ESTIMATING NATURAL FREQUENCIES AND MODES OF ARCH DAMS WITH THE THEORY OF PLATES ON ELASTIC FOUNDATION

by

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SYNOPSIS

The complex partial differential equation of the free vibration of thin arch dams is reduced to that of a plate on elastic foundation. By an extension of the variational method, the natural frequencies pertinent to the first (symmetrical) and to the second (antisymmetrical) modes are obtained in closed forms. These modes are represented as the products of the deflection functions of equivalent beams on elastic foundation. A numerical example based on the dimensions of an actual arch dam, for which measured natural frequencies and pertinent modes of free vibration are available, illustrates the feasibility and accuracy of the method introduced.

INTRODUCTION

In the design of thin arch dams located in seismicly active zones, it is mandatory to include the effects of a strong motion earthquake. The dynamic response of any structure to seismic excitations depends to a considerable degree on its structural-dynamical characteristics, which can be best given in the form of natural frequencies. Furthermore, a number of currently used approaches to aseismic design of arch dams(1) involve the response spectra method, which also requires the determination of the natural frequencies. While the use of finite difference or finite element methods(2) are highly recommended for the final static and dynamic analyses, they are less suited for the first appraisal of the dynamical characteristics of various alternative designs because of the rather extensive computational effort required.(3) In addition, it is always recommended to ascertain the results of a computerized solution by means of independent checks. Consequently, the objective of this paper is to develop a relatively simple method of estimating the required natural frequencies pertinent to the first (symmetrical) and to the second (antisymmetrical) modes with reasonable accuracy. To achieve this purpose, the rigorous theory of the free vibration of thin arch dams has been considerably simplified. The simplifying assumptions used in this study have been introduced after a careful determination of their effects. Because of the space limitations, these comparative studies are not discussed here, but the obtainable accuracy is demonstrated by comparing the results of the proposed method with actual measurements.

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THE DIFFERENTIAL EQUATIONS OF FREE VIBRATION

Applying the rigorous bending theory of double curved thin shells, Ganev[4] derived the differential equation of equilibrium, which includes the effect of the varying curvature and thickness in both directions. Substituting the inertia forces for the lateral forces, the differential equation of free, undamped vibration of thin arch dams becomes

$$\nabla^2[D\nabla^2 w] - (1-\nu)\left[\frac{\partial^2 D}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 D}{\partial y^2}\cdot\frac{\partial^2 w}{\partial x^2} - 2\,\frac{\partial^2 D}{\partial x\partial y}\cdot\frac{\partial^2 w}{\partial x\partial y}\right] + k_x{}^2 D\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right) + k_y{}^2 D\left(\frac{\partial^2 w}{\partial y^2} + \nu\,\frac{\partial^2 w}{\partial x^2}\right) + (1-\nu)\left[k_x{}^2\,\frac{\partial}{\partial y}\left(D\,\frac{\partial w}{\partial y}\right) + k_y{}^2\,\frac{\partial}{\partial x}\left(D\,\frac{\partial w}{\partial x}\right)\right] + \left[(k_x{}^4 + k_y{}^4 + 2\nu k_x{}^2 k_y{}^2)D + k_1\right]w + \bar{m}\,\frac{\partial^2 w(x,y,t)}{\partial t^2} = 0, \qquad (1)$$

where ∇^2 is the two-dimensional Laplacian operator, $w(x,y,t)$ is the time-dependent lateral deflection, k_x and k_y the curvatures in the X and Y direction, respectively. Furthermore

$$\left.\begin{aligned} D(x,y) &= \frac{Ed^3(x,y)}{12(1-\nu^2)} && a)\\ k_1(y) &= \frac{Ed_o(y)}{r_x{}^2(y)\mu(y)} && b)\\ \mu(y) &= \frac{1}{0.6\alpha(y)}\{\sin\alpha(y) - \frac{\vartheta^2(y)}{6}\,2[\alpha(y)\cos\alpha(y) + (\alpha^2-2)\sin\alpha(y)]\} && c)\end{aligned}\right\} \qquad (2)$$

and E is the modulus of elasticity, ν is the Poisson's ratio, $\bar{m}$ represents the mass of the dam per unit area. The other notations are listed in Figure 1. The validity of Eq. (1) is limited to dams with ratio of wall thickness to radius less than 0.20.

The differential equation of motion of a plate of constant thickness lying on a "Winkler-type" elastic foundation can be written as[3]

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^2 w}{\partial x^2\partial y^2} + \frac{\partial^2 w}{\partial y^4}\right) + cw + \bar{m}\,\frac{\partial^2 w}{\partial t^2} = 0, \qquad (3)$$

where c is the "bedding" constant.

A comparison between Eqs. (1) and (3) shows that they are basically of the same structure. Thus, utilizing the analogy which exists between shallow shells and plates on elastic foundations[3], a considerably simplified form of the differential equation of free vibration is obtained:

$$D\nabla^2\nabla^2 w + [(k_x^4 + k_y^4 + 2\nu k_x^2 k_y^2)D + k_1]w + \bar{m}\frac{\partial^2 w}{\partial t^2} = 0. \qquad (4)$$

In Eq. (4) a constant, "equivalent" shell thickness is assumed.

Comparative studies indicated that, for arch dams of the type under consideration, the governing term in the expression within brackets in Eq. (4) is k_1 and the contribution of the other terms, for an approximate analysis, is of negligible order of magnitude. Therefore, Eq. (4) can be given in the form of

$$\boxed{D\nabla^2\nabla^2 w + k_1 w + \bar{m}\frac{\partial^2 w}{\partial t^2} = 0.} \qquad (5)$$

Although is is a greatly simplified equation it contains all essential terms and is capable (as it will be shown later) of describing the free vibration of thin arch dams effectively.

APPLICATION OF A VARIATIONAL METHOD

The irregular boundary of an arch dam makes the use of an approximate or numerical method in the solution of the simplified differential equation of free vibration(5) mandatory. The general variational method of Vlasov, developed initially for cylindrical and prismatic shells(5), has been extended in this paper to obtain the required natural frequencies.

In essence, the variational method applied to a static problem states that the sum of the work performed by the external and internal forces is zero. To express the total work, the differential equation of the equilibrium is used instead of actually determining the potential energy. In the case of free vibration the lateral force is the inertia force $p_z = \bar{m}\ddot{w}$.

Considering Eq. (5) and a small, virtual displacement, ∂w, the basic variational equation is

$$\iint_A (D\nabla^2\nabla^2 w + k_1 w + \bar{m}\ddot{w})\ \delta w\ dxdy = 0. \qquad (6)$$

In this equation the intergration is over the whole area of the dam surface.

Let us express the lateral displacements $w(x,y,t)$ as the product of three functions each of which depends on a single argument

$$w(x,y,t) = W\ \Phi(x)\ \Psi(y)\ \Theta(t), \qquad (7)$$

where W is the amplitude of the free vibration, $\Phi(x)$ and $\Psi(y)$ are orthogonal shape functions representing the pertinent modes in polynomial forms, while $\Theta(t)$ describes the time dependency of the displacements. Assuming a harmonic type motion, Eq. (7) can be written as

$$w(x,y,t) = W\,\Phi(x)\,\Psi(y)\,\sin\omega t, \tag{8}$$

where W is the circular frequency of the free vibration pertinent to the mode given by the shape functions.

Expressing the arbitrary variation of the displacement function by

$$\delta w = \delta W\,\Phi(x)\,\Psi(y)\,\sin\omega t \tag{9}$$

and substituting this expression along with Eq. (7) into Eq. (6), we obtain (after canceling the trigonometric term) the following equation:

$$\iint_A \{D[W(\Phi''''(x)\Psi(y) + 2\Phi''(x)\Psi^{\cdot\cdot}(y) + \Phi(x)\Psi^{\cdot\cdot\cdot\cdot}(y)] + (k_1 - \bar{m}\omega^2)W\Phi(x)\Psi(y)\}\delta W\Phi(x)\Psi(y)\,dxdy = 0, \tag{10}$$

where primes and dots indicate differentiation with respect to x and y, respectively.

Using the arithmetic mean values of D, $\bar{m}$ and k_1, Eq. (10) yields

$$\omega^2 = \frac{k_1}{\bar{m}} + \frac{[D\iint_A \Phi''''(x)\Psi(y) + 2\Phi''(x)\Psi^{\cdot\cdot}(y) + \Phi(x)\Psi^{\cdot\cdot\cdot\cdot}(y)]\Phi(x)\Psi(y)\,dxdy}{\bar{m}\iint_A \Phi^2(x)\Psi^2(y)\,dxdy} \tag{11}$$

which represents the general expression of the circular frequencies.

Provided that the shape functions are known, the solution of the required circular frequencies has been reduced to the evaluation of definite integrals, which can be obtained numerically. In many cases, however, it is possible to use an equivalent rectangular plate on elastic foundation, as shown in Figure 1, which permits the analytical determination of the definite integrals in Eq. (11).

SELECTION OF PROPER SHAPE FUNCTIONS

The obtainable accuracy of the above described variational approach depends to a considerable degree on the proper choice of the two orthogonal shape functions $\Phi(x)$ and $\Psi(y)$ used in Eq. (7). Vibration studies of arch dams(6)(7) indicate that the first and second modes have a considerable similarity with the elastic lines of finite and infinite beams on elastic foundation subjected to symmetrical and antisymmetrical lateral loadings representing the translational inertia forces. While the boundary conditions of the finite beams on elastic foundation rigorously satisfy the statical and geometrical boundary conditions of the actual dam, the infinitely long beams on elastic foundation can approximate the actual boundary conditions quite closely. The hydrodynamic mass participating in the vibration due to the inherent complexity of the problem are not considered in this approximate analysis.

a) First Mode

The two orthogonal shape functions pertinent to the first mode of free vibration are obtained using the deflection line of a finite beam (Fig. 2a) or an infinite beam on elastic foundation (Fig. 2b) and that of a cantilever beam on elastic foundation (Fig. 2c) subjected to uniformly distributed loads[8]. Although the thickness of the dam varies in both directions, comparative studies have indicated that the use of an "arithmetic mean thickness" yields a fairly close solution for the deflections. If the length of the arch dam at the crest is approximately $1.5\pi/\lambda$, the deflection curve of the uniformly loaded finite beam on elastic foundation with fixed boundary condition can be used for the shape function in the X direction. The "characteristic length", λ, of the arch dam is

$$\lambda = \sqrt[4]{\frac{k_1}{4EI}} \ . \tag{12}$$

Thus, we can write[II]

$$\Phi_1(x) = 1 - \frac{1}{R}[S_1 + S_2] \ , \tag{13}$$

where, after introduction of

$$\xi = \lambda(\frac{L_x}{2} + x) \quad \text{and} \quad \xi = \lambda(\frac{L_x}{2} - x) \tag{14}$$

the individual expressions are:

$$R = \sinh \lambda L_x + \sin \lambda L_x; \qquad S_1 = \sinh \xi \cos \xi' + \cosh \xi \sin \xi'$$

$$S_2 = \sinh \xi' \cos \xi + \cos \xi' \sin \xi. \tag{15}$$

The selected shape function in the Y direction is

$$\Psi_1(y) = 1 - \frac{1}{N}(\cosh \lambda L_y M_1 - \cos \lambda L_y M_2), \tag{16}$$

where

$$N = \cosh^2 \lambda L_y + \cos^2 \lambda L_y; \quad M_1 = \sinh \eta' \sin \eta + \cosh \eta' \cos \eta$$

$$M_2 = \sinh \eta \sin \eta' - \cosh \eta \cos \eta' \tag{17}$$

and

$$\eta = \lambda y \ ; \qquad \eta' = \lambda(L_y - y) \tag{18}$$

The above given shape function can also be used when a substitute structure, in the form of an equivalent rectangular plate on elastic foundation, is introduced provided that $L_x \leq 1.5\pi/\lambda$. In most of the cases, the developed elevation of an arch dam is of a trapezoidal shape which can be transformed into an equivalent rectangular plate of the following dimensions (Fig. 1):

$$L_x = \frac{2}{3}(2a_1 + a_2)\frac{a_2}{a_1 + a_2}$$

$$L_y = H - \frac{a_2(a_2 - a_1)}{6(a_1 + a_2)} \ ; \qquad \text{if } \frac{a_1}{a_2} > 0.5 \tag{19}$$

(II) $1/k_1^4$ cancels out from the second term of (11)

If the length of the arch dam at the crest is considerably larger than $1.5\pi/\lambda$, the recommended shape function in the X direction is

$$\Phi_1(x) = \frac{\lambda}{2} e^{-\lambda x} (\cos \lambda x + \sin \lambda x); \quad x \geqq 0 \tag{20}$$

which corresponds to the deflection curve of an infinitely long beam on elastic foundation[8].

b) Second Mode

Similarly, the two orthogonal shape functions describing the second mode of vibration are obtained using the deflection lines of the same type of beams on elastic foundation but now subjected to an antisymmetric uniformly distributed load as shown in Figure 2a. The pertinent shape functions are:

$$\begin{aligned} \Phi_2(x) &= \frac{1}{4} + (AF_1 + BF_2 - \frac{1}{4}F_3) \qquad \text{for } 0 \leqq x \leqq \frac{L_x}{2} \\ \Phi_2(x) &= -[\frac{1}{4} + (A\ _1 + B\ _2 - \frac{1}{4}\varphi_3)] \ \text{for } 0 \geqq x \geqq -\frac{L_x}{2} \end{aligned} \tag{21}$$

where

$$\begin{aligned} &A = \frac{1}{\sigma}[\frac{1}{4} f_2(1-f_1) - f_3 f_4]; \quad B = \frac{1}{\sigma}[f_3{}^2 - \frac{1}{4}(1-f_1)f_1] \\ &\sigma = f_1 f_4 - f_2 f_3 \end{aligned} \tag{22}$$

and

$$\begin{aligned} f_1 &= \cosh \lambda\frac{L_x}{2} \cos \lambda\frac{L_x}{2} \\ f_2 &= \frac{1}{2}(\cosh \lambda\frac{L_x}{2} \sin \lambda\frac{L_x}{2} + \sinh \lambda\frac{L_x}{2} \cos \lambda\frac{L_x}{2}) \\ f_3 &= \frac{1}{2}\sinh \lambda\frac{L_x}{2} \sin \lambda\frac{L_x}{2} \\ f_4 &= \frac{1}{4}(\cosh \lambda\frac{L_x}{2} \sin \lambda\frac{L_x}{2} - \sinh \lambda\frac{L_x}{2} \cos \lambda\frac{L_x}{2}). \end{aligned} \tag{23}$$

Using Eq. (14) the additional constants can be given in the following forms:

$$\begin{aligned} &F_1 = \frac{1}{2} \sinh \xi' \sin\xi'; \quad F_3 = \cosh \xi' \cos \xi' \\ &F_2 = \frac{1}{4} (\cosh \xi' \sin \xi' - \sinh \xi' \cos \xi') \\ &\varphi_1 = \frac{1}{2} \sinh \xi \sin \xi; \quad \varphi_3 = \cosh \xi \cos \xi \\ &\varphi_2 = \frac{1}{4} (\cosh \xi \sin \xi - \sinh \xi \cos \xi). \end{aligned} \tag{24}$$

The shape function in the Y direction is identical to the one used for the first mode; thus, $\Psi_1(y) = \Psi_2(y)$.

EVALUATION OF THE DEFINITE INTEGRALS

Assuming that the actual form of the arch dam permits the use of an equivalent rectangular plate as shown in Figure 1, the definite integrals indicated in Eq. (11) can be evaluated analytically. For this purpose, Eq. (11) can be written as

$$\omega_1{}^2 = \frac{1}{\bar{m}} \left(k_1 + D \frac{I_1 I_2 + 2I_3 I_4 + I_5 I_6}{I_1 I_6}\right), \tag{25}$$

where

$$I_1 = \int_{x_0}^{x_1} \Phi_1{}^2 dx = L_x - \frac{1}{\lambda R} \left(2C_1 - \frac{C_2 + C_3}{R}\right) \tag{26}$$

$$I_2 = \int_{y_0}^{y_1} \Psi_1{}^{''''} \Psi_1 dy = \frac{4\lambda^3}{N} \left(C_6 - \frac{C_7}{N}\right) \tag{27}$$

$$I_3 = \int_{x_0}^{x_1} \Phi_1{}^{''} \Phi_1 dx = \frac{1}{R^2} (C_4 + C_5) \tag{28}$$

$$I_4 = \int_{y_0}^{y_1} \Psi_1{}^{''} \Psi_1 dy = \frac{\lambda}{N} \left[2C_8 + \frac{1}{N}(C_9 + C_{10})\right] \tag{29}$$

$$I_5 = \int_{x_0}^{x_1} \Phi_1{}^{''''} \Phi_1 dx = \frac{4\lambda^3}{R} \left(C_1 - \frac{C_2 + C_3}{R}\right) \tag{30}$$

$$I_6 = \int_{y_0}^{y_1} \Psi_1{}^2 dy = L_y - \frac{1}{N\lambda} \left(2C_6 - \frac{C_7}{N}\right). \tag{31}$$

The constants R and N in these expressions are given in Eqs. (15) and (17), respectively. The other constants are:

$$C_1 = 2(\cosh \lambda L_x - \cos \lambda L_x); \quad C_2 = \frac{1}{4}(3\sinh 2\lambda L_x - \sin 2\lambda L_x)$$

$$C_3 = \frac{1}{2}(2\lambda L_x \sinh \lambda L_x \sin \lambda L_x + 2\cosh \lambda L_x \sin \lambda L_x - 3\sinh \lambda L_x \cos \lambda L_x)$$

$$C_4 = -2\lambda L_x \cosh \lambda L_x \cos \lambda L_x + 2\cosh \lambda L_x \sin \lambda L_x + \cos \lambda L_x \sinh \lambda L_x$$

$$C_5 = \frac{1}{2}(4\lambda L_x + \sin 2\lambda L_x - \sinh 2\lambda L_x)$$

$$C_6 = \sinh \lambda L_y \cosh \lambda L_y + \sin \lambda L_y \cos \lambda L_y$$

$$C_7 = \frac{1}{8}[\cosh^2\lambda L_y(\sin 2\lambda L_y + 3\sinh 2\lambda L_y) + 2\cosh\lambda L_y \cos\lambda L_y(2\lambda L_y \cosh\lambda L_y \cos\lambda L_y +$$
$$+ \cosh\lambda L_y \sin\lambda L_y + \sinh\lambda L_y \cos\lambda L_y) + \cos^2\lambda L_y(3\sin 2\lambda L_y + \sinh 2\lambda L_y)]$$

$$C_8 = \cosh \lambda L_y \sin \lambda L_y - \sinh \lambda L_y \cos \lambda L_y$$

$$C_9 = \frac{1}{4}(\cosh^2\lambda L_y - \cos^2\lambda L_y)(4\lambda L_y + \sin 2\lambda L_y + \sinh 2\lambda L_y)$$

$$C_{10} = \cosh \lambda L_y \cos \lambda L_y (2\lambda L_y \sinh \lambda L_y \sin \lambda L_y - C_8). \qquad (32)$$

Similarly, the circular frequency pertinent to the second mode of vibration is obtained from

$$\omega_2{}^2 = \frac{1}{m}\left[k_1 + D\,\frac{I_7 I_2 + 2I_8 I_4 + I_9 I_6}{I_7 I_6}\right] \qquad (33)$$

where the newly introduced definite integrals are:

$$I_7 = \int_{-\frac{L_x}{2}}^{+\frac{L_x}{2}} \Phi_2{}^2 dx = 2\left[\frac{L_x}{32} + \frac{C_{11}}{2} + C_{12}\right] \qquad (34)$$

$$I_8 = \int_{-\frac{L_x}{2}}^{+\frac{L_x}{2}} \Phi_2{}''\Phi_2 dx = 2\lambda^2\left[\frac{1}{4}C_{13} + C_{14}\right] \qquad (35)$$

$$I_9 = \int_{-\frac{L_x}{2}}^{+\frac{L_x}{2}} \Phi_2{}''''\Phi_2 dx = -2\lambda^4[C_{11} + 4C_{12}]. \qquad (36)$$

The new constants appearing in these equations represent

$$C_{11} = AG_3 + BG_4 - \frac{1}{4}G_1; \qquad C_{13} = AG_1 + BG_2 + G_3$$

$$C_{12} = A^2G_3{}^2 + 2ABG_3G_4 + B^2G_4{}^2 - \frac{1}{2}(AG_1G_3 + BG_1G_4 - \frac{1}{8}G_1{}^2)$$

$$C_{14} = A^2G_1G_3 + AB(G_1G_4 + G_2G_3) + B^2G_2G_4 + A(G_3{}^2 - \frac{1}{4}G_1{}^2) + $$
$$+ B(G_3G_4 - \frac{1}{4}G_1G_2) - \frac{1}{4}G_1G_3. \qquad (37)$$

The coefficients A and B are defined in Eq. (22). Furthermore,

$$G_1 = \frac{1}{2\lambda}\left(\sinh\lambda\frac{L_x}{2}\cos\lambda\frac{L_x}{2} + \cosh\lambda\frac{L_x}{2}\sin\lambda\frac{L_x}{2}\right)$$

$$G_2 = \frac{1}{2\lambda}\left(\sinh\lambda\frac{L_x}{2}\sin\lambda\frac{L_x}{2}\right); \qquad G_4 = \frac{1}{4\lambda}\left(1-\cosh\lambda\frac{L_x}{2}\cos\lambda\frac{L_x}{2}\right)$$

$$G_3 = \frac{1}{4\lambda}\left(\cosh\lambda\frac{L_x}{2}\sin\lambda\frac{L_x}{2} - \sinh\lambda\frac{L_x}{2}\cos\lambda\frac{L_x}{2}\right). \qquad (38)$$

NUMERICAL EXAMPLE

In order to prove the feasibility of the method introduced and to obtain information concerning the accuracy of the solution, the natural frequencies pertinent to the first and second modes of the Monticello Dam[6] have been computed and compared with the actual frequencies measured by the Bureau of Reclamation, Denver, Colorado. The average dimensions of the Monticello Dam are shown in Figure 3. The units ft, lb and sec have been used in all numerical computations.

The substitution of these values into Eq. (2) gives

$$D = \frac{2.99 \cdot 144 \cdot 38^3}{12(1-0.17^2)} = 1.966 \cdot 10^{12} \qquad \mu = \frac{\sin 45°}{0.6(\pi/4)} = 1.5$$

$$\vartheta_1 = \frac{\cos^{-1} 1^3}{\pi/4} = 0 \qquad k_1 = \frac{2.9 \cdot 10^6 \cdot 144 \cdot 38}{470^2 \cdot 1.5} = 4.78 \cdot 10^4 \text{ and } \bar{m} = 1.72 \cdot 10^2$$

From Eqs. (2), (12) we obtain:

$$I = \frac{1}{12} 38^3 = 4.57 \cdot 10^3 \quad \text{and} \quad \lambda = \sqrt[4]{\frac{4.78 \cdot 10^4}{4 \cdot 4.18 \cdot 10^8 \cdot 4.57 \cdot 10^3}} = 8.88 \cdot 10^{-3}$$

The dimensions of the "equivalent" rectangular plate on elastic foundation are (13):

$$L_x = \frac{2}{3}[2 \cdot 495 + 990]\frac{990}{495+990} \cong 880 \text{ and } L_y = 220 - \frac{990(990-495)}{6(990+495)} \cong 165.$$

The constants required for evaluation of the definite integrals $I_{n=1,2...6}$ are calculated from Eqs. (15), (17) and (32), respectively:

$N = 5.16$; $R = 1221$

$C_1 = 2440$; $C_2 = 2.28 \cdot 10^6$; $C_3 = 1.057 \cdot 10^4$; $C_4 = -1.52 \cdot 10^6$; $C_5 = 17.12$;

$C_6 = 4.74$; $C_7 = 18.4$; $C_8 = 2.04$; $C_9 = 19.8$; $C_{10} = 0.950$

Substituting these quantities into the expressions of the definite integrals given in Eqs. (26) to (31), we obtain:

$I_1 = 603$; $I_2 = 6.34 \cdot 10^{-7}$; $I_3 = -9.06 \cdot 10^{-3}$; $I_4 = 1.4 \cdot 10^{-2}$

$I_5 = 1.292 \cdot 10^{-6}$; $I_6 = 36.2$.

Using Eq. (25), we can write

$$\omega_1^2 = \frac{1}{1.712\ 10^2}\Big[4.78\ 10^4 + 1.96\ 10^{12}\ \frac{603 \times 6.34 \cdot 10^{-7} + 2(-9.06 \cdot 10^{-3} \times 1.4 \cdot 10^{-2}) + 1.292 \cdot 10^{-6} \times 36.2}{603 \times 36.2}\Big]$$

$$= 3.74 \cdot 10^2 \quad \text{and} \quad \omega = 19.34 \text{ rad/sec},$$

from which the natural frequency of the dam is

$$f = \frac{\omega}{2\pi} = \frac{19.34}{2\pi} = 3.08 \text{ cps.}$$

A comparison of this computed frequency with the measured one indicates an error of 1.6 percent. In view of the introduced simplifying assumptions, however, such a high accuracy should be considered as a coincidence and not a rule.

The natural frequency pertinent to the second mode was obtained from Eq. (33) in a similar manner. The discrepancy between the estimated and measured natural frequencies was below 1 percent. The measured and estimated first and second modes of free vibration are illustrated in Figure 4.

CONCLUSIONS AND RECOMMENDATIONS

It has been demonstrated that it is possible to reduce the inherently complex problem of free vibration of thin arch dams to that of an equivalent plate or elastic foundation. Closed form solutions for estimating the natural frequencies pertinent to the first and second modes of free vibration have been derived which yield numerical results in a relatively short time. The validity of the closed form solution is limited to "equivalent" crest length $L_x \leqq 1.5\pi/\lambda$, provided that the developed elevation is of a trapezoidal form which can be transformed into equivalent rectangular plate on elastic foundation. Although an excellent agreement between calculated and measured frequencies has been obtained, the obtainable accuracy is estimated to be between 10 and 15 percent. More arch dams for which measured or calculated data on free vibration is available will be investigated using the method presented. The results of these studies will be reported orally at the 4WCEE in Santiago.

For arch dams which do not satisfy the above mentioned limitation, a general expression has been derived requiring numerical evaluation of certain definite integrals. It is possible, however, to obtain closed form solutions for arch dams of trapezoidal and triangular shapes also.

For final design, when high accuracy is mandatory, the use of an improved finite difference technique for solution of the rigorous differential equation of motion (1) is highly recommended[10].

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REFERENCES

(1) Crawford, C. C.: "Earthquake Design Loadings for Thin Arch Dams," Proceedings of the 3rd World Conference on Earthquake Engineering, held in New Zealand, Jan. 22 - Feb. 1, 1965, Vol. III, pp. IV. 172 - IV. 185.

(2) Rydzewski, J. R.: "Theory of Arch Dams," Proceeding of the International Symposium held at Southampton University, April 1964, Pergamon Press, Oxford, 1965.

(3) Szilard, R.: "Statics and Dynamics of Plates, Classical and Numerical Methods," a text and reference book, in preparation for the Macmillan Co.

(4) Ganev, H. G.: "A New Method of Calculating Arch Dams as Plates Lying on an Elastic Foundation," paper in Proceedings of the International Symposium on Theory of Arch Dams, held at Southampton University, April 1964, Pergamon Press, Oxford, 1965, pp. 275-311.

(5) Wlassow, W. S.: "Allgemeine Schalentheorie und ihre Anwendung in der Technik," Akademie-Verlag, Berlin, 1958.

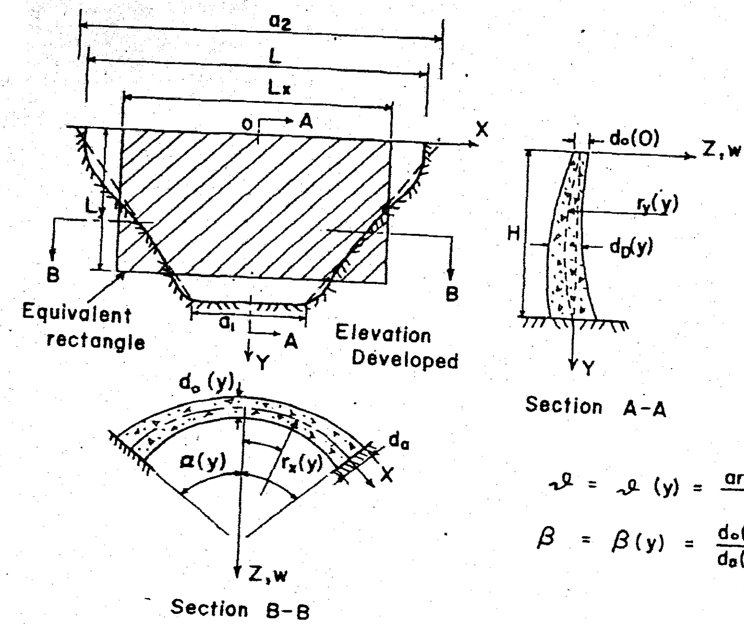
(6) Rouse, G. C. and Bouwkamp, J. G.: "Vibration Studies of Monticello Dam," Research Report No. 9, Bureau of Reclamation, U.S. Gov. Printing Office, Washington, D.C., 1967.

(7) Yoshida, T. and Baba, K.: "Dynamic Response of Dams," Proceedings of the 3rd World Conference on Earthquake Engineering, held in New Zealand, Jan. 22 - Feb. 1, 1965, Vol. II, pp. II-748 - II-764.

(8) Hetényi, M.: "Beams on Elastic Foundation," The University of Michigan Press, 6th Printing, Ann Arbor, 1961.

(9) Gontkewitsch, W. S.: "Eigenschwingungen von Platten und Schalen," (German translation of the original Russian text), VEB Fachbuchverlag, Leipzig, 1967.

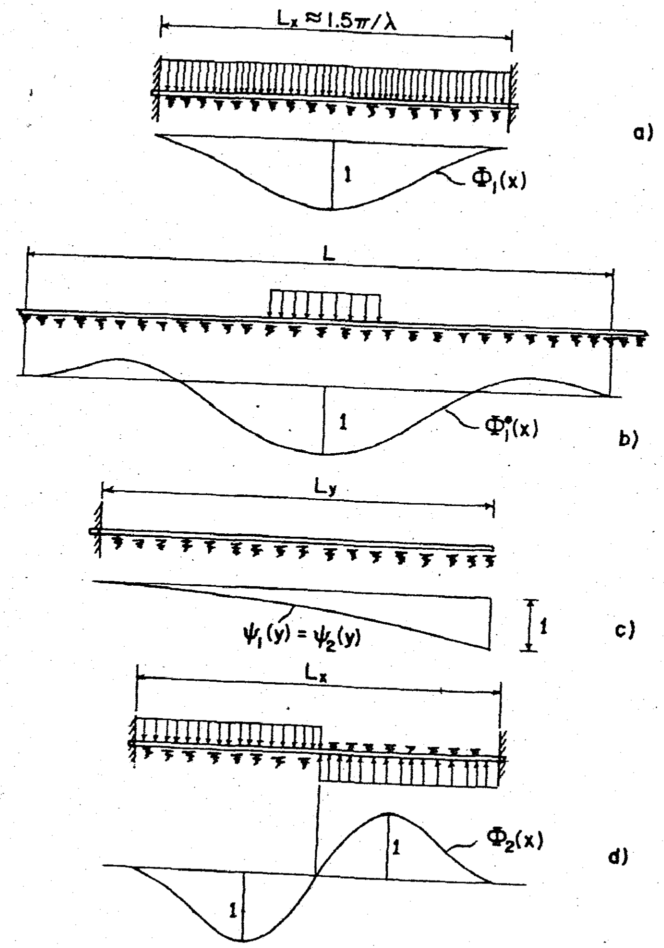
(10) Szilard, R. and Nomura, E.: "Free and Forced Vibrations of Thin Arch Dams," paper in preparation.



$$\vartheta = \vartheta(y) = \frac{\arccos \beta^3(y)}{\alpha(y)}$$

$$\beta = \beta(y) = \frac{d_o(y)}{d_a(y)}$$

<u>Figure 1:</u> Geometry of an Arch Dam



<u>Figure 2</u>

Shape Functions

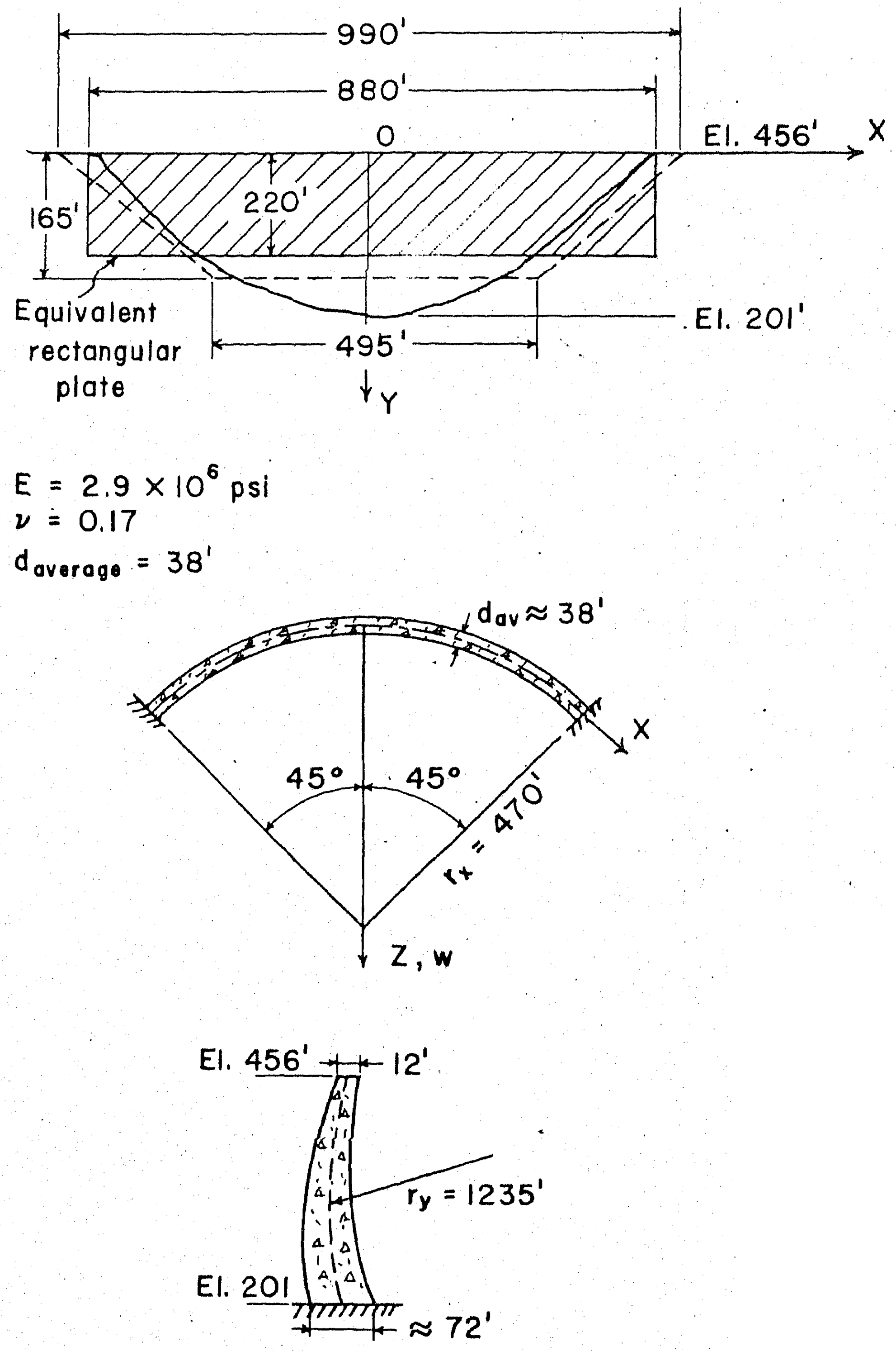
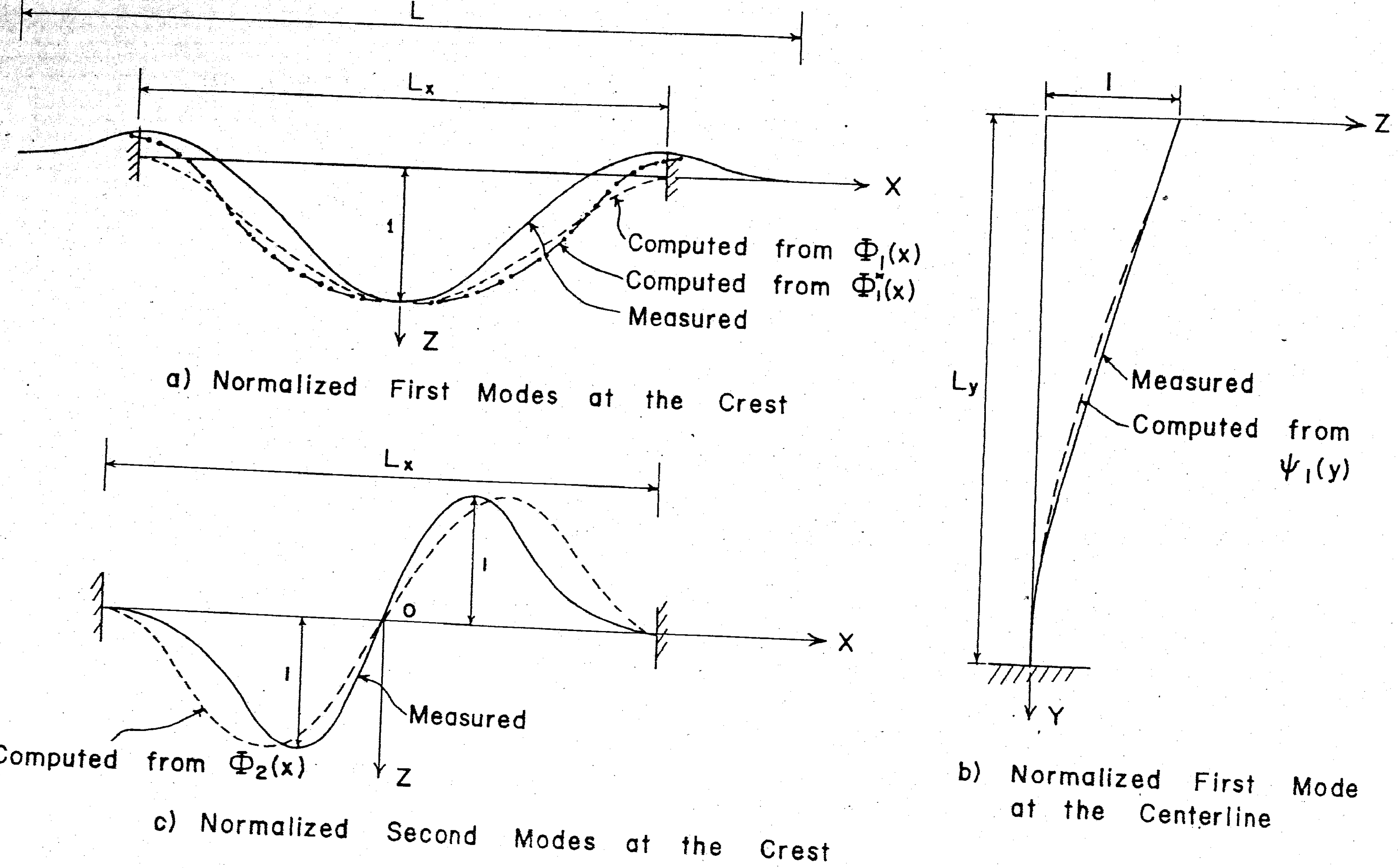


Figure 3: Average Dimensions of Monticello Dam



a) Normalized First Modes at the Crest

b) Normalized First Mode at the Centerline

c) Normalized Second Modes at the Crest

Figure 4: Free Vibration Modes of Monticello Dam