A MATHEMATICAL MODEL FOR DYNAMIC RESPONSE OF A RIGID BLOCK ON A CIRCULAR ARC SLIDING SURFACE

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ABSTRACT :

This paper presents a mathematical model to describe dynamic response of a rigid block on a circular arc sliding surface during an earthquake. The rigid block is under excitations of both horizontal and vertical accelerations. The model can be considered an extension of the well-known Newmark sliding block model. A numerical technique based on a fourth-order Runge-Kutta step-by-step time integration scheme was used to solve the derived dynamic differential equation. A computer program was developed following this numerical technique. Under a given earthquake motion, the sliding acceleration, sliding velocity, and sliding displacement can be obtained for the rigid block on a pre-defined circular arc sliding surface. The model proposed in this paper can find many applications, such as seismic analyses of dams and embankments, protection of historic monuments, and maintenance of art items in museums.

KEYWORDS: Seismic response, dynamic analysis, Newmark method, sliding displacement

1. INTRODUCTION

Lots of engineering problems can be modeled as a rigid body on a pre-defined sliding surface. The well-known Newmark sliding block model has been widely used in evaluation of seismically-induced displacements of earth dams and embankment (Newmark, 1965). In the original Newmark model, the sliding mass is rested on an inclined plane sliding surface under the excitation of a horizontal acceleration of the base motion. To extend the Newmark's concept, a mathematical model is developed for a rigid block on a circular arc sliding surface. Both horizontal and vertical components of the base motion are incorporated in the model.

2. FORMULATION

As illustrated in Figure 1, a rigid block on a circular arc sliding surface is subjected to a horizontal ground acceleration A_h and a vertical ground acceleration A_v. The block has mass M and mass moment of inertia I_c. The mass center is at C. The block slides along the circle arc O_C, the center of the circle. The radius of the circle is denoted by R and the distance between the mass center C and O_C is denoted by R_C. Displacements of the block relative to the base are denoted by x(t) along the arc and y(t) perpendicular to the arc. x(t) is positive in the downslope direction and y(t) is positive in the direction towards O_C. Angular rotation about O_C is denoted by $\Psi(t)$, positive in the counter-clockwise (i.e., the block slides downslope) direction. Let $\theta(t)$ denote, at time t, the inclined angle of the tangential line through point b, which is the point at the bottom of the block on the extended line $O_C C$. Suppose at t = 0, $\theta(0) = \theta_0$.

The inertia forces from the ground excitations and the gravity force are exerted on the block at the mass center, C. They can be expressed as F_t in the tangential direction and F_r in the radial direction (positive as shown in Figure 1).

$$F_t = MG\sin\theta + MA_e\cos\theta \tag{2.1}$$

$$F_r = MG\cos\theta - MA_e\sin\theta \tag{2.2}$$

where $G = A_v + g$, $A_e = -A_h$; and g is acceleration due to gravity.

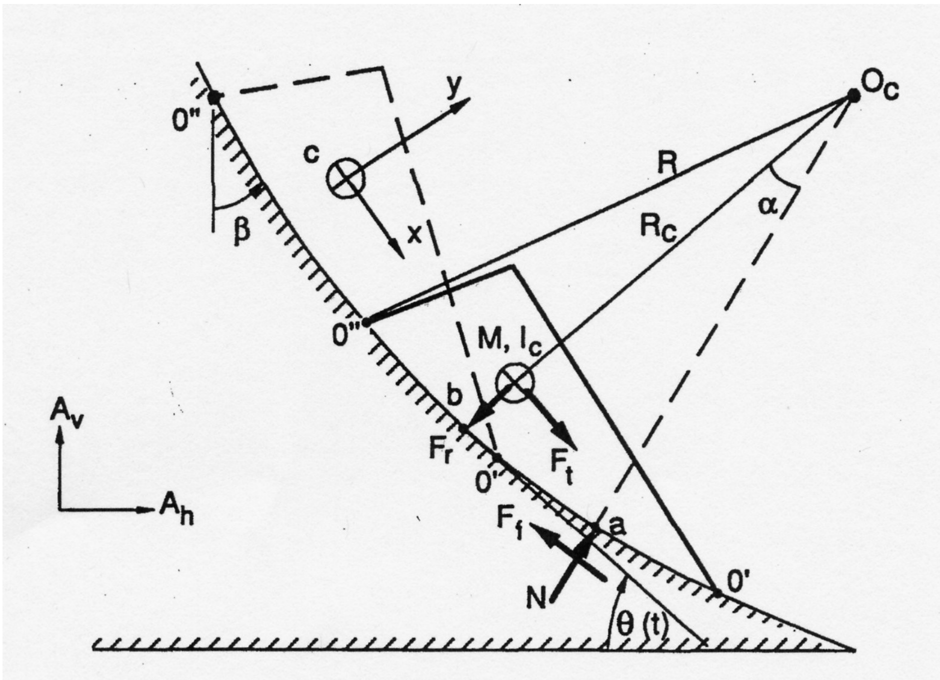


Figure 1 Block on a circular arc

Resistance on the interface between the block and the sliding surface is characterized as a Coulomb-type friction. The coefficients of static friction and dynamic friction are denoted as μ_s and μ_k, respectively. The static friction angle and dynamic friction angle for the interface are denoted as ϕ_s and ϕ_k, respectively. The resultant normal reaction force N from the arc and the frictional resistance force F_f are assumed to be exerted at point a, the location of which will be determined later. The angel between O_CC and O_Ca is α.

2.1. Rest Mode

Equilibrium of force and moment about O_C leads to

$$MG\cos(\theta-\alpha)-MA_e\sin(\theta-\alpha)=N \tag{2.3}$$

$$MG\sin(\theta-\alpha)+MA_e\cos(\theta-\alpha)=F_f \tag{2.4}$$

$$F_fR=(MG\sin\theta+MA_e\cos\theta)R_C \tag{2.5}$$

So

$$G\sin(\theta-\alpha)+A_e\cos(\theta-\alpha)=\frac{R_C}{R}(G\sin\theta+A_e\cos\theta) \tag{2.6}$$

Let

$$\sin\delta=\frac{A_e}{\sqrt{G^2+A_e{}^2}} \tag{2.7}$$

$$\cos\delta=\frac{G}{\sqrt{G^2+A_e{}^2}} \tag{2.8}$$

Then

$$\sin(\theta-\alpha+\delta)=\frac{R_C}{R}\sin(\theta+\alpha) \tag{2.9}$$

$$\delta = \arcsin\left(\frac{A_e}{\sqrt{G^2 + A_e^{\,2}}}\right) \quad (2.10)$$

Therefore

$$\alpha = (\theta + \delta) - \arcsin\left[\frac{R_C}{R}\sin(\theta + \alpha)\right] \quad (2.11)$$

2.2. Onset of Sliding

Assuming the body is initially at rest, a slide mode is initiated once the following expressions holds:

$$N\cos\alpha - F_f \sin\alpha = F_r \quad (2.12)$$

$$N\sin\alpha + F_f \cos\alpha = F_t \quad (2.13)$$

$$F_t R_C = F_f R \quad (2.14)$$

$$\left|F_f\right| \geq N\mu_s \quad (2.15)$$

Therefore

$$N^2 + F_f^{\,2} = F_r^{\,2} + F_t^{\,2} \quad (2.16)$$

$$\left|\frac{R_C}{R}F_f\right| \geq \tan\phi_s \sqrt{F^2 - \left(\frac{R_C}{R}F_t\right)^2} \quad (2.17)$$

or

$$\left|F_t\right| \geq \frac{R}{R_C} F\sin\phi_s \quad (2.18)$$

where

$$F = \sqrt{F_r^{\,2} + F_t^{\,2}} \quad (2.19)$$

Let

$$f_r = F_r / M = G\cos\theta - A_e\sin\theta \quad (2.20)$$

$$f_t = F_t / M = G\sin\theta + A_e\cos\theta \quad (2.21)$$

$$f = F / M = \sqrt{f_r^{\,2} + f_t^{\,2}} \quad (2.22)$$

Then Eqn. 2.18 becomes

$$\left|f_t\right| \geq \frac{R}{R_C} f\sin\phi_s \quad (2.23)$$

or

$$f_t \geq \frac{R}{R_C} f\sin\phi_s;\ \text{for downslope sliding} \quad (2.24)$$

$$f_t \leq -\frac{R}{R_C} f\sin\phi_s;\ \text{for upslope sliding} \quad (2.25)$$

At the instant of onset of sliding, Eqn. 2.12 through 2.14 hold and

$$\left|F_f\right| = N\mu_s \quad (2.26)$$

So

$$N\sin\alpha + N\tan(\phi_s^*)\cos\alpha = \frac{R}{R_C}N\tan(\phi_s^*) \tag{2.27}$$

i.e.

$$\sin(\alpha+\phi_s^*) = \frac{R}{R_C}\sin(\phi_s^*) \tag{2.28}$$

where

$$\phi_s^* = \phi_s \text{ ; if Eqn. 2.24 holds} \tag{2.29}$$

$$\phi_s^* = -\phi_s \text{ ; if Eqn. 2.25 holds} \tag{2.30}$$

Thus

$$\alpha = \arcsin[\frac{R_C}{R}\sin(\phi_s^*)] - \phi_s^* \tag{2.31}$$

2.3. Sliding Mode

Once the block is sliding, the dynamic governing equations of motion are:

$$F_t R_C - F_f R = I_O \ddot{\Psi} \tag{2.32}$$

$$N\cos\alpha - F_f \sin\alpha - F_r = MR_C\dot{\Psi}^2 \tag{2.33}$$

$$F_t - N\sin\alpha - F_f\cos\alpha = M\ddot{\Psi}R_C \tag{2.34}$$

where

$$I_O = I_C + MR_C{}^2 \tag{2.35}$$

$$F_f = N\tan(\phi_k^*) \tag{2.36}$$

$$\phi_k^* = \phi_k S(\dot{\Psi}) \tag{2.37}$$

$$N = M(A_v + g)\cos\theta + MA_h\sin\theta + MR_C\dot{\Psi}^2 \tag{2.38}$$

$$\theta = \theta_0 - \Psi \tag{2.39}$$

$S(\dot{\Psi})$ is the signum function in angular rotation velocity $\dot{\Psi}$.

Since

$$\Psi = \frac{x(t)}{R_C} \tag{2.40}$$

and

$$I_C = MR_i{}^2 \tag{2.41}$$

therefore

$$[1+(\frac{R_i}{R_C})^2]\ddot{x} = f_t - f_n\tan(\phi_k^*)\frac{R}{R_C} \tag{2.42}$$

$$f_n\cos\alpha - f_n\tan(\phi_k^*)\sin\alpha = f_r + \frac{\dot{x}^2}{R_C} \tag{2.43}$$

$$f_t = f_n\sin\alpha + f_n\tan(\phi_k^*)\cos\alpha + \ddot{x} \tag{2.44}$$

where

$$f_n = N/M \tag{2.45}$$

$$\phi_k^* = \phi_k S(\dot{x}) \tag{2.46}$$

From Eqn. 2.43 and Eqn. 2.44, we have

$$\cos(\alpha + \phi_k^*) = \frac{f_r^*}{f_n} \tag{2.47}$$

$$\sin(\alpha + \phi_k^*) = I^* + \frac{f_t^*}{f_n} \tag{2.48}$$

where

$$f_r^* = \cos(\phi_k^*)(f_r + \frac{\dot{x}^2}{R_C}) \tag{2.49}$$

$$I^* = \sin(\phi_k^*)\frac{R}{R_C}/[1+(\frac{R_i}{R_C})^2] \tag{2.50}$$

$$f_t^* = f_t \cos(\phi_k^*)[1 - \frac{1}{1+(R_i/R_C)^2}] \tag{2.51}$$

Eqn. 2.47 and Eqn. 2.48 lead to

$$(\frac{f_r^*}{f_n})^2 + (I^* + \frac{f_t^*}{f_n})^2 = 1 \tag{2.52}$$

or

$$(1-I^{*2})f_n^2 - 2I^* f_t^* f_n - (f_r^{*2} + f_t^{*2}) = 0 \tag{2.53}$$

Solving Eqn. 2.53 yields

$$f_n = \frac{2I^* f_t^* + \sqrt{4I^{*2} f_t^{*2} + 4(1-I^{*2})(f_r^{*2} + f_t^{*2})}}{2(1-I^{*2})} \tag{2.54}$$

Substituting Eqn. 2.54 into Eqn. 2.42 and solving yield the equation for $\ddot{x}$. Also

$$\alpha = \arcsin(I^* + \frac{f_t^*}{f_n}) - \phi_k^* \tag{2.55}$$

The absolute accelerations of the block at the mass center C in the X and Y directions are

$$A_{bh} = A_h + \ddot{x}\cos\theta + \frac{\dot{x}^2}{R_C}\sin\theta \tag{2.56}$$

$$A_{bv} = A_v - \ddot{x}\sin\theta + \frac{\dot{x}^2}{R_C}\cos\theta \tag{2.57}$$

The sliding displacement in the tangential direction, and the absolute accelerations of the block at O" at the upper corner of the block on the sliding surface in the X and Y directions are:

$$d_{O''} = \frac{R}{R_C}x \tag{2.58}$$

$$A_{O''h} = A_h + \frac{R}{R_C}(\ddot{x}\sin\beta + \frac{\dot{x}^2}{R_C}\cos\beta) \tag{2.59}$$

$$A_{O"v} = A_v - \frac{R}{R_C}(\ddot{x}\cos\beta - \frac{\dot{x}^2}{R_C}\sin\beta) \qquad (2.60)$$

where β is the angle of the tangential at O” to the vertical.

Note that the system is not only characterized by the mass and coefficients of friction but also by physical dimensions relating to size and shape of the block and the sliding surface, and the initial position of the block.

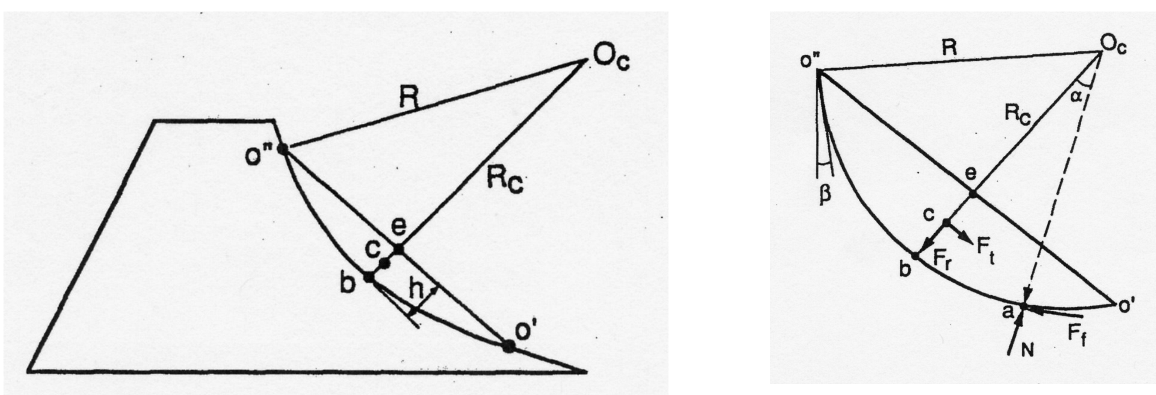
3. COMPUTER PROGRAM DEVELOPMENT

In the development of the computer program, *SLIP-C*, the dynamic coefficient of friction is assumed to be equal to the static coefficient and is taken as a simple constant (ϕ) during sliding. When the base on which the sliding surface is resting is subjected to excitations, A_h and A_v, no displacement will take place until the thresholds Eqn. 2.24 and Eqn. 2.25 are satisfied. From that time until the sliding stops, the motion is obtained by solving the dynamic equations (2.42) through (2.54). Integration of the velocity-time plot yields the sliding displacement of the block.

A fourth-order Runge-Kutta step-by-step time integration scheme (Hildebrand, 1974) was used to solve the dynamic equations. In doing so, these equations are written as a set of two first-order, ordinary differential equations (ODE) in the x and $\dot{x}$.

4. PARAMETRIC STUDIES

The problem concerned is shown in Figure 2, where the sliding block is a circle segment o”bo’eo”.



(a) Geometrical configuration (b) Force diagram

Figure 2 Problem concerned in parametric studies

The independent geometric parameters are R and h/R. Once h/R is known, R/R_C and R_i/R_C can be obtained from

$$\frac{R}{R_C} = \frac{C_a}{C_s} \qquad (2.61)$$

$$(\frac{R_i}{R_C})^2 = \frac{C_i}{C_a}(\frac{R}{R_C})^2 - 1 \quad (2.62)$$

where

$$C_a = \frac{\pi}{2} - (\frac{a}{R})\sqrt{1-(\frac{a}{R})^2} - \arcsin(\frac{a}{R}) \quad (2.63)$$

$$C_s = \frac{2}{3}[1-(\frac{a}{R})^2]^{1.5} \quad (2.64)$$

$$C_i = \frac{\pi}{4} - (\frac{a}{R})\sqrt{1-(\frac{a}{R})^2}[\frac{1}{6}+\frac{1}{3}(\frac{a}{R})^2] - \frac{1}{2}\arcsin(\frac{a}{R}) \quad (2.65)$$

$$\frac{a}{R} = 1 - \frac{h}{R} \quad (2.66)$$

In this problem, the input motion is horizontal excitation only defined by a sinusoidal function, $A_h(t) = -A_m \sin(2\pi ft)$; amplitude $A_m = 0.5g$, frequency f = 2 Hz, and duration $T_d = 0.5$ sec.

R is set to 120 m to study the effect of $\frac{h}{R}$, which is chosen as: 0.05, 0.1, and 0.2. Using the computer program *SLIP-C*, the response of the block is computed and the results are shown in Figure 3. It is found that they are sensitive to $\frac{h}{R}$, especially the angle α (or Alpha in Figure 3a), sliding velocity and sliding displacement.

5. CONCLUSIONS

A mathematical model has been developed to describe dynamic response of a rigid block on a circular arc sliding surface during an earthquake. All the equations have been derived in details and a computer program have been developed. Parametric studies have been performed using the computer program to investigate the effect of a key parameter h/R on the overall dynamic behavior. The developed mathematic model can represent the dynamic response of a rigid block on a circular arc sliding surface during an earthquake. It can find many applications, such as seismic analyses of dams and embankments, protection of historic monuments, and maintenance of art items in museums.

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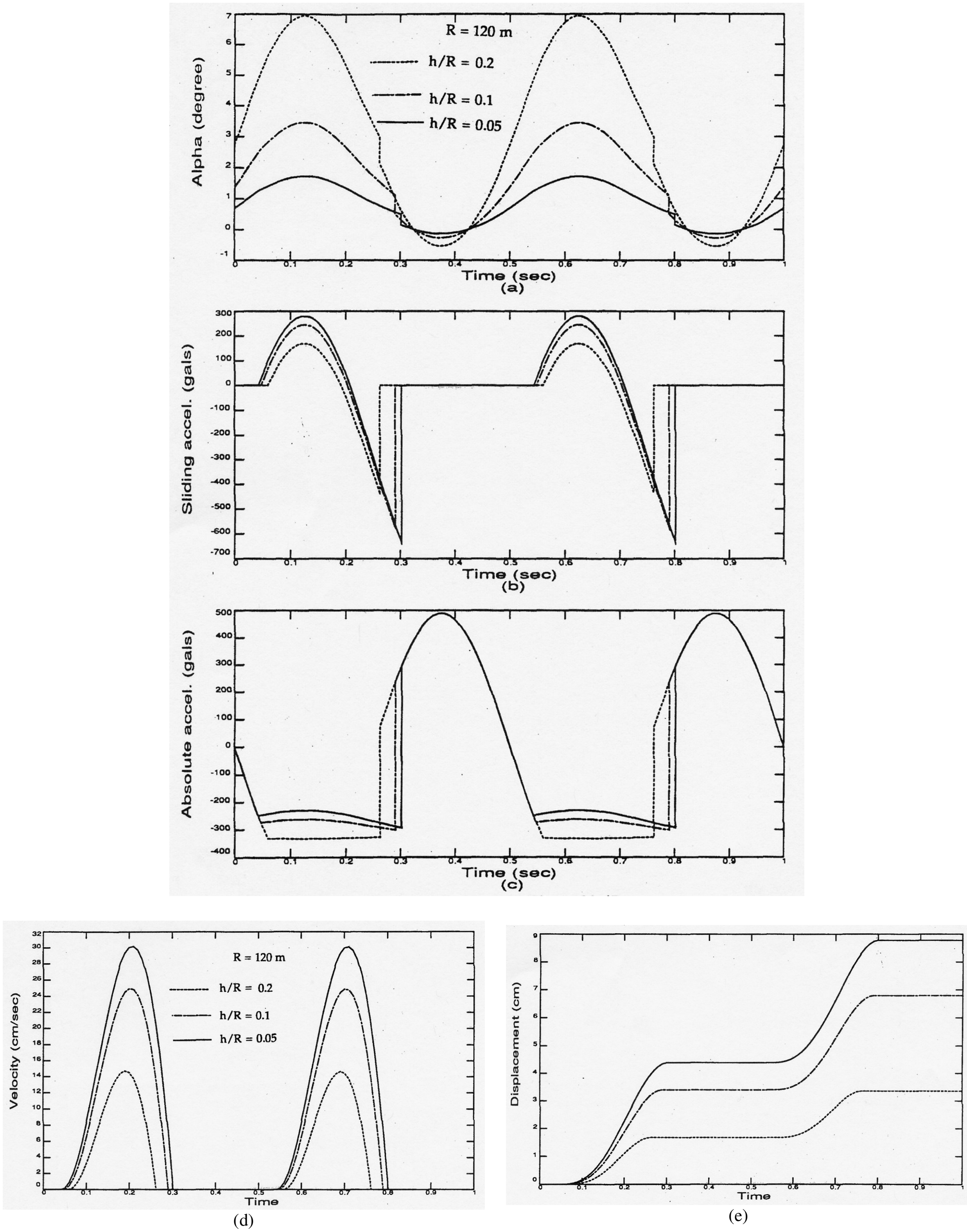

Figure 3 Parametric studies: effect of h/R