DEVELOPMENT OF A 3D FORCE-BASED BEAM ELEMENT FOR NONLINEAR ANALYSIS OF RC FRAME STRUCTURES CONSIDERING FLEXURE-SHEAR COUPLING

J.P. Almeida[1], A.A. Correia[1] and R. Pinho[2]

[1] *PhD Student, ROSE School, Pavia, Italy*
[2] *Assistant Professor, Structural Mechanics Department, University of Pavia, Italy*
Email: jpacheco@roseschool.it, aaraujo@roseschool.it, rui.pinho@unipv.it

ABSTRACT :

The applicability of fully 2D or 3D refined finite element models of reinforced concrete structures subjected to seismic loading is questionable, especially due to their high computational burden and inherent complexities involved in developing and running the model and interpreting the results. On the other hand, beam-column type finite elements developed in the framework of distributed plasticity, where the nonlinear response of the basic system is found from the integration of the response at control sections along the element axis, seem to provide the best compromise between the desired accuracy and computational efficiency.

The present work starts with a review of the main features of the state-of-the-art distributed inelasticity beam-column formulations that include shear effects and presents the grounds on which the motivation for the development of a new model settles. One of such elements is then developed with a view to the seismic assessment of reinforced concrete frame structures. The proposed model is able to deal with arbitrary loading conditions and involves the interaction of axial force, shear, bending moment and torsion. Since shear-normal stress interaction is explicitly included, the current model is deemed suitable for shear critical member analysis.

KEYWORDS: shear, torsion, beam-column finite element, reinforced concrete, shear-normal stress interaction, distributed inelasticity

1. INTRODUCTION

The strong seismic shaking of common reinforced concrete (RC) structures, such as buildings and bridges, unveils the improbable interaction of an amount of phenomena of the most diverse materially and geometrically nonlinear nature. To the present time, there is not a single model capable of suitably considering the complexity of all such mechanisms and thus predicting with the desired generality, accuracy and reliability the response of these structures. It is then of the outmost importance to correctly identify and prioritize the contribution of such phenomena to the global (and local) level responses, so that an appropriate model can be developed.

The most immediate and obvious decision that has to be taken is whether the shear deformation of the member under analysis (beam, column or wall) is of relevance or not. While the consideration of the flexural deformation (the one caused by axial force and bi-directional bending moments, for the 3D case) is a crucial component to correctly model the global structural deformation, the relative importance of the shear counterpart (bi-directional shear and torsion, for 3D) should be critically assessed. In view of its significance for the behaviour of some types of RC members as short bridge piers and walls, the modelling strategies for this type of mechanism constitute the main issue of the present paper.

When the shear response (in addition to the flexural behaviour) of the member is relevant, then the interaction between the flexural and shear components becomes of immediate concern. It is well-known that such forces are effectively coupled and disregarding such situation is a crude assumption. Additionally, it should also be noted that if, instead of the two-dimensional coupling (N-M-V), the complete three-dimensional interaction is considered (N-M_y-M_z-V_y-V_z-T), then the complexity of the problem increases considerably.

Beam-column elements where the inelastic behaviour is lumped at certain locations of the member – also known as plastic hinge approaches, are not able to capture such phenomena. Detailed 2D or 3D finite elements correspond to the opposite end of the modelling spectrum and have been used by the research community to reproduce the behaviour of RC walls, but are not a practical tool for design engineers and for generalized analyses of beam and column members. Additionally, they require a rare combination of considerable expertise in numerical modelling and powerful processing capabilities. Overall, beam-column elements of the distributed inelasticity type seem to offer the best compromise between output accuracy and computational cost and thus will be the sole focus of this paper. In particular, the authors' attention will fall on the development of a general three-dimensional beam-column formulation where both the beam equilibrium and the inter-fiber equilibrium are respected, given the almost complete absence of proposals in this topic up to the present date.

2. EXISTING MODELS

In the current chapter some of the most important features of beam-column formulations will be briefly addressed, underlining those characteristics that are suitable for earthquake engineering modelling. The classical finite element method is based on the exact satisfaction of the strain-displacement and displacement boundary conditions, while the arguably even more fundamental differential equations of equilibrium in the interior and the natural boundary conditions are only satisfied in the limit as the number of elements increase. It is recalled that the local form equations governing the spatial variation of the stress tensor **S** in any portion of a three-dimensional body are:

$$\begin{cases} \operatorname{div}\mathbf{S} + \mathbf{b} = 0 \\ \mathbf{S}^{\mathrm{T}} = \mathbf{S} \end{cases} \tag{2.1}$$

where **b** is the vector of body forces. It will be shown in this paper that models accounting for the shear behaviour of members, even recent proposals, frequently disregard equilibrium considerations. Such simplifying assumptions obviously imply a loss of accuracy of the model and will be analysed in the following two sections.

2.1. Element Formulation and Inter-section Equilibrium

A beam is a 3D body which, due to its geometrical characteristics, is amenable to a reduction from three to one dimension, in terms of the governing differential equations. In what concerns equilibrium, the well-know beam differential equations are usually obtained from the equilibrium of an infinitesimal portion of the beam, subjected to a general loading combination. This approach has the advantage of providing a direct, simple and physically meaningful interpretation of the equations of equilibrium of a beam, in function of common engineering quantities such as axial force, moment and shear force. The drawback of such procedure is that the relation with the local form of equilibrium – Eqn. 2.1 – is lost. To find an explicit connection between the beam equations and the previous solid mechanics relations two different paths can be followed.

The first one is by enforcing the local form of equilibrium in an average sense over the cross section:

$$\begin{gathered} \int_{\Omega} \left(\operatorname{div}\mathbf{S} + \mathbf{b}\right) dA = \mathbf{0} \\ \int_{\Omega} \mathbf{p} \times \left(\operatorname{div}\mathbf{S} + \mathbf{b}\right) dA = \mathbf{0} \end{gathered} \tag{2.2}$$

where $\mathbf{p}_{(y,z)} = y\mathbf{e}_y + z\mathbf{e}_z$ is the sectional position vector relative to the beam (x) axis and $\mathbf{\Omega}$ stands for the cross section. If the previous expressions are developed using some basic mathematical tools and the customary definition of section generalized forces, it can be proved that the original beam equilibrium equations are completely recovered.

The previous approach, which shows that the beam equilibrium equations can be obtained by an "averaging" operation over the cross section of its more general three-dimensional counterpart, can be further and more thoroughly understood through the projection of the local equilibrium equations in a space of a pre-determined displacement field.

In fact, note that for a general $\delta\mathbf{u}$ the following equivalence holds:

$$\operatorname{div}\mathbf{S}=0 \Leftrightarrow \int_V \left(\operatorname{div}\mathbf{S}\cdot\delta\mathbf{u}\right)dv=0 \Leftrightarrow \int_L \left(\int_\Omega \left(\operatorname{div}\mathbf{S}\cdot\delta\mathbf{u}\right)dA\right)dx=0 \tag{2.3}$$

The term between parentheses is the so-called equilibrium residual $R_{(x)}$ of a differential element of beam. Now consider the following displacement field corresponding to the plane-section (PS) hypothesis of the Euler-Bernoulli beam theory:

$$\mathbf{u}^{PS}_{(x,y,z)}=\mathbf{N}^{PS}_{(y,z)}\mathbf{d}_{(x)}=\begin{bmatrix}1&0&0&0&z&-y\\0&1&0&-z&0&0\\0&0&1&y&0&0\end{bmatrix}\begin{bmatrix}u_0\\v_0\\w_0\\\theta_x\\\theta_y=-w_0'\\\theta_z=v_0'\end{bmatrix} \tag{2.4}$$

where $\mathbf{d}_{(x)}$ is the vector of the generalized section displacements. If this displacement field is used as the projection space of Eqn. 2.3 it can be checked that zeroing the beam equilibrium residual yields:

$$\boxed{R^{PS}_{(x)}=0 \Leftrightarrow \delta\mathbf{d}^T\int_\Omega\left(\begin{bmatrix}\sigma_x'\\\tau_{xy}'\\\tau_{xz}'\\-z\tau_{xy}'+y\tau_{xz}'\\z\sigma_x'\\-y\sigma_x'\end{bmatrix}-\begin{bmatrix}0\\0\\0\\0\\\tau_{xz}\\-\tau_{xy}\end{bmatrix}\right)dA=0} \tag{2.5}$$

Making use of the traditional definitions of section generalized forces, it is readily noted that the previous set of equations corresponds to the beam equilibrium, but limited to the case of constant shear. It is noted, however, that the full original beam equilibrium equations are recovered if the projection space is the one corresponding to the Timoshenko beam theory.

Independently of how these equilibrium relations are obtained, it is to be underlined that the development of element formulations that satisfy them is of the biggest concern for earthquake engineering modelling purposes. This is so since RC frame structures under seismic action are expected to undergo highly inelastic behaviour, which cannot be accurately captured by the fixed displacement shape functions of the classical finite element method (stiffness-based approach). Such intent of verifying this "inter-section" equilibrium has been pursued more acutely throughout the last 15 years, leading to the advancement of the so-called nonlinear flexibility methods. The comparison between these two different approaches is discussed elsewhere [Correia *et al.*, 2008].

2.2. Sectional Formulation and Shear-Normal Stress Interaction

The verification of an "average measure" of the local form of equilibrium, examined before, obviously does not guarantee that it will hold at a pointwise level within the section. Unfortunately, it has been shown and it is now generally recognized that such "intra-section" or "inter-fibre" equilibrium strongly influences the shear stress and strain distribution in the section ([Vecchio and Collins, 1988], [Ranzo, 2000], [Mazars *et al.*, 2006], [Gregori *et al.*, 2007]). Thus, if an accurate modelling of the shear (or shear-torsional, for the 3D case) behaviour of RC members is to be achieved, than this shear-normal stress interaction has to be taken into account. The simpler two-dimensional beam serves as an illustration of the state-of-the-art shear modelling approaches regarding this issue.

Unfortunately, the most common assumption – even in recent publications ([Mazars *et al.*, 2006], [Ceresa *et al.*, 2006], [Gregori *et al.*, 2007]) – is to assume a fixed shear strain γ_{xy} or shear stress τ_{xy} pattern along the section, which effectively means that the intra-section equilibrium is neglected. In the latter situation a constant stress pattern is frequently considered. In the former case, the two alternatives that are customarily adopted are those of a constant shear strain profile (coinciding with the hypotheses of the Timoshenko beam theory) or a parabolic one (which satisfies the inter-fibre equilibrium in the case of a beam with a rectangular section and linear elastic constitutive behaviour). There are some advantages in the consideration of a fixed shear strain pattern over the fixed shear stress one, like the possibility of verifying the compatibility of the section warping. More importantly, it is highlighted that the numerical implementation of the sectional model only requires a direct procedure, unlike the formulations that account for the shear-normal stress interaction. This issue will become clearer in the next two paragraphs.

The state-of-the-art 2D beam formulations that consider the local form of equilibrium are based on the following rationale:

$$\operatorname{div}\mathbf{S}=0 \Leftrightarrow \begin{cases} \dfrac{\partial\sigma_x}{\partial x}+\dfrac{\partial\tau_{xy}}{\partial y}=0 & \rightarrow \quad \tau_{xy(y)}=-\dfrac{1}{b(y)}\displaystyle\int_{-c}^{y}\dfrac{\partial\sigma_x}{\partial x}b(s)\,ds \\ \dfrac{\partial\tau_{yx}}{\partial x}+\dfrac{\partial\sigma_y}{\partial y}=0 & \rightarrow \quad \left(\sigma_y=0\right) \end{cases} \tag{2.6}$$

The second equation of equilibrium above is disregarded based on the common assumption of no total stress in the transverse direction (implying that the components ε_y can be statically condensed). However, the first condition shows that the shear stress can be computed through the integration of the derivatives of the normal stress in relation to the beam axis. Such shear stress distribution has to be the same than the one resulting from the assumed shear strain pattern through the RC constitutive relations. This effectively means that iterations in the shear strain profile should be performed until both conditions are satisfied. The first proposal accounting for this intra-section equilibrium was that of Vecchio and Collins [1988], who computed the integrand in Eqn. 2.6. by a finite difference estimation obtained from a flexural analysis of two different sections. This "dual section analysis" was again employed by Ranzo [2000], while simultaneously Bentz [2000] implemented the truly sectional version of it, which became known as "longitudinal stiffness method".

2.3. Comparative Review and Motivation for the Development of a New 3D Model

The previous section showed how the inter-fibre equilibrium was included in some limited research studies regarding the planar behaviour of beams. Naturally, an accurate structural assessment requires the consideration of the behaviour of RC members subjected to the more general combination of axial force, bi-directional shear, bi-directional bending moment and torsional moment. It should be rather obvious that the importance of the continuous adjustment of the shear distribution throughout the loading range greatly increases for the previous 3D situation. In addition, the inclusion of cyclic modelling capabilities further adds to this observation.

Unfortunately, it can be seen in Table 2.1 that, up to now, there is only one model that simultaneously considers the full 3D behaviour along with the inter-fibre equilibrium [García, 2005]. The table also depicts those models that verify the inter-section equilibrium, of relevance for earthquake engineering modelling purposes as mentioned in section 2.1.

Table 2.1 Comparison of different shear modelling proposals

	2D: N - M_z - V_y 3D: N - M_y - M_z - V_y - V_z - T	**INTER-SECTION EQUIL.** FB: Flexibility-based SB: Stiffness-based	**INTRA-SECTION EQUILIBRIUM**
Vecchio and Collins [1988]	N - M_z - V_y	Not addressed	YES ("Dual-section")
Rahal and Collins [1995, 1996, 2003]	N - M_y - M_z - V_y - V_z - T	Not addressed	YES (simplified)
Ranzo and Petrangeli [1998]	N - M_z - V_y	FB	NO
Martinelli [1998, 2000]	N - M_y - M_z - V_y - V_z - T	SB	NO
Petrangeli *et al.* [1999a, 1999b]	N - M_z - V_y	FB	NO
Ranzo [2000]	N - M_z - V_y	FB	YES ("Dual-section")
Bentz [2000]	N - M_z - V_y	Not addressed	YES ("Longitudinal Stiff. Met.")
Remino [2004]	N - M_z - V_y	FB	NO
Mazars and Kotronis [2005, 2006]	N - M_y - M_z - V_y - V_z - T	SB	NO
Marini and Spacone [2006]	N - M_z - V_y	FB	NO
Ceresa *et al.* [2006]	N - M_z - V_y	SB	NO
Gregori *et al.* [2007]	N - M_y - M_z - V_y - V_z - T	SB	NO
García [2005, 2006a, 2006b, 2007a, 2007b]	N - M_y - M_z - V_y - V_z - T	Not clear	YES

The model proposed in the present work is based on the framework of García [2005], although a significant number of different features are introduced.

3. PROPOSED MODEL

3.1. Displacement Decomposition and Equilibrium Equations

The model of García [2005] is based on the equilibrium residual of a differential beam element approach outlined in section 2.1. As it was referred, projecting the equations of equilibrium on the reduced space of the displacement field corresponding to the Euler-Bernoulli beam theory is equivalent to the reduction from the 3D set of equations to the usual one-dimensional beam equilibrium. Thus, if the full 3D interaction is to be respected a completely general displacement space has to be considered. The warping-distortion component field adding to $\mathbf{u}^{PS}$ to form the more general three-dimensional displacement vector $\mathbf{u}$ is denoted as $\mathbf{u}^{w}$:

$$\mathbf{u} = \mathbf{u}^{PS} + \mathbf{u}^{w} = \mathbf{N}^{PS}_{(y,z)}\mathbf{d}_{(x)} + \mathbf{N}^{w}_{(y,z)}\mathbf{d}^{w}_{(x)} \quad (3.1)$$

where $\mathbf{N}^{w}$ is the (3 x n_d) matrix containing the n_d orthogonalized sectional deformation mode shapes $\mathbf{N}_i^{w}{}_{(y,z)}$ corresponding to the adopted cross-section mesh and $\mathbf{d}^{w}{}_{(x)}$ is the associated vector of n_d modal amplitudes. This is a similar approach to the one adopted in the generalized beam theories [Davies and Leach, 1994].

So that trivial solutions for our problem are avoided, the information contained by the component $\mathbf{u}^{w}$ should not repeat that which is already included in the component $\mathbf{u}^{PS}$. From a mathematical perspective, this corresponds to saying that both vectors should be orthogonal within the element section. However, recalling that the rotations θ_y and θ_z in the plane section hypothesis (Eqn. 2.4) are dependent of the derivatives of the transversal displacements of the section w_0 and v_0, the former degrees of freedom should not be included in the orthogonalization procedure. That is, $\mathbf{u}^{w}$ should comprise the displacement field corresponding to rigid rotations around axes y and z, which will then obviously add to the displacements associated to the classical Euler-Bernoulli assumption, $\theta_y = -w_0'$ and $\theta_z = v_0'$. In other words, only the rigid body movements of translation along the three axes and the rotation around the longitudinal beam axis should not be enclosed in $\mathbf{u}^{w}$.

In the proposed model, and unlike the one of García [2005], the shape functions $\mathbf{N}_i^w$ that define $\mathbf{u}^w$ are computed such that the previous conditions are satisfied *a priori*, which means that each $\mathbf{N}_i^w$ verifies:

$$\int_\Omega \left(\mathbf{N}_{(y,z)}^{Rigid}\right)^T \mathbf{N}_i^w dA = \mathbf{0} \Leftrightarrow \int_\Omega \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -z \\ 0 & 0 & 1 & y \end{bmatrix}^T \mathbf{N}_i^w dA = \mathbf{0} \tag{3.2}$$

It can be shown that this can be accomplished by subtracting the mentioned rigid body displacements from the original sectional deformation mode shapes $\mathbf{N}_i^{w@}$ in the following way:

$$\mathbf{N}_i^w = \mathbf{N}_i^{w@} - \mathbf{N}_{(y,z)}^{Rigid}\left[\int_\Omega \left(\mathbf{N}_{(y,z)}^{Rigid}\right)^T \mathbf{N}_{(y,z)}^{Rigid} dA\right]^{-1} \int_\Omega \left(\mathbf{N}_{(y,z)}^{Rigid}\right)^T \mathbf{N}_i^{w@} dA \tag{3.3}$$

It is now possible to apply the equilibrium residual approach of section 2.1 using the more general displacement field depicted in Eqn. 3.1 as the projection space of Eqn. 2.3, which means that a residual R^w on the warping-distortion space will add to that on the plane-section space (R^{PS}, Eqn. 2.5):

$$R_{(x)} = \int_\Omega \left(\operatorname{div}\mathbf{S} \cdot \delta\mathbf{u}\right) dA = \int_\Omega \left(\operatorname{div}\mathbf{S} \cdot \delta\mathbf{u}^{PS}\right) dA + \int_\Omega \left(\operatorname{div}\mathbf{S} \cdot \delta\mathbf{u}^{w}\right) dA = R_{(x)}^{PS} + R_{(x)}^{w} \tag{3.4}$$

If both residuals R^w and R^{PS} are set to be equal to zero then obviously the complete set of equilibrium equations is recovered. This is the ultimate goal of the present formulation and it will be briefly presented in the remaining pages of the paper.

3.2. Constitutive Relations and Residual in the Warping-Distortion Field

Although discussions on the chosen smeared constitutive model for RC are avoided due to space limitations, it is assumed that a tangent constitutive tensor $\mathbf{D}^{tan}$ is available so that the following relation holds between the stress vector increment $\Delta\boldsymbol{\sigma}$ and the strain vector increment $\Delta\boldsymbol{\varepsilon}$:

$$\Delta\boldsymbol{\sigma} = \Delta\begin{bmatrix}\sigma_x & \sigma_y & \sigma_z & \sigma_{xy} & \sigma_{xz} & \sigma_{yz}\end{bmatrix} = \mathbf{D}^{tan}\Delta\boldsymbol{\varepsilon} = \mathbf{D}^{tan}\Delta\begin{bmatrix}\varepsilon_x & \varepsilon_y & \varepsilon_z & \gamma_{xy} & \gamma_{xz} & \gamma_{yz}\end{bmatrix} \tag{3.5}$$

It is also noted that the strain vector corresponding to the displacement $\mathbf{u}$ can be written down as:

$$\boldsymbol{\varepsilon} = \begin{bmatrix}\varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz}\end{bmatrix} = \mathbf{L}\mathbf{u} = \left(\mathbf{L}_x + \mathbf{L}_{yz}\right)\mathbf{u} = \left(\mathbf{E}_x \frac{\partial}{\partial x} + \mathbf{L}_{yz}\right)\mathbf{u} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\frac{\partial}{\partial x} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & 0 & 0 \\ \partial/\partial z & 0 & 0 \\ 0 & \partial/\partial z & \partial/\partial y \end{bmatrix}\right)\mathbf{u} \tag{3.6}$$

Making use of the displacement decomposition shown in Eqn. 3.1, the previous expression can be developed into:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{PS} + \boldsymbol{\varepsilon}^{w} = \mathbf{L}\mathbf{u}^{PS} + \mathbf{L}\mathbf{u}^{w} = \mathbf{B}^{PS*}_{(y,z)}\mathbf{e}^{*}_{(x)} + \mathbf{E}_{x}\mathbf{N}^{w}_{(y,z)}\frac{\partial}{\partial x}\mathbf{d}^{w}_{(x)} + \mathbf{L}_{yz}\mathbf{N}^{w}_{(y,z)}\mathbf{d}^{w}_{(x)} = \mathbf{B}^{PS*}_{(y,z)}\mathbf{e}^{*}_{(x)} + \mathbf{B}^{w}_{x}\mathbf{d}^{w}_{(x)}{}' + \mathbf{B}^{w}_{yz}\mathbf{d}^{w}_{(x)} \quad (3.7)$$

where

$$\mathbf{B}^{PS*}_{(y,z)} = \begin{bmatrix} 1 & 0 & z & -y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -z & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}^{*}_{(x)} = \begin{bmatrix} \varepsilon_0 \\ \phi_x \\ \phi_y \\ \phi_z \end{bmatrix} = \begin{bmatrix} u_0{}' \\ \theta_x{}' \\ \theta_y{}' \\ \theta_z{}' \end{bmatrix} \quad (3.8)$$

Taking the derivative in order to the beam coordinate axis x, one obtains:

$$\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}^{PS}{}' + \boldsymbol{\varepsilon}^{w}{}' = \mathbf{B}^{PS*}_{(y,z)}\mathbf{e}^{*}_{(x)}{}' + \mathbf{B}^{w}_{x}\mathbf{d}^{w}_{(x)}{}'' + \mathbf{B}^{w}_{yz}\mathbf{d}^{w}_{(x)}{}' \quad (3.9)$$

Coming back to the residual in the warping-distortion field, R^w, note that it can be recasted as:

$$R^{w}_{(x)} = \int_{\Omega}\left(\operatorname{div}\mathbf{S}\cdot\delta\mathbf{u}^{w}\right)dA = \int_{\Omega}\left(\delta\mathbf{u}^{w}\right)^{T}\mathbf{L}^{T}\boldsymbol{\sigma}\,dA = \int_{\Omega}\left(\delta\mathbf{u}^{w}\right)^{T}\mathbf{E}^{T}_{x}\boldsymbol{\sigma}'\,dA - \int_{\Omega}\left[\mathbf{L}_{yz}\left(\delta\mathbf{u}^{w}\right)\right]^{T}\boldsymbol{\sigma}\,dA \quad (3.10)$$

The previous equation can be expanded making use of Eqn.'s 3.5, 3.7 and 3.9, originating a final expression for the nullification of R^w with the following form:

$$\boxed{R^{w}_{(x)} = 0 \Leftrightarrow \mathbf{A}_{e0}\mathbf{e}^{*} + \mathbf{A}_{e1}\mathbf{e}^{*}{}' + \mathbf{A}_{d0}\mathbf{d}^{w} + \mathbf{A}_{d1}\mathbf{d}^{w}{}' + \mathbf{A}_{d2}\mathbf{d}^{w}{}'' = \mathbf{0}} \quad (3.11)$$

It can be readily proven that the matrices $\mathbf{A}$ in the above equation are integrals over the cross-section of the product of a combination of the following matrices: $\mathbf{N}^w$, $\mathbf{E}_x$, $\mathbf{D}^{tan}$, $\mathbf{B}^{PS*}$, $\mathbf{B}_x{}^w$ and $\mathbf{B}_{yz}{}^w$. If one considers Eqn. 3.11 one obtains the linear combination of modal amplitudes that must be respected in order to represent a displaced configuration that is both compatible and in equilibrium,

3.3. Adopted Formulation for the Element and Sectional Level

As it was mentioned in the end of section 3.1, the objective of the present formulation is to verify both Eqn.'s 2.5 and 3.11, thus guaranteeing the full equilibrium. In the present work, the common definition of section generalized forces is adopted, as well as a nonlinear flexibility method. Such equilibrium approach implies that the beam equilibrium is immediately satisfied. The purposed element state determination centres on an iterative solution algorithm based on the transference of residual deformations from the section level to the element level. This iterative nested phase was first proposed by Taucer et al. [1991], being also explained in Correia *et al.* [2008]. For the verification of Eqn. 3.11 an additional hypothesis is considered, which is the assumption that $\mathbf{d}^w$ is locally constant, i.e. $\mathbf{d}^w$ is a step function of x, being constant in each integration point. Hence, $\mathbf{d}^{w}{}' = \mathbf{d}^{w}{}'' = \mathbf{0}$ and thus the vector of warping-distortion nodal values $\mathbf{d}^w$ can be obtained from the knowledge of $\mathbf{e}^*$ and $\mathbf{e}^*{}'$:

$$\mathbf{d}^{w} = -\mathbf{A}^{-1}_{d0}\left(\mathbf{A}_{e0}\mathbf{e}^{*} + \mathbf{A}_{e1}\mathbf{e}^{*}{}'\right) \quad (3.12)$$

The equivalent shear distortions for the cross-section γ_{0y} and γ_{0z} are obtained at each iteration by imposing the power conjugacy between the vector of generalized forces $\mathbf{s} = [N\ V_y\ V_z\ T\ M_y\ M_z]^T$ and the complete vector of generalized deformations $\mathbf{e} = [\varepsilon_0\ \gamma_{0y}\ \gamma_{oz}\ \phi_x\ \phi_y\ \phi_z]^T$ so that $\mathbf{s}\cdot\mathbf{e} = \int_{\Omega} \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\, dA$. The implementation of the present model is still in progress. However, due to space restrictions, not even simple results of linear elastic applications are included herein.

4. CONCLUSIONS

Distributed inelasticity line elements, the behaviour of which is obtained from the integration of the response at several sectional locations, offer the best compromise between model accuracy and computational cost for the analysis of RC structures. The review of the state-of-the-art models of this category that account for shear deformation has shown that equilibrium considerations are frequently neglected, either from the 'inter-section' perspective or the 'intra-section' viewpoint. Additionally, the few proposals that account for shear-normal stress interaction focus solely on bi-dimensional models, which are considerably simpler. In the present paper a theoretical formulation of a three-dimensional beam-column element verifying the local form of equilibrium and power conjugacy is developed. Equilibrium is strictly verified at the element level through the use of a flexibility based formulation, while at the section level a combination of warping-distortion shape functions is determined in order to nullify the corresponding equilibrium residual.

5. ACKOWLEDGMENTS

The authors wish to thank the financial support provided by the Fundação para a Ciência e Tecnologia through the funding program Programa Operacional Ciência e Inovação 2010.

REFERENCES

Bentz, E.C. (2000). Section analysis of RC members. *Ph.D.Thesis*. University of Toronto, Toronto, Canada.

Ceresa, P., Petrini, L., Pinho, R. and Auricchio, F. (2006). Development of a flexure-shear fibre beam-column element for modelling of frame structures under seismic loading. *Proceedings of the First European Conference on Earthquake Engineering and Seismology,* Paper no. 1425.

Correia, A.A., Almeida, J.P. and Pinho, R. (2008). Force-based versus displacement-based formulations in the cyclic nonlinear analysis of reinforced concrete frames. *Proceedings of the 14 World Conference on Earthquake Engineering*, Paper no. 14-0164.

Davies, J.M., Leach, P. (1994). First-order generalised beam theory. *Journal of Constructional Steel Research* **31:2**, 187-220.

García, J.B. (2005). A non-linear coupled model for the analysis of reinforced concrete sections under bending, shear, torsion and axial forces. *Ph.D. Thesis.* Universitat Politècnica de Catalunya, Barcelona, Spain.

Gregori, J.N., Sosa, P.M., Prada, M.A.F. and Filippou, F.C. (2007). A 3D numerical model for reinforced and prestressed concrete elements subjected to combined axial, bending, shear and torsion loading. *Engineering Structures* **29**, 3404-3419.

Mazars, J., Kotronis, P., Ragueneau, F. and Casaux, G. (2006). Using multifiber beams to account for shear and torsion. Applications to concrete structural elements. *Computer Methods in Applied Mechanics and Engineering* **195**, 7264-7281.

Rahal, K.N. and Collins, M.P. (2003). Combined torsion and bending in reinforced and prestressed concrete beams. *ACI Structural Journal* **100:2**, 157–165.

Ranzo, G. (2000). Experimental and numerical studies on the seismic performance of beam–column RC structural members subjected to high shear. *Ph.D. thesis.* University of Roma "La Sapienza", Roma, Italy.

Taucer, F., Spacone, E. and Filippou, F. (1991). A fiber beam-column element for seismic response analysis of reinforced concrete structures. *Report No. UCB/EERC-91/17*, Earthquake Engineering Research Center, College of Engineering, University of California, Berkeley, USA.

Vecchio, F.J. and Collins, M.P. (1988). Predicting the response of reinforced concrete beams subjected to shear using modified compression field theory. *ACI Structural Journal* **85:3**, 258–268.