NO-TENSION THEORY FOR SEISMIC ANALYSIS OF MASONRY STRUCTURES

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SUMMARY

The paper presents a consistent theoretical formulation for structural objet made by no-tension materials. The existence of the solution strongly depends on the loading pattern. Necessary and sufficient conditions are discussed and a procedure to check the existence of the solution is illustrated. Next the search of the solution is approached, based on the analysis of the Complementary and Potential energy functionals. In the Complementary Energy approach it is proved that the solution stress field obeys a constrained minimum condition of the complementary energy functional on the set of all stress fields in equilibrium with the applied loads and keeping the material in pure compression. In the Potential Energy approach, the primary solution is composed by displacement field plus the fracture strain field. The latter must be positively semi-definite in each point of the body; such a strain field is named an *admissible fracture* field. It is proved that the solution, in this case, obeys a constrained minimum principle of the Potential Energy functional over the set of all displacement field and on the set of admissible fracture fields. A number of computer codes have been implemented allowing to analyse by a F.E. procedure the behaviour of structural systems acted on by forces simulating the action of earthquakes.

INTRODUCTION

The material organisation of the masonry tissue that one encounters in old buildings is very different from the one that is commonly manufactured at present days in modern masonry buildings. Therefore, in dealing with ancient buildings, it is worthwhile to enhance some basic features that are peculiar of such typology.

Without entering into the details of the many types of old masonry, it can be assessed, following Heymann's work [Heyman, 1966], that in many cases and for a number of structural typologies the prevalent feature that characterises such structures, and makes them dissimilar from actual concrete and steel structures, is quite definitely poor capacity to resist tensile stresses. In a few words, the *no-tension* masonry model assumes that the material follows a fully elastic behaviour in compression, but cannot resist tension stress. In a solid the model requires that equilibrium against external loads can satisfied by stress fields called here *admissible* stress fields, that imply pure compression at every point of the solid. Assuming *stability* of the material in the Drucker's sense, compatibility of the strain field can be ensured by superposing to the elastic strain field an additional *fracture* field, that does not admit contraction in any point and along any direction. In other words the stress tensor in any point must be negative semi-definite, while the fracture strain field is required to be positive semi-definite.

Analysis of no-tension structures proves that the stress, strain and displacement fields obey extremum principles of the basic energy functionals. So the solution stress field is found as the constrained minimum of the Complementary Energy functional, under the condition that it is *admissible* (i.e. negatively semi-definite at every point and in equilibrium with the applied loads); on the other side solution displacement and fracture strain fields yield in solution the constrained minimum of the Potential Energy functional, under the condition that the fracture field is positively semi-definite at any point.

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Both the above functionals are *convex* and subject to a set of *convex* constraints. With these conditions the existence and uniqueness of the extremal points are guaranteed provided that at least one solution exists for the constraint equalities and/or inequalities. This latter event occurs, according to the fundamental theorems of Limit Analysis for no-tension structures [Como and Grimaldi, 1983], if the structure does not collapse under the given loads. So the Limit Analysis of a no-tension structure acts as a tool to check both the static safety of the structure and the existence of the solution of the equilibrium problem. After some Finite Elements model of structure has been produced. Limit Analysis can be set, like in perfect plasticity, as a Linear Programming problem, and can be approached from the numerical point of view through the relevant algorithms like the Simplex one. After this preliminary step, stresses, strains, fractures and displacements under the assumed non-collapse load pattern can be searched by optimisation of the energy functionals. The *stress approach*, taking as primary unknown the stress field, is founded on the minimisation of the Complementary Energy functional, while the *displacement approach* is based on the minimisation of the Potential Energy functional and takes as primary unknowns the joint displacement and fracture fields [Baratta, 1991]. Both methods have been implemented to work on some F.E. model of uni- and bi-dimensional structures and can be operatively executed through appropriate procedures for solving convex constrained optimisation problems. Practical experience has proved that search methods are better implemented with reference to the Complementary Energy functional, while more sophisticated descent methods, requiring derivatives of the objective function and the constraints, are more effective to treat the Potential Energy, that enjoys more mathematical regularity than the Complementary functional. After the solution has been obtained, a picture of the stress distribution, of the structure deformation and the localisation and orientation of fractures can be observed, yielding data for safety evaluation by comparing the compressive-shear stress with admissible ones, and a basis to verify the reasons of observed diseases and the effectiveness of proposed reinforcements [Baratta and al., 1996].

In previous papers the above introduced procedure have been referred to for the typical structural systems for masonry buildings under earthquakes loading; numerical examples, concerned with a NT bi-dimensional panel with steel inclusions, have been lead out; a finite-element constant-stress/constant strain version of the problem have been presented and a two-steps relaxation procedure have been implemented for the solution of the discretised problem by the Potential Energy approach [Baratta and Voiello, 1996 –1997)]. On the other hand, for portal arches or multiple-arch frame structures, where the stress field is described by stress resultant at every cross section (shear and normal force, and bending moment), the complementary energy approach has been considered. In this case the stress-approach can be managed with some ease, and proves to be more effective and less cumbersome than the displacement-approach [Baratta and al., 1998].

APPLICATION OF THE TOTAL POTENTIAL ENERGY METHOD : Masonry WALLS LOADED BY IN-PLANE FORCES

Let $\mathcal{E}$ be the Total Potential Energy functional defined on displacement field $\mathbf{u}(\mathbf{x})$ and the fracture field $\boldsymbol{\varepsilon}_f(\mathbf{x})$ of the system, the solution ($\mathbf{u}_o$, $\boldsymbol{\varepsilon}_{fo}$) in kinematic variables obeys to the conditions [Baratta 1991]

$$E\ (\mathbf{u}_0, \boldsymbol{\varepsilon}_{f_0}) = \min_{\substack{u \in R^N \\ \boldsymbol{\varepsilon}_f \in K_\varepsilon}} E\ (\mathbf{u}, \boldsymbol{\varepsilon}_f) \tag{1}$$

subject to the condition that the fracture strain tensor $\boldsymbol{\varepsilon}_f$ is positive semidefinite, (i.e. $J_{1f} \geq 0\,; J_{2f} \geq 0$, with J_{1f} and J_{2f} the first and second strain invariants). The Kuhn-Tucker necessary conditions for optimality in every element are, on one side [Baratta and Voiello, 1997]

$$\frac{\partial \mathrm{E}}{\partial \mathrm{u}_{\mathrm{i}}} = 0 \;\; (\mathrm{i} = 1,..., \mathrm{N})\,; \frac{\partial \mathrm{E}}{\partial \varepsilon_{\mathrm{fi}}^{(e)}} = \lambda_1^{(e)} \frac{\partial \mathrm{J}_{1\mathrm{f}}^{(e)}}{\partial \varepsilon_{\mathrm{fi}}^{(e)}} + \lambda_2^{(e)} \frac{\partial \mathrm{J}_{2\mathrm{f}}^{(e)}}{\partial \varepsilon_{\mathrm{fi}}^{(e)}}\,; \frac{\partial \mathrm{E}}{\partial \gamma_{\mathrm{f}}^{(e)}} = \lambda_1^{(e)} \frac{\partial \mathrm{J}_{1\mathrm{f}}^{(e)}}{\partial \gamma_{\mathrm{f}}^{(e)}} + \lambda_2^{(e)} \frac{\partial \mathrm{J}_{2\mathrm{f}}^{(e)}}{\partial \gamma_{\mathrm{f}}^{(e)}} \tag{2}$$

with $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\lambda_1 J_{1f} = \lambda_2 J_{2f} = 0$, and should be satisfied at the optimal point ($\mathbf{u}_o$, $\boldsymbol{\varepsilon}_{fo}$). Considering that in any generic element "*e*" one has

$$\frac{\delta \mathrm{E}}{\delta \varepsilon_{\mathrm{fi}}^{(e)}} = -\sigma_i^{(e)}; \frac{\delta \mathrm{E}}{\delta \gamma_{\mathrm{f}}^{(e)}} = -\tau^{(e)}; \frac{\delta \mathrm{J}_{1\mathrm{f}}^{(e)}}{\delta \varepsilon_{\mathrm{fi}}^{(e)}} = 1; \frac{\delta \mathrm{J}_{1\mathrm{f}}^{(e)}}{\delta \gamma_{\mathrm{f}}^{(e)}} = 0; \frac{\delta \mathrm{J}_{2\mathrm{f}}^{(e)}}{\delta \varepsilon_{\mathrm{fx}}^{(e)}} = \varepsilon_{\mathrm{fy}}^{(e)}; \frac{\delta \mathrm{J}_{2\mathrm{f}}^{(e)}}{\delta \varepsilon_{\mathrm{fy}}^{(e)}} = \varepsilon_{\mathrm{fx}}^{(e)}; \frac{\delta \mathrm{J}_{2\mathrm{f}}^{(e)}}{\delta \gamma_{\mathrm{f}}^{(e)}} = \frac{1}{2} \gamma_{\mathrm{f}}^{(e)} \tag{3}$$

From eqs.(2) e (3) the stress tensor (at the solution point) and the stress invariants are easily calculated. Since it result $I_1^{(e)} \leq 0$ and $I_2^{(e)} \geq 0$, one can conclude that **the stress tensor in every element is *negative semidefinite* in solution.** Moreover, consider the *fracture work* $L_f^{(e)} = \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}_f$ one can conclude that **the fracture work in every element is *null* in solution**. It can be proved, moreover, that **the stress tensor in every element is *coaxial to the fracture strain tensor* in solution**. Since the minimum point of the Total Potential Energy fulfills the requirements on the stress and fracture-strain fields, characteristics of the No-tension material, it remains concretely confirmed that the conditions for the minimum of the relevant functional coincide with the solution of the equilibrium problem of a No-tension solid.

The search for the solution of the structural problem can be pursued by employing the two-steps relaxation procedure. [Baratta and Voiello, 1997)]:

- Constrained minimization ($\varepsilon_f^{(e)} \geq 0$) of the energy functionalwith respect to the components of the fracture strain *under prescribed nodal displacements*;

- Free minimization of the energy functional with respect to the displacement components, *under fixed fractures*;

The first step is executed by reducing the stress in every element to respect the condition $\boldsymbol{\sigma}^{(e)} \geq 0$ (*relaxation* of the tensile stresses). The scope is attained by introducing suitable fracture strains ($\varepsilon_f^{(e)} \geq 0$), thus violating the equilibrium of the solid; the second step aims at restoring the global and punctual equilibrium of the system by modifying the nodal displacements. The procedure starts from a reference equilibrium configuration that in the present formulation is assumed to coincide with the linearly elastic solution Il procedimento muove a partire da una soluzione di riferimento che si assume coincidente con la soluzione elastica lineare. More precisely:

STEP 1: The minimum of the energy is searched for varying fractures and for the temporary displacement field

$$E\ (\mathrm{u}, \varepsilon_f') = \min_{\boldsymbol{\varepsilon}_f \in K_{\mathcal{E}}} E\ (\mathrm{u}, \varepsilon_f) \tag{4}$$

Considering that the Total Potential Energy is the sum of the values the functional assumes in the single elements and the energy in the *"e-th"* element depends only on the tensor $\boldsymbol{\varepsilon}_f^{(e)}$ relevant to the same element, apart the nodal displacements that in this phase cannot be modified, the minimization of the functional with respect to the strain field can be pursued by minimizing the relevant partial energy $\mathcal{E}^{(e)}$ in every element. It is proved that the minimum of the Energy under such conditions for the assigned displacement field corresponds to the fracture strain tensor that leads the stress tensor in every element, inependently from each other, to verify the admissibility conditions.

STEP 2: In this phase the system is re-equilibrated. In fact, the stress field corresponding to the assumed displacements **u** and to the fractures, as updated in the previous step, are no more in equilibrium with the applied loads. It is necessary therefore to update also the displacements **u** that, coupled with the previously modified $\boldsymbol{\varepsilon}_f'$, should yield an equilibrated solution

$$\mathrm{E}\ (\mathbf{u}', \boldsymbol{\varepsilon}_f') = \min_{\mathbf{u} \in R^N} \mathrm{E}\ (\mathbf{u} \mid \boldsymbol{\varepsilon}_f) \rightarrow \mathbf{u}' = \mathrm{F}\ (\boldsymbol{\varepsilon}_f') \tag{5}$$

The scope is attained by a classical free optimization (for instance, the conjugate gradient method), keeping fixed the fracture strains $\boldsymbol{\varepsilon}_f$.

This procedure is applied to a NT material panel, loaded with forces acting in its plane, modelled by a finite element mesh. The structural model is a 560x840 cm stone wall, 0.50 cm thick, with steel reinforcing inclusions consisting of flat arches and string courses. The considered seismic coefficient varies from 0 to 0.6, according to the current italian seismic engineering code.
The convergence of the procedure is shown in Fig.1; one can notice that, as the structure gets nearer to the collapse condition, while increasing the number of iterations, the rate of convergence decreases.

As, in the investigated cases, the stress tension approximates the zero value following a rule of the type $\sigma^+ = \sigma_o \cdot i^{-\alpha}$ where "i" is the iterations number and σ_o is the order of the maximum traction stress in the linear elastic field, one can point out that the exponent "α" decreases rather quickly when "c" tends to the collapse value (Fig. 1d), showing that, in such a situation, the convergence turns more difficult as one can

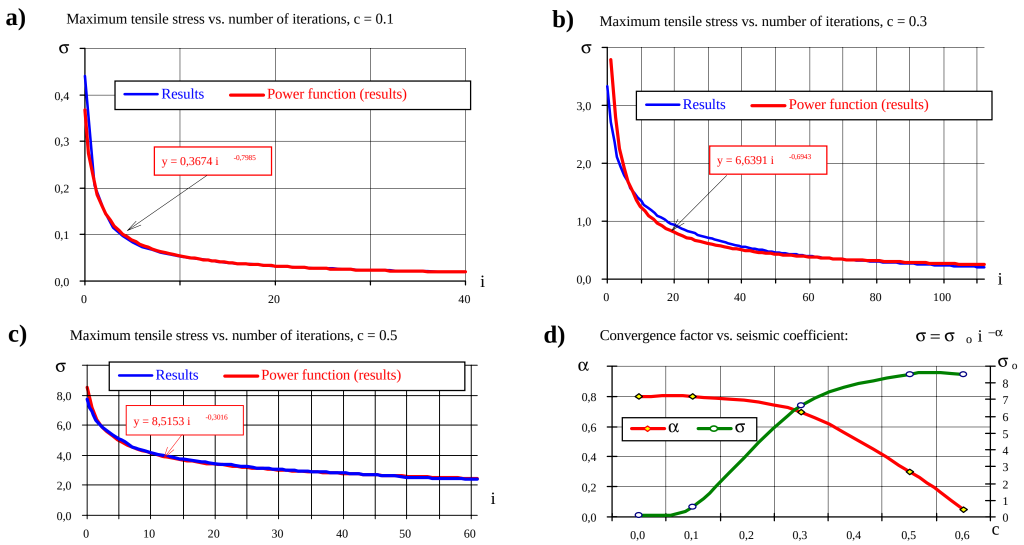
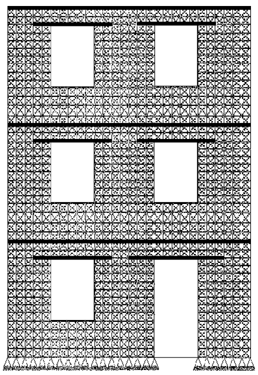


Figure 1: Traction stress convergence



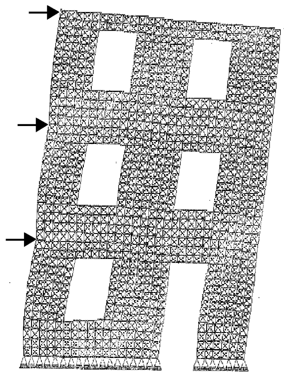
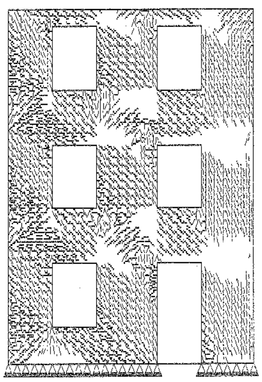


Figure 2a: The considered masonry wall

Figure 2b: The deformed configuration (Displacement Magnification. 40:1)



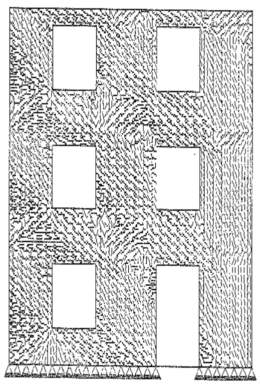


Figure 2c: The fractures distribution

Figure 2d: Compression isostatics

In Fig. 2a the discretised model of the considered masonry wall is depicted, while its instantaneous deformed configuration, for a seismic coefficient of 0.3, is shown in Fig. 2b; in Figs. 2c and 2d the possible distribution of the fractures and the compression isostatics are reported.

In Fig. 3 the first floor displacement is diagrammed versus the seismic coefficient and a comparison is reported with the corresponding displacement in the linear elastic phase and with the displacement (U*) calculated by following the procedure shown in the mentioned standard-code (POR-type approach).

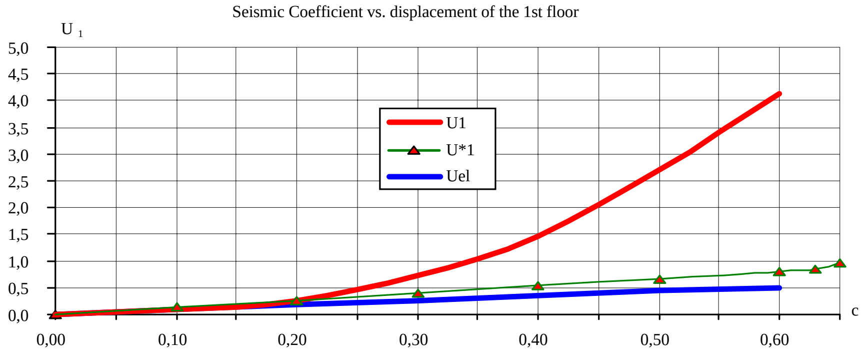


Figure 3 : 1st floor displacements

3. APPLICATION OF THE COMPLEMENTARY ENERGY METHOD : NO-TENSION MULTIPLE ARCH-SYSTEM

The procedure of calculus for the verification in exercise and/or for the interpretation of a pronounced fracture multiple arch-system is based on the minimisation of the Complementary Energy functional.
In the following an example is illustrated with reference to a three span structural system as in Fig. 4. The no-tension assumption requires that in solution the joint field of bending moment and normal force yields a pressure line (i.e. a *funicular curve*) that is interior to the arch profile at every cross section [Baratta and al., 1998].

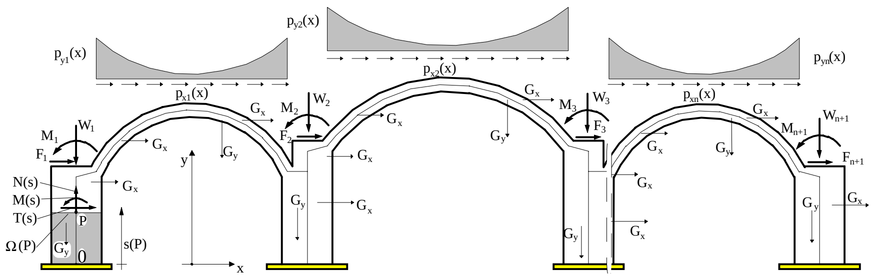


Figure 4: The load pattern

With reference to the scheme in Fig.5 the stress fields can be expressed as a function of the static redundancies

$$\left.\begin{array}{l}
N_{0y}(s)+c\cdot N_{0x}(s)+\sum_{i=1}^{3N} X_i N_i(s)\leq 0\\
M_{0y}(s)+c\cdot M_{0x}(s)+\sum_{i=1}^{3N} X_i M_i(s)\leq -h'(s)\left[N_{0y}(s)+c\cdot N_{0x}(s)+\sum_{i=1}^{3N} X_i N_i(s)\right]\\
M_{0y}(s)+c\cdot M_{0x}(s)+\sum_{i=1}^{3N} X_i M_i(s)\geq h''(s)\left[N_{0y}(s)+c\cdot N_{0x}(s)+\sum_{i=1}^{3N} X_i N_i(s)\right]
\end{array}\right\}\forall s\in \mathsf{S} \qquad (6)$$

In eq.(6) , with reference to the equivalent isostatic structure (Fig. 5), N_{oy}, M_{oy} are the characteristics due to vertical permanent loads; N_{ox}, M_{ox} are the characteristics due to load components deriving from a horizontal acceleration equal to the gravity acceleration, and N_i, M_i are the characteristics induced by the unitary static redundancies, and "c" the seismic factor affecting horizontal loads

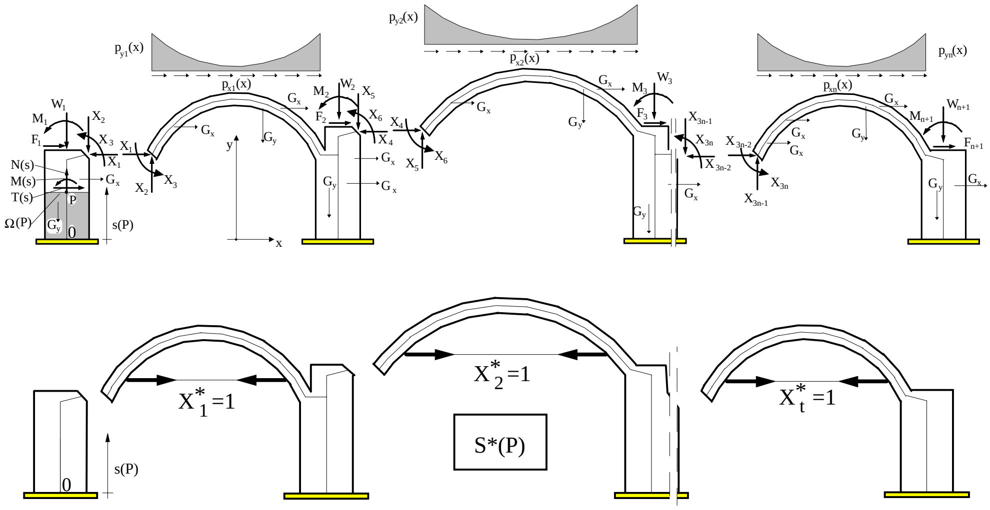


Figure 5: The statically determinate equivalent scheme: a) Structural redundancies b) Tie-rods redundancies

Ineqs. (6) mean that the funicular curve is everywhere interior to the arch profile, a necessary condition for the existence of equilibrated stress fields of the type in Fig. 7.

As shown by eq. (6), the constraints on the redundant force variables (X_1, X_2,....X_{3N}) are all of a linear type. If any solution of such inequalities exists, the set D_0 of admissible stress fields is not empty (the classical bilinear distribution of normal stres on the cross-section exs), and equilibrium of the structure is possible. The minimum of the *convex* function U(X_1, X_2, ..., X_{3N}), which represents the Complementary Energy obtained by adding to the Energy Elastic term to the potential produced by the possible settlements of the foundation basis, over the *convex* set **X** defined by inequalities (6), is a problem of *convex optimisation*, and the equilibrium NT solution exists and is unique. It follows that, by the principles of Masonry Limit Analysis [Como and Grimaldi,1983], if no solution exists to ineqs. (6), the structure is over the failure condition and it is condemned; if, on the contrary, any solution exists, an equilibrium configuration, possibly fractured, exists and must be found. The problem of the research of the initial value of the static redundancies is solved by a Linear Programming approach. Inequalities (6) are as 3m linear inequalities in the 3N+1 redundant force variables "X_1, ..., X_{3N}" and "c", while the objective function to be maximised (minimised), is reduced to the coefficient "c". The solution of this optimisation problem yields the *seismic collapse factor* c_f of the structure and an admissible stress field for any value of the seismic factor lower than c_o. If any factor $c \leq c_f$ is assumed, the previous search yields an initial *feasible* set of redundant unknowns X_i corresponding to an *admissible* stress field, whose compatibility is consequently ensured through minimisation of the Complementary Energy. The latter optimisation is pursued by a procedure of the "search" kind, which operates only on the energetic functional values and does not require the introduction of derivatives. The displacements are obtained in a further step by integration, ignoring the shear strain. A calculus code has been implemented and applied to some examples, including in the analysis some elements that have not been explicitly introduced in the basic scheme, and the relevant solutions are discussed with reference to the classically known features of the behaviour of masonry structures.

Consider the portal arch in Fig. 4, with N = 3, the elasticity modulus, E_c , constant and the following data, referred to a Fig. 5 with rectangular cross-section with basis b= 100 cm.

L_1 (m)	L_2 (m)	L_3 (m)	F_1 (m)	f_2 (m)	F_3 (m)	R_1 (m)	R_2 (m)	R_3 (m)
6.5	8.85	6.5	2	2.5	2	3.64	5.17	3.64
b_1 (m)	b_2 (m)	b_3 (m)	B_4 (m)	h_1 (m)	H_2 (m)	h_3 (m)	h_4 (m)	S (m)
1.5	2	2	1.5	4	5	5	4	0.6
W_1(Kg)	W_2(Kg)	W_3(Kg)	W_4(Kg)	M_1 (Kgm)	M_2 (Kgm)	M_3 (Kgm)	M_4 (Kgm)	
40000	40000	40000	40000	5000	5000	5000	5000	

Fig. 6.a presents the structure's state at the failure threshold, for $c = c_f = 0.12$. It is possible to verify (Fig.6.a) that the funicular is everywhere interior to the arch profile, and that the procedure yields perfectly compatible rotations

and displacements (Fig.6.b). The displacements in Fig. 6.b are amplified by a factor $\alpha = 50$. Collapse takes place at the level $c_f = 0.12$.

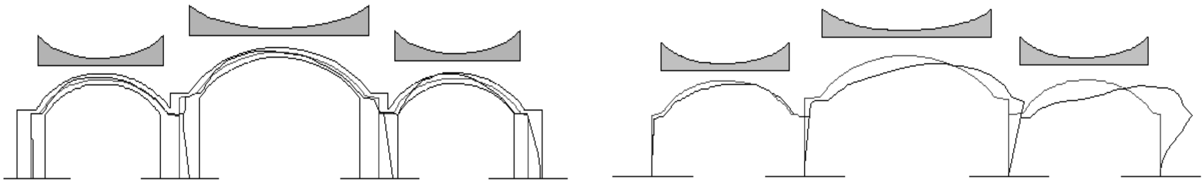


Figure 6: a) The funicular curve (c = 0.12) b) Deformation (c = 0.12, u_{max} = 0.0164 m)

STRUCTURAL SYSTEM WITH THREE SPANS REINFORCED BY TWO TIE RODS AT THE INTRADOS

In this section, the behaviour of the portal is demonstrated, after the installation of two tie rods. Fig. 7.a shows the condition of the structure, under the same value of c = 0.12 as

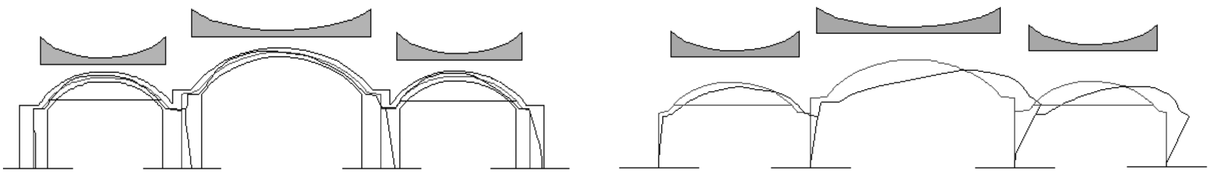


Figure 7: a) The funicular curve (c = 0.12) b) Deformation (c = 0.12, u_{max} = 0.0093 m)

the former case. It is possible to observe that the tie rods produce a very significant improvement in the structure's performance, curvatures are very moderate, and displacements much smaller. Note that displacements in Fig. 7b are amplified by a factor $\alpha = 300$. The efficiency of the installation of the tie is confirmed by the limit behaviour.

Collapse takes place at the level $c_f = 0.20$, much larger than for the unreinforced arch.

STRUCTURAL SYSTEM WITH THREE SPANS REINFORCED BY THREE TIE RODS AT THE INTRADOS

In this section, the behaviour of the portal is demonstrated, after three tie rods have been installed. Fig.8.a shows the condition of the structure, under the same value of c = 0.12 as in

the former cases. It is possible to observe that the three tie rods produce, obviously, a further improvement in the structure's performance: sections are not partialised, curvatures are improved and displacements are smaller. Note that displacements in Fig. 8.b are amplified by a factor $\alpha = 300$. The collapse factor reaches the level of c_f = 0.24, larger than previous ones.

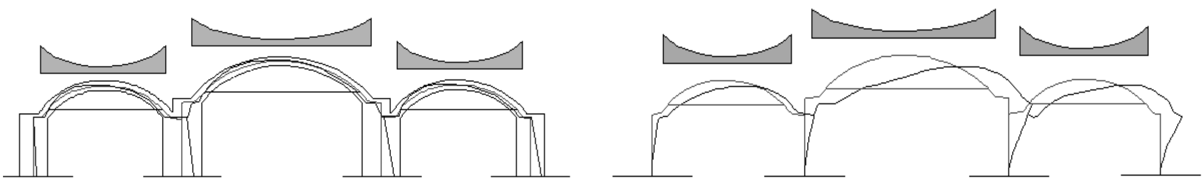


Figure 8: a) The funicular curve (c = 0.12) b) Deformation (c = 0.12, u_{max} = 0.0079 m)

Finally, in Fig. 9 the behaviour of the portal arch, for each of the three cases investigated is summarised. Every diagram plots the proceeding of the maximum horizontal displacements, and of the maximum compressive stress versus the lateral load factor.

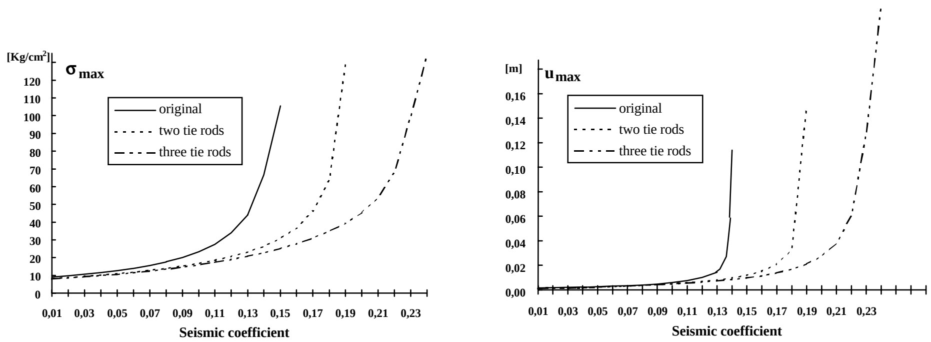


Figure 9: a) Stress; b) Drift , vs. seismic factor

CONCLUSIONS

In the paper, a theoretical support to the analysis of structures made by NT material has been presented, aiming at a gross modelling of the behaviour of masonry structures.

The approach is rather effective, in view of the current analysis of the safety of masonry buildings, under ordinary vertical loads or even horizontal actions intended to simulate the effects of earthquake shaking. The problem, very frequent in the practice of the old masonry buildings has been treated in detail, showing how the model gives account of the influence of intentional reinforcements like ties in the arches or flat arches and string courses in laterally loaded walls. In particular, the effectiveness of such measures has been proved, both producing a very significant attenuation of stresses for moderate loading and yielding a high improvement in the collapse value of the lateral load factor.

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