 0.6$ (for pseudo-earthquakes). The response is presented for one value of damping only, as it was found that moderate change in damping causes no significant change in the effects of interaction. The three values of γ correspond to the yield accelerations 0.247g, 0.099g and 0.049g for the Taft earthquake record of 30 sec. duration. These are expected to cover the range of values likely to occur in real structures. The pseudo-earthquakes were multiplied by the factor 1.61, the ratio of the average r.m.s. of Taft earthquake to the average r.m.s. of pseudo-earthquakes, in order to match the Taft earthquake. The average and the range of values of various response parameters were computed and are shown in Figs. 7b, 8b and 9b. The effect of interaction on each of the response parameters is summarized below:

1. The total energy input denotes the energy input to the structure by the two components of the earthquake and is the sum of the energy dissipated by yielding, damping and residual energy. The curves in Figs. 7 indicate that the interaction has the effect of reducing the energy input to the frame by as much as 20 percent.

2. The curves labelled $25(\gamma = 0.2)$ and $6.25(\gamma = 0.2)$, in Fig. 7a, were obtained by multiplying the curve for $\gamma = 0.2$ by 25 and 6.25 respectively. These curves indicate that energy input to a structure decreases as its yield strength decreases.

3. The maximum velocity (Figs. 8) is reduced significantly by yielding and interaction reduces it still more.

4. The ratio of the maximum radial displacement to the yield displacement is analogous to the ductility ratio. For bending about two axes it is a better measure of the total plastic strain occurring at a section than the ductility ratios along the two directions alone or taken together. It is seen from Fig. 9 that interaction does not have a consistent effect on this parameter.

DESIGN OF STRUCTURES

The significance of the concepts presented in this paper to the aseismic design of structures can be summarized as follows:

1. Elastic design. -The margin of safety against yielding must be obtained in relation to the initial yield surface under the simultaneous action of the three components of ground motion. If the design is based on the superposition of the spectral values for the three components it will be, in general, conservative. If this is not done the structure may actually yield under the combined action of the three components, although the response to individual components is elastic.

2. Simplified inelastic design. -Simplified methods of inelastic design (12, 13) depend upon estimates of energy input to the structure during an earthquake and the capacity of the structure to store and to dissipate energy by damping and hysteresis. The inelastic interaction affects all these aspects of the response. Most significantly it reduces the energy input to a structure.

3. Design parameters. -The ductility ratio for a single component of motion has been commonly used as a criterion for inelastic design. As pointed out earlier, it is better instead to use the ratio of the maximum radial displacement to yield displacement. Such a ratio is directly related to maximum flexural strain occurring at a section during an earthquake and is, therefore, meaningful in relation to structural damage. It is also a convenient parameter to measure in an experimental program designed to correlate structural damge with permissible design-parameter values. It suffers, however, with the following disadvantages: i) Due to the random nature of plastic-drift during yielding, estimates of maximum displacement involve a large element of uncertainty. ii) The maximum displacement is a transient phenomenon and as such its correlation with structural failure may not be good, and iii) The maximum displacement has no correlation with the number of times yielding occurs at a section which is important in view of the deterioration in strength with repeated alternating loading into the plastic range.

The ratio of the total energy dissipated by yielding during an earthquake to the elastic energy capacity of the structure (hereafter called the plastic energy ratio) is another parameter which can be used for purposes of design, particularly when fatigue failure is a consideration. Since plastic energy ratio represents the cumulative yield behavior of the structure during an earthquake, it does not suffer from some of the disadvantages pointed out above for maximum displacement. The disadvantage with plastic energy ratio is the difficulty of measuring it in an experimental investigation. The plastic energy ratio can be used either as an alternative to maximum radial displacement ratio or as an additional parameter to govern the design.

Building frames. -In building frames columns provide the primary resistance to bilateral and vertical motion. In view of this the effect of inelastic interaction should be expected to be more in columns than in girders.

CONCLUSIONS

The dynamic response of framed structures to an earthquake must be obtained under the simultanecus action of the three components. For elastic behavior such an approach is needed to estimate the true margin of safety against yielding. For inelastic behavior it is necessary to do so because the yielding at a section depends upon the interaction between forces acting at the section. The effect of such interactions on the response of a structure can be significant.

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APPENDIX-I

FORCE-DISPLACEMENT RELATIONS FOR INELASTIC BEHAVIOR

If a typical one-dimensional element of a framed structure is assumed to be free of intermediate loading, the maximum values of the forces acting on the element occur at its end sections and yielding will be initiated at these sections as the loads are increased. Since a section is either elastic or fully-plastic, there are four possible states of the element:

1. Both ends α and β are elastic;
2. End α is yielding and β is elastic;
3. End α is elastic and β is yielding;
4. Both α and β are yielding.

Let the equations of the yield surfaces at the ends α and β be given by

$$\Phi_\alpha(\bar{Q}_\alpha) = 1 \quad \text{and} \quad \Phi_\beta(\bar{Q}_\beta) = 1 \qquad \text{(AI.1)}$$

The force-displacement relations and the necessary conditions for each of the four states are then given by (1):

State-1
Necessary conditions:

$$\Phi_\alpha(\bar{Q}_\alpha) < 1 \quad \text{or} \quad \Phi_\alpha(\bar{Q}_\alpha) = 1 \quad \text{and} \quad dW^P_\alpha < 0 \qquad \text{(AI, 2a)}$$

$$\Phi_\beta(\bar{Q}_\beta) < 1 \quad \text{or} \quad \Phi_\beta(\bar{Q}_\beta) = 1 \quad \text{and} \quad dW^P_\beta < 0 \qquad \text{(AI, 2b)}$$

Force-displacement relations:

$$\Delta\bar{Q}_\alpha = K_{\alpha\alpha}\Delta\bar{q}_\alpha + K_{\alpha\beta}\Delta\bar{q}_\beta \qquad \text{(AI, 3a)}$$

$$\Delta\bar{Q}_\beta = K_{\beta\alpha}\Delta\bar{q}_\alpha + K_{\beta\beta}\Delta\bar{q}_\beta \qquad \text{(AI, 3b)}$$

State-2
Necessary conditions:

$$\Phi_\alpha(\bar{Q}_\alpha) = 1 \quad \text{and} \quad dW^P_\alpha > 0 \qquad \text{(AI, 4a)}$$

$$\Phi_\beta(\bar{Q}_\beta) < 1 \quad \text{or} \quad \Phi_\beta(\bar{Q}_\beta) = 1 \quad \text{and} \quad dW^P_\beta < 0 \qquad \text{(AI, 4b)}$$

Force-displacement relations:

$$\Delta\bar{Q}_\alpha = K_{\alpha\alpha}\left[1 - \Omega_\alpha K_{\alpha\alpha}\right]\Delta\bar{q}_\alpha + \left[K_{\alpha\beta} - K_{\alpha\alpha}\Omega_\alpha K_{\alpha\beta}\right]\Delta\bar{q}_\beta \qquad \text{(AI, 5a)}$$

$$\text{and} \quad \Delta\bar{Q}_\beta = K_{\beta\alpha}\left[1 - \Omega_\alpha K_{\alpha\alpha}\right]\Delta\bar{q}_\alpha + \left[K_{\beta\beta} - K_{\beta\alpha}\Omega_\alpha K_{\alpha\beta}\right]\Delta\bar{q}_\beta \qquad \text{(AI, 5b)}$$

where 1 = identity matrix

$$\Omega_\alpha = \frac{\dfrac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\left(\dfrac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\right)^T}{\left\langle K_{\alpha\alpha}\dfrac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}, \dfrac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\right\rangle}$$

State-3
Necessary conditions:

$$\Phi_\alpha(\bar{Q}_\alpha) < 1 \quad \text{or} \quad \Phi_\alpha(\bar{Q}_\alpha) \quad \text{and} \quad dW^P_\alpha < 0 \qquad \text{(AI, 6a)}$$

$$\Phi_\beta(\bar{Q}_\beta) = 1 \quad \text{and} \quad dW^P_\beta > 0 \qquad \text{(AI, 6b)}$$

Force-displacement relations:

$$\Delta\bar{Q}_\alpha = \left[K_{\alpha\alpha} - K_{\alpha\beta}\Omega_\beta K_{\beta\alpha}\right]\Delta\bar{q}_\alpha + K_{\alpha\beta}\left[1 - \Omega_\beta K_{\beta\beta}\right]\Delta\bar{q}_\beta \qquad \text{(AI, 7a)}$$

$$\Delta\bar{Q}_\beta = \left[K_{\beta\alpha} - K_{\beta\beta}\Omega_\beta K_{\beta\alpha}\right]\Delta\bar{q}_\alpha + K_{\beta\beta}\left[1 - \Omega_\beta K_{\beta\beta}\right]\Delta\bar{q}_\beta \qquad \text{(AI, 7b)}$$

where

$$\Omega_\beta = \frac{\dfrac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\left(\dfrac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\right)^T}{\left\langle K_{\beta\beta}\dfrac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}, \dfrac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\right\rangle}$$

State-4
Necessary conditions:

$$\Phi_\alpha(\bar{Q}_\alpha) = 1 \quad \text{and} \quad dW^P_\alpha > 0 \qquad \text{(AI, 8a)}$$

$$\Phi_\beta(\bar{Q}_\beta) = 1 \quad \text{and} \quad dW^P_\beta > 0 \qquad \text{(AI, 8b)}$$

Force-displacement relations:

$$\begin{aligned}
\Delta\bar{Q}_\alpha = \Big[K_{\alpha\alpha} &- \Gamma_{\beta\beta}K_{\alpha\alpha}\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\left(\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\right)^T K_{\alpha\alpha} \\
&- \Gamma_{\beta\alpha}K_{\alpha\alpha}\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\left(\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\right)^T K_{\beta\alpha} - \Gamma_{\alpha\beta}K_{\alpha\beta}\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\left(\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\right)^T K_{\alpha\alpha} \\
&- \Gamma_{\alpha\alpha}K_{\alpha\beta}\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\left(\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\right)^T K_{\beta\alpha}\Big]\Delta\bar{q}_\alpha + \Big[K_{\alpha\beta} \\
&- \Gamma_{\alpha\beta}K_{\alpha\beta}\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\left(\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\right)^T K_{\alpha\beta} - \Gamma_{\alpha\alpha}K_{\alpha\beta}\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\left(\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\right)^T K_{\beta\beta} \\
&- \Gamma_{\beta\beta}K_{\alpha\alpha}\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\left(\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\right)^T K_{\alpha\beta} - \Gamma_{\beta\alpha}K_{\alpha\alpha}\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\left(\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\right)^T K_{\beta\beta}\Big]\Delta\bar{q}_\beta
\end{aligned} \qquad \text{(AI, 9a)}$$

$$\begin{aligned}
\Delta\bar{Q}_\beta = \Big[K_{\beta\alpha} &- \Gamma_{\beta\beta}K_{\beta\alpha}\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\left(\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\right)^T K_{\alpha\alpha} - \Gamma_{\beta\alpha}K_{\beta\alpha}\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\left(\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\right)^T K_{\beta\alpha} \\
&- \Gamma_{\alpha\beta}K_{\beta\beta}\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\left(\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\right)^T K_{\alpha\alpha} - \Gamma_{\alpha\alpha}K_{\beta\beta}\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\left(\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\right)^T K_{\beta\beta}\Big]\Delta\bar{q}_\alpha \\
+ \Big[K_{\beta\beta} &- \Gamma_{\beta\beta}K_{\beta\alpha}\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\left(\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\right)^T K_{\alpha\beta} - \Gamma_{\beta\alpha}K_{\beta\alpha}\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\left(\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\right)^T K_{\beta\beta} \\
&- \Gamma_{\alpha\beta}K_{\beta\beta}\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\left(\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\right)^T K_{\alpha\beta} - \Gamma_{\alpha\alpha}K_{\beta\beta}\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\left(\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\right)^T K_{\beta\beta}\Big]\Delta\bar{q}_\beta
\end{aligned} \qquad \text{(AI, 9b)}$$

where

$$\Gamma = \left\langle K_{\alpha\alpha}\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}, \frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\right\rangle \times \left\langle K_{\beta\beta}\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}, \frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\right\rangle - \left\langle K_{\alpha\beta}\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}, \frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\right\rangle \times \left\langle K_{\beta\alpha}\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}, \frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\right\rangle$$

$$\Gamma_{\alpha\alpha} = \left\langle K_{\alpha\alpha}\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}, \frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\right\rangle \Big/ \Gamma, \qquad \Gamma_{\alpha\beta} = -\left\langle K_{\beta\alpha}\frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}, \frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\right\rangle \Big/ \Gamma,$$

and

$$\Gamma_{\beta\alpha} = -\left\langle K_{\alpha\beta}\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}, \frac{\partial\Phi_\alpha}{\partial\bar{Q}_\alpha}\right\rangle \Big/ \Gamma, \qquad \Gamma_{\beta\beta} = \left\langle K_{\beta\beta}\frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}, \frac{\partial\Phi_\beta}{\partial\bar{Q}_\beta}\right\rangle \Big/ \Gamma$$

The force-displacement relationship derived above for each of the four states of the element can be expressed in the form

$$\begin{Bmatrix} \Delta\bar{Q}_\alpha \\ \Delta\bar{Q}_\beta \end{Bmatrix} = \begin{bmatrix} S_{\alpha\alpha} & S_{\alpha\beta} \\ S_{\beta\alpha} & S_{\beta\beta} \end{bmatrix} \begin{Bmatrix} \Delta\bar{q}_\alpha \\ \Delta\bar{q}_\beta \end{Bmatrix} \qquad \text{(AI, 10)}$$

Dividing both sides by Δt and taking limit as $\Delta t \to 0$, gives

$$\begin{Bmatrix} \dot{\bar{Q}}_\alpha \\ \dot{\bar{Q}}_\beta \end{Bmatrix} = \begin{bmatrix} S_{\alpha\alpha} & S_{\alpha\beta} \\ S_{\beta\alpha} & S_{\beta\beta} \end{bmatrix} \begin{Bmatrix} \dot{\bar{q}}_\alpha \\ \dot{\bar{q}}_\beta \end{Bmatrix} \qquad \text{(AI, 11)}$$

in which $\dfrac{d\bar{Q}_\alpha}{dt} = \dot{\bar{Q}}_\alpha$ and $S_{\alpha\alpha}$, $S_{\alpha\beta}$, $S_{\beta\alpha}$ and $S_{\beta\beta}$ are nxn matrices defined by Eqs. AI. 3, AI. 5, AI. 7 and AI. 9 *for each state of the element.*

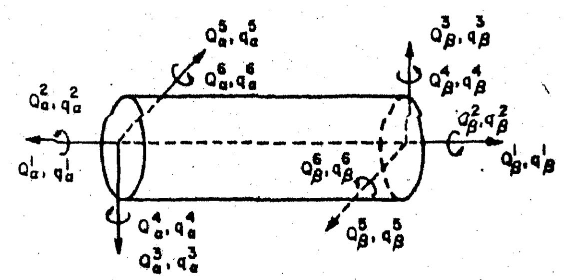

FIG. 1 ONE-DIMENSIONAL FRAME ELEMENT

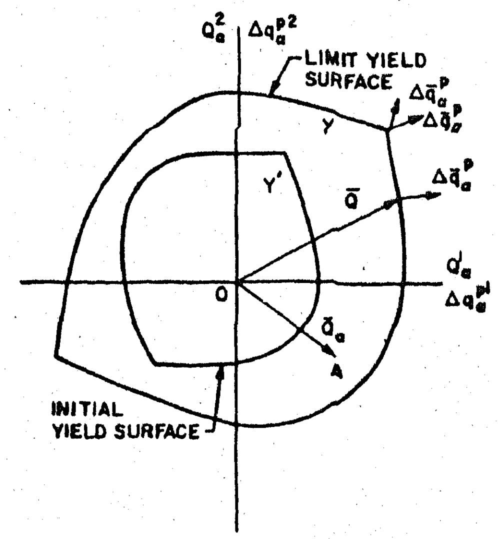

FIG. 2 YIELD BEHAVIOR IN TWO-DIMENSIONAL FORCE-SPACE

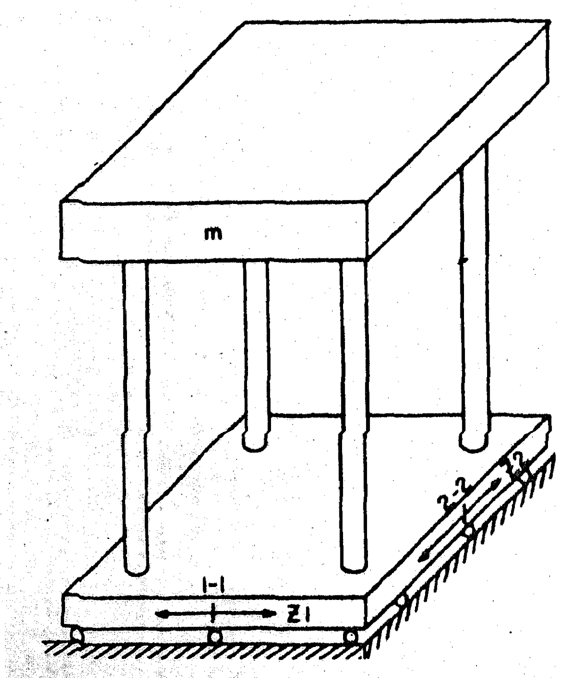

FIG. 3 A SIMPLE SPACE-FRAME

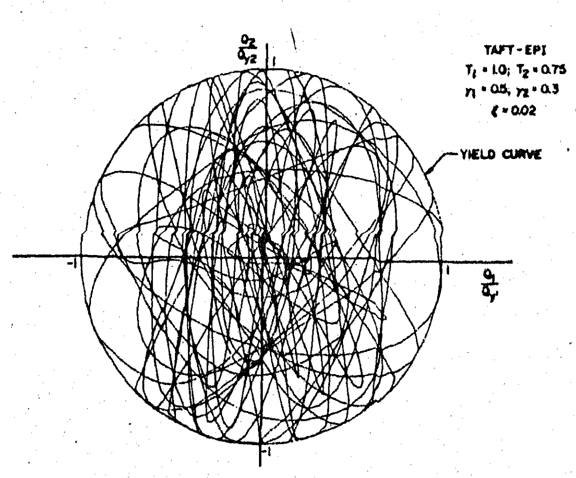

FIG. 6 RESPONSE IN THE FORCE-SPACE

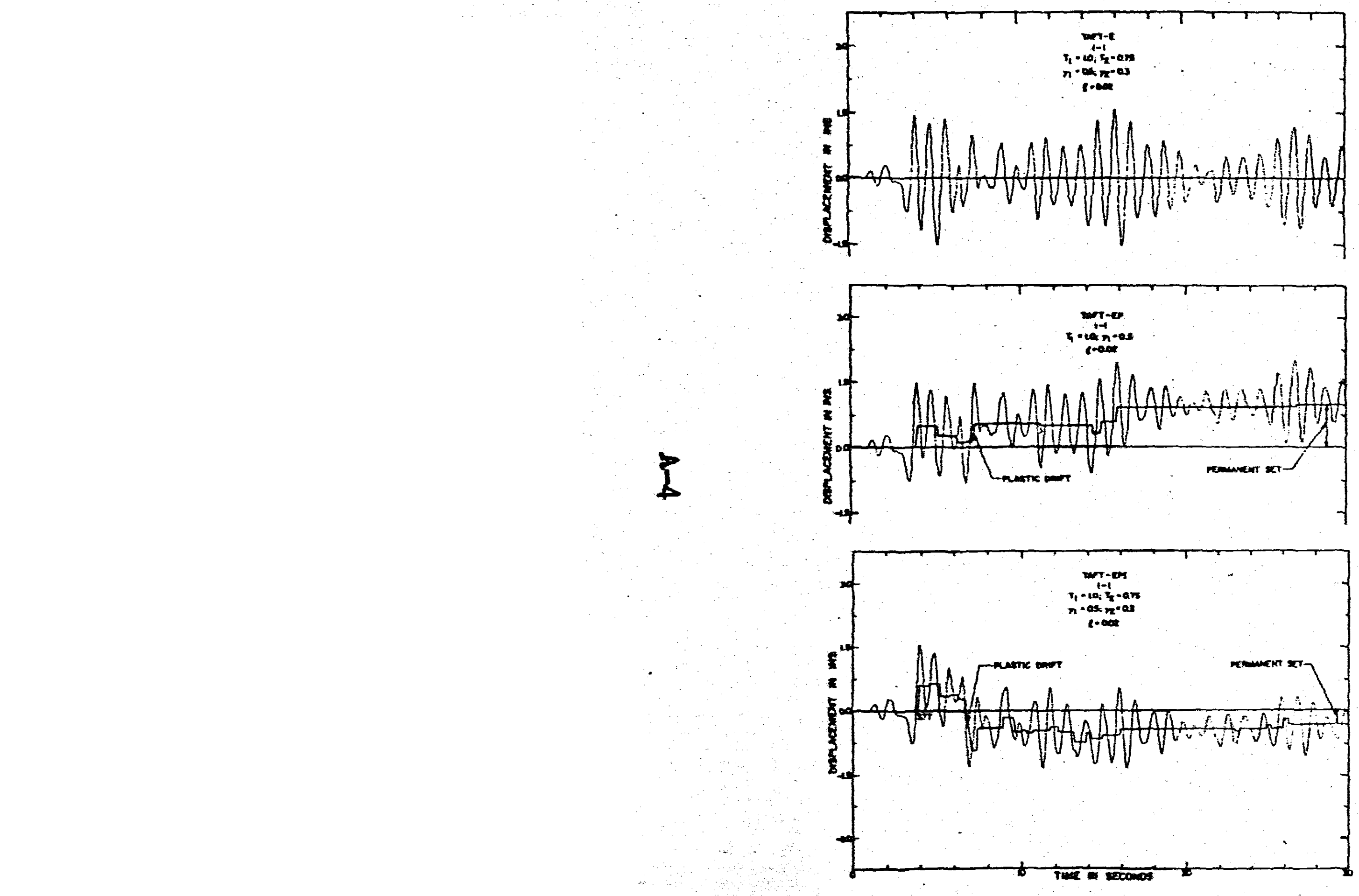

FIG. 4 DISPLACEMENT-TIME RESPONSE FOR ELASTIC (E), ELASTO-PLASTIC (EP) AND ELASTO-PLASTIC BEHAVIOR WITH INTERACTION (EPI)

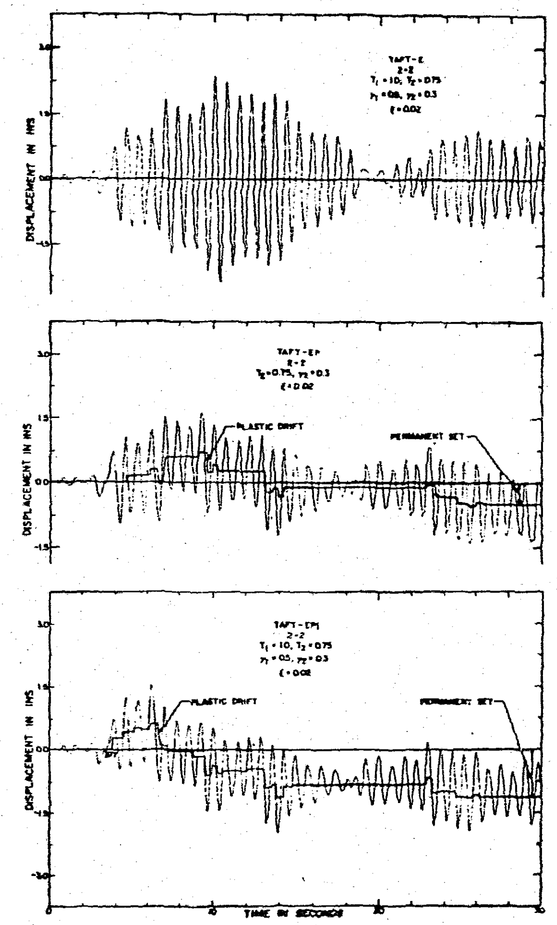

FIG. 5 DISPLACEMENT-TIME RESPONSE FOR ELASTIC (E), ELASTO-PLASTIC (EP) AND ELASTO-PLASTIC BEHAVIOR WITH INTERACTION (EPI)

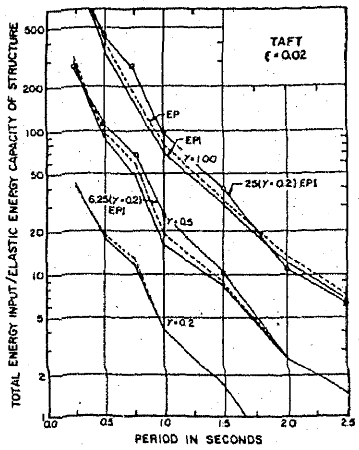

FIG. 7a ENERGY INPUT RATIO FOR TAFT EARTHQUAKE

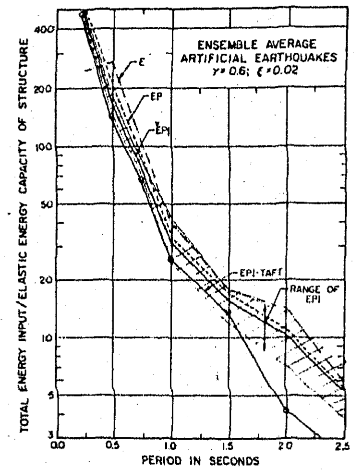

FIG. 7b ENERGY INPUT RATIO FOR AN ENSEMBLE OF ARTIFICIAL EARTHQUAKES

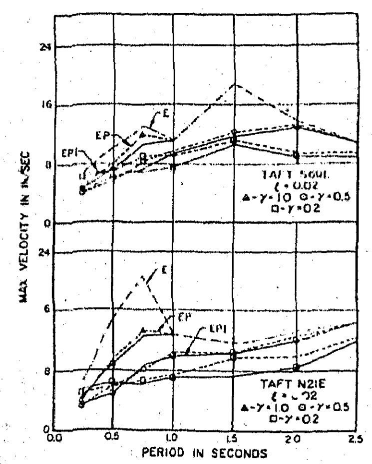

FIG. 8a VELOCITY SPECTRA FOR TAFT EARTHQUAKE

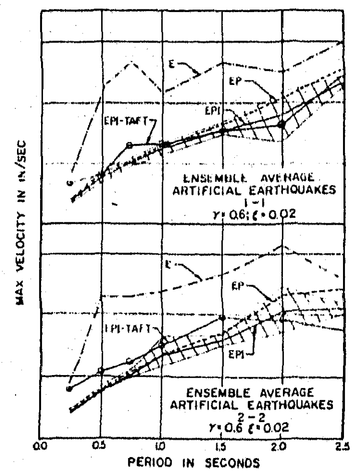

FIG. 8b VELOCITY SPECTRA FOR AN ENSEMBLE OF ARTIFICIAL EARTHQUAKES

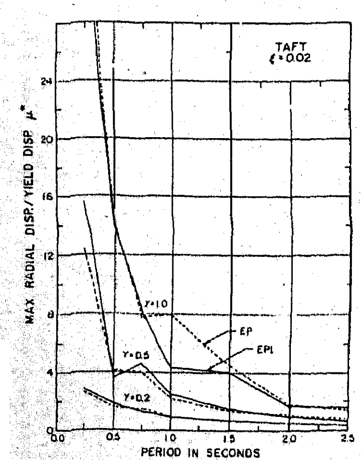

FIG. 9a RADIAL DISPLACEMENT RATIO FOR TAFT EARTHQUAKE

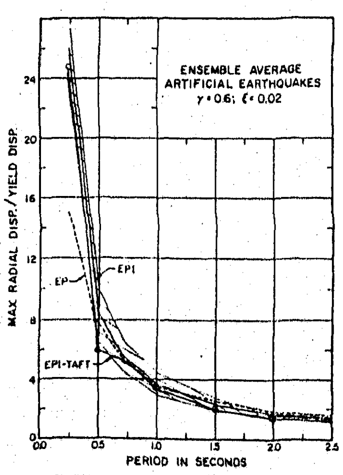

FIG. 9b RADIAL DISPLACEMENT RATIO FOR AN ENSEMBLE OF ARTIFICIAL EARTHQUAKES