STOCHASTIC METHOD TO EVALUATE MAXIMUM RESPONSE VALUE FOR SDOF EXCITED BY INPUT MOTION EXPRESSED AS ARBITRARY SHAPE SPECTRUM

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ABSTRACT:
A stochastic method is proposed to evaluate root mean square (RMS) response displacement, expectation and standard deviation of maximum value of the single degree of freedom (SDOF) system under transient conditions excited by input motion expressed as an arbitrary shape power spectrum density function. Accuracy of the proposed method is examined by comparing with the results of Monte Carlo Simulation (MCS).

KEYWORDS: Threshold Level, Poisson Process, Maximum Response, Non-Stationary Input, Random Vibration Theorem

1. Introduction

The RMS response value generally used in random vibration theorem is difficult to use for the intuitive recognition of analytical results compared with the maximum response value. This is one of the reasons that random vibration theorem does not apply to the design of a structure even though there is high potential. Three methods to evaluate maximum response value are: (1) A method to use non-across probability derived from the Poisson process[1)]. (2) A method to analytically obtain the peak-factor as a ratio of the maximum response value for RMS response value[2,3)]. (3) A method to solve simultaneous differential equations concerning RMS response value and maximum response value defined as state variables[4)]. (1) and (2) are easily handled, however accuracy may not be sufficient under non-stationary conditions because they are based on stationary conditions. (3) has high accuracy and it is possible to obtain covariances excluding mean values, however its theorem is complex and calculation time is too long. In this paper, a new method is proposed that focuses on related numbers of across and non-across probability based on Cartwright's method (1) referenced with the results of MCS. This method may be applied in cases when input motion is expressed as a non-stationary power spectral density function and when the system has strong non-linearity. The SDOF system excited by non-stationary input motion is examined.

2. Statistics of Maximum Displacement for SDOF

2.1 Outline of Cartwright's Method

The probability of absolute displacement across threshold level x_a with number n between duration T is expressed with the following equation (1) using the Poisson process.

$$P(x_a) = \frac{\left(2 \cdot \nu^+(x_a) \cdot T\right)^n}{n!} \cdot \exp\left(-2 \cdot \nu^+(x_a) \cdot T\right) \qquad \text{..... (1)}$$

$\nu^+(x_a)$ is the number of across over threshold level x_a during unit time. Cartwright[1)] denotes that the Cumulative Distribution Function (CDF) of maximum displacement $P(X_{max})$ is approximately equal to the probability of the non-across threshold level x_{max}. i.e. the number of across n set to 0 in above.

$$P(x_{max}) = \exp\left[-\frac{2\cdot T}{T_0}\cdot\exp\left(-\frac{x_{max}^2}{2\cdot\sigma_x^2}\right)\right] \quad ..... (2)$$

σ_x, σ_v are the RMS response values of displacement and velocity. T_0 is the natural period of the system. Davenport[2)] proposed a simple method to obtain the mean and variance of maximum response using the above feature. Vanmarcke[3)] attempted to improve this method to introduce a shape factor. However, these methods both have the same weak point in that accuracy declines when correlativity in each peak occurs over long periods as these theorems are based on the Poisson process. To solve this problem, a method to use envelope distribution is proposed. It focuses on the accuracy of the across rate and does not focus on the probability of across. These methods are subject to stationary conditions and are not adopted for non-stationary conditions including transient response. In this paper, a new method to estimate statistics concerning maximum displacement with high accuracy over long periods is proposed.

2.2 Estimation of RMS Response Value for Linear SDOF System

The RMS Response Value of the Linear SDOF System excited by input motion with an arbitrary shape power spectral density function $S_0(\omega)$ is obtained as follows:

$$\begin{cases} \dot{E}[xx] = 2\cdot E[xv] \\ \dot{E}[xv] = E[vv] - 2\cdot h\cdot\omega\cdot E[xv] - \omega^2\cdot E[xx] - E[xf] \\ \dot{E}[vv] = -4\cdot h\cdot\omega\cdot E[vv] - 2\cdot\omega^2\cdot E[xv] - 2\cdot E[vf] \end{cases} \quad ..... (3)$$

x,v are the response displacement and velocity of the SDOF system, f is input motion acceleration, E[ ] is the operator of expectation, h, ω_0 are the damping factor and natural frequency of the SDOF system and dot denotes differentials for time. RMS responses are $\sigma_x = \sqrt{E[xx]}$, $\sigma_v = \sqrt{E[vv]}$ respectively. E[xf], E[vf] are expressed as product stationary response and envelope function e(t) shown as follows.

$$E[xf] = \frac{1}{2\pi}\cdot e(t)^2\cdot\int_{-\infty}^{\infty} S_0(\omega)\cdot H(i\omega)\cdot d\omega, \quad E[vf] = \frac{1}{2\pi}\cdot e(t)^2\cdot\int_{-\infty}^{\infty} S_0(\omega)\cdot i\omega\cdot H(i\omega)\cdot d\omega \quad ..... (4)$$

$H(i\omega)$ is the displacement transfer function of the SDOF system $(=(\omega_0^2-\omega^2+2\cdot i\cdot h\cdot\omega\cdot\omega_0)^{-1})$. The integrals of the above equations will be performed only once before the calculation of the equation (3). The imaginary part of the integral term is symmetrical to $\omega=0$ and its integral value is real. To evaluate the accuracy of this procedure, analysis of the SDOF system with T=1s, h=0.02 should be conducted. RMS response displacement σ_x is shown in Fig.1 compared with the results of MCS. Both the results of analysis and MCS match well.

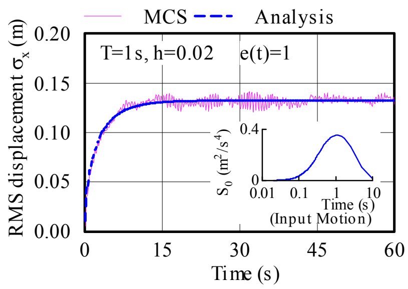


Fig.1 RMS response displacement of SDOF compared with MCS

2.3 Estimation of Statics for Maximum Response Value

(1) Method

The probability that displacement does not go across threshold level x_a during duration is shown as follows[5)], if the Poisson process is available.

$$P(x_a) = \exp\left(-2\cdot\int_0^{t_d} v^+(x_a)\cdot dt\right), \qquad v^+(x_a) = \frac{\sigma_v}{2\pi\cdot\sigma_x}\cdot\exp\left(-\frac{x_a^2}{2\cdot\sigma_x^2}\right) \quad ..... (5)$$

It is known that non-across probability for the maximum response displacement $P(x_a)$ as shown above tends to be close to the CDF of maximum response displacement. To use this feature, the Probabilistic Mass Function (PMF) of maximum response displacement are obtained as shown below. The PMF is defined at discrete points where its amount is 1 and its shape is analogous to the probabilistic density function (PDF).

$$p_m(x_{max}) = P(x_i) - P(x_{i-1}), \qquad i = 1 \cdots n, \qquad x_i = x_0 + \Delta x \cdot (i-1) \qquad ..... (6)$$

x_i is the displacement of discrete points and the start point x_0 and the interval Δx are selected in an appropriate manner. To use this procedure, the mean of maximum response displacement $E[X_{max}]$ and standard deviation $\sigma[X_{max}]$ are obtained as follows. If this procedure is used for a non-linear system, it is necessary that the PDF of displacement and the velocity is close to normal distribution.

$$E[x_{max}] = \sum_{i=1}^{n} x_i \cdot p_m(x_i), \qquad \sigma[x_{max}] = \sqrt{\sum_{i=1}^{n} x_i^2 \cdot p_m(x_i) - E[x_{max}]^2} \qquad ..... (7)$$

(2) Comparison with MCS

The number of across over mean maximum response displacement $N_c(E[x_{max}])$ examined by MCS for 5 different input motions is shown in Fig.2. Power spectrum density function of input motion is indicated at the corner of each graph. Analytical models are the Linear SDOF system with a damping factor of h=0.05. The duration of input motion is 60s and Jennings's envelope function is used. The results of analytical solutions derived from the equation below are shown in Fig.2 compared with MCS to verify the accuracy of the analytically derived $N_c(E[x_{max}])$. Slight flicker of analytical value is caused from scatter of threshold level $E[x_{max}]$ calculated by MCS.

$$N_c(E[x_{max}]) = 2 \cdot \int_0^{t_d} \nu^+(E[x_{mzx}], t) \cdot dt \qquad ..... (8)$$

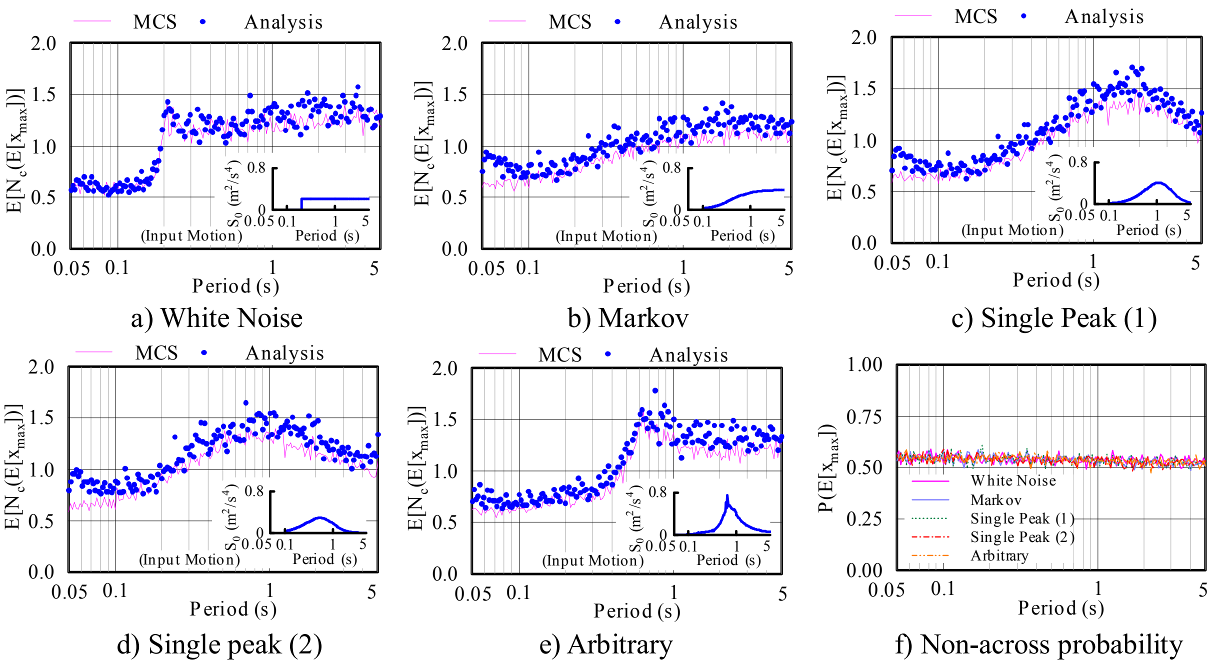


Fig.2 Number of across over mean response displacement $E[N_c(E[x_{max}])]$ and Non-across probability

Equation (8) is equivalent to the inner part of the exponential term in equation (5) and directly relates to the accuracy of non-across probability $P(x_a)$. In Fig.2 a - e, the number of across over mean maximum response displacement $N_c(E[x_{max}])$ is about 0.6 over a short period and increases to 1.0~1.5 as the period lengthens. Both the results of analytical solutions and MCS generally agree. Non-across probability $P(x_a)$ shown in Fig.2 f is about 0.55 and does not have period dependency under any circumstances. The relationship between non-across probability $P(E[x_{max}])$ and number of across $N_c(E[x_{max}])$ is shown in Table.1. $N_c(E[x_{max}])$ is about 0.6 and corresponds to the value 0.598 substituting $P(x_a)=0.55$ to equation (5) for any cases over a short period. However, this relationship is not recognized over long periods because the Poisson process is not approved. The shape of $N_c(E[x_{max}])$ is proportional to the shape of the spectral density function $S_0(\omega)$ of input motion. It is considered that there is some correlation between $N_c(E[x_{max}])$ and input motion.

Table.1 Relation between $P(E[x_{max}])$ and $N_c(E[x_{max}])$ from Eq.(5),(8)

Non-across Probability for $E[x_{max}]$ $P(E[x_{max}])$	Number of Exceed for $E[x_{max}]$ $N_c(E[x_{max}])$
0.368	1.000
0.550	0.598

(3) Correction for Number of Across $N_c(E[x_{max}])$

In acceleration time history when the long period component is dominant, across over threshold level continuously occurs. The Poisson process is applied on the assumption that each across that independently occurs is not approved. In this case, to avoid a reduction in precision, envelope distribution is sometimes used as the PDF of the extreme value of peaks[5)]. However, it is not possible to improve non-across probability $P(x_a)$. In this paper, non-across probability as shown in equation (5) is modified to fit the results of MCS. To use coefficient α as shown in equation (10), a large value of $N_c(E[x_{max}])$ over long periods maintains the same level over short periods. Non-across probability $P(E[x_{max}])$ as shown in equation (5) will be approximately available in all periods. These are shown as follows and in Fig.3.

$$P(x_a) = \exp\left(-2\cdot\int_0^{t_d}\alpha\cdot\nu^+(x_a,t)\cdot dt\right) \quad \text{..... (9)}$$

$$\begin{cases} \alpha = 1.00 & (T_{eq} < 0.1s) \\ \quad (\text{Interpolation } 1.00 \sim 0.45) & (0.1s < T_{eq} < 0.8s), \\ \alpha = 0.45 & (0.8s < T_{eq}) \end{cases} \quad \text{..... (10)}$$

$$T_{eq} = 2\pi\cdot\sigma_x/\sigma_v = 2\pi\cdot\sqrt{E[xx]/E[vv]} \quad \text{..... (11)}$$

The number of across $N_c(E[x_{max}])$ for each of the above cases and coefficient α divided by 0.598 corresponds to the value over short periods. The Poisson process is available as shown in Fig.4. To apply coefficient α, the number of across $N_c(E[x_{max}])$ is modified from a solid line to a broken line.

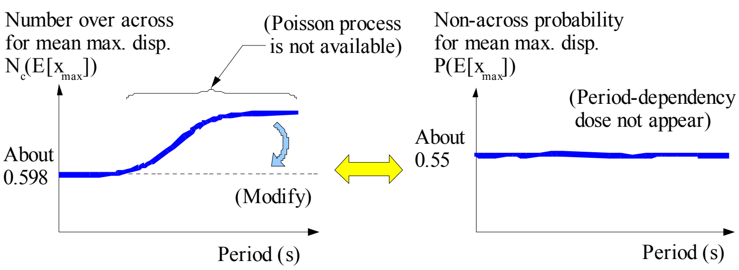


Fig.3 Modifications to number of across

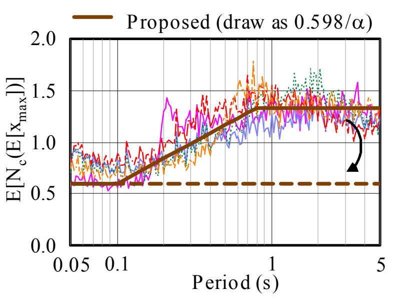
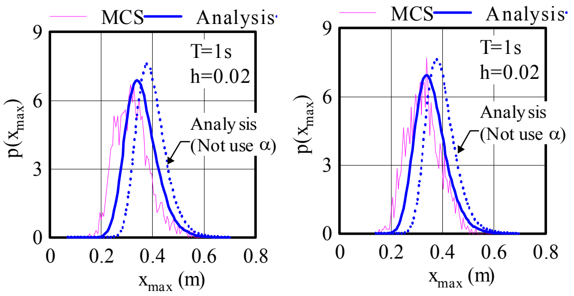
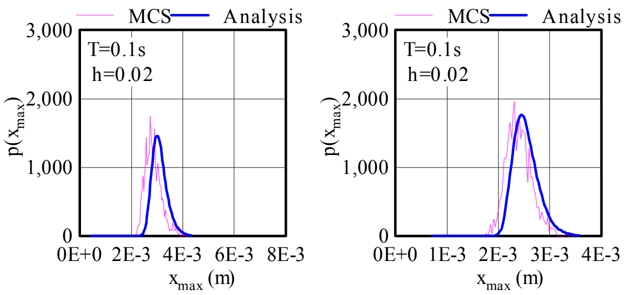


Fig.4 coefficient α

The PDF of maximum displacement converted from PMF for the SDOF system with T=0.1s, 1.0s and h=0.02 using this procedure is shown in Fig.5 and 6. The result of analysis is slightly higher than MCS and both PDF shapes fit well in each case. In Fig.5, the case that is not used with the above coefficient α is also shown. If coefficient α is not used, the differences between analysis and MCS are larger than cases when coefficient α is used.



a) Single peak b) Arbitrary shape
Fig.5 PDF of maximum displacement (T=1s)



a) Single peak b) Arbitrary shape
Fig.6 PDF of maximum displacement (T=0.1s)

3. Statistics of Maximum Displacement for SDOF under Non-stationary Input Motion

3.1 Numerical Results utilizing Past Observed Records

The above procedure to evaluate the mean and variance of maximum displacement will be available for non-stationary input motion and non-linear systems even if the PDF of response displacement and velocity are remarkably different to those of normal distribution. In the case of non-stationary input motion, the precision of this procedure is examined. In this paper, non-stationary property of input motion is expressed as the evolutionary power spectrum (EPS) function as shown below[6].

$$S_0(\omega, t) = 2 \cdot h_0 \cdot \omega^3 \cdot \left\{x^2(t) + v^2(t)/\omega^2\right\} \qquad ..... (12)$$

x(t), v(t) are the displacement and velocity time history of input motion. h_0 is the parameter related to the smoothness of the EPS function and is normally 0.05. The RMS response of the system is obtained to perform the integration of equation (3) after equations (4) replace equations (13) as shown below.

$$E[xf(t)] = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} S_0(\omega, t) \cdot H(i\omega) \cdot d\omega, \quad E[vf(t)] = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} S_0(\omega, t) \cdot i\omega \cdot H(i\omega) \cdot d\omega \qquad ..... (13)$$

Amplitude components extracted from past observed records are used for input motion Hachinohe NS (HCN) that was observed in the 1968 Miyagiken-oki earthquake and Fukushima EW (FKS) that was observed in the 1997 Hanshin great earthquake. EPS function for HCN and FKS obtained from the above procedure is shown in Fig.7 and 8. Non-stationary property where some peaks occur at different times or frequencies is clearly recognized in both records. 3 Samples of acceleration time history synthesized from these EPS functions to use MCS using the equation below are shown in Fig.9.

$$f(t) = \sum_{i=1}^{N} \sqrt{2 \cdot S_0(\omega_i, t) \cdot \Delta\omega/\pi} \cdot \cos(\omega_i \cdot t + \phi_i) \qquad ..... (14)$$

f(t) is the acceleration time history, ω_i is discrete frequency, $\Delta\omega$ is the frequency interval, ϕ_i is phase as a uniform random number with a range of 0~2π, N is the amount of summation cos waves. Time history obtained from this procedure shown in Fig.9 is very similar to the original observed time history shown in Fig.7. Max acceleration value 2.059~2.270m/s^2 is similar to the value of the original record of 2.327m/s^2.

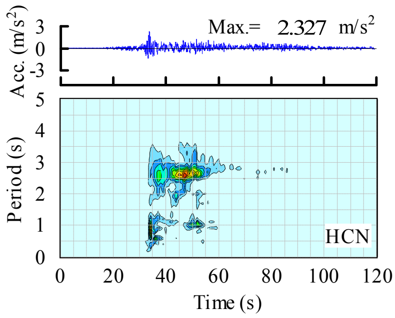


Fig.7 EPS Function (HCN)

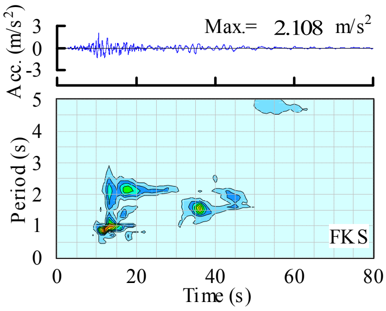


Fig.8 EPS Function (FKS)

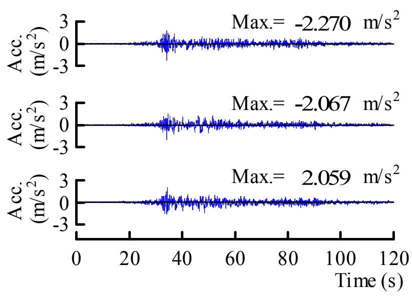
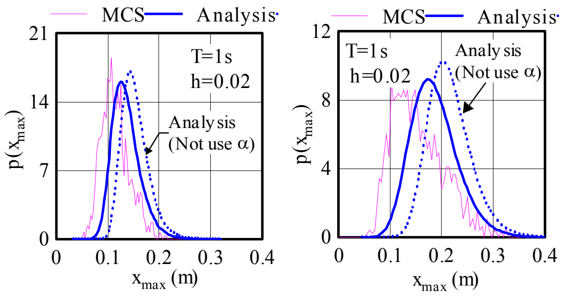


Fig.9 Sample of input (HCN)

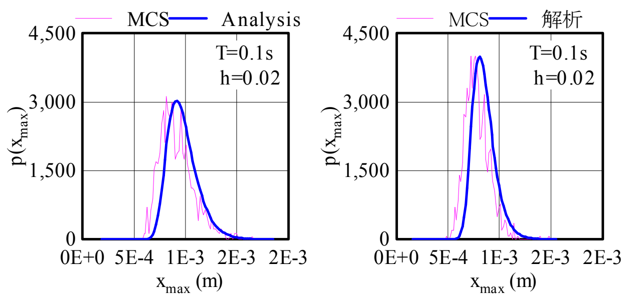
The PDF of maximum displacement converted from PMF for the SDOF system is shown in Fig.10 and 11. The results of analysis are slightly higher than MCS in all cases and the shape of the PDF fits well in all periods. However, when input motion is FKS and the natural period T=1.0s as shown in Fig.10, differences between analysis and MCS for the shape of the PDF are large compared with other cases. In Fig.2, $N_c(E[x_{max}])$ has some small peaks when the period input motion is dominant. Small errors occur at the points of these small peaks because coefficient α is smoothly set as the average of MCS for various input motion spectrums.

The mean and variance of the acceleration response spectrum obtained from the above procedure is shown in Fig.12 compared with the results of MCS. Both the results of analysis and MCS fit well in any period but slight errors occur at the top of peaks. Peak factor is shown in Fig.13. Values of the peak factor for input motions HCN and FKS is about 2 - 3 and tend to decrease with the period the same as in stationary input cases.



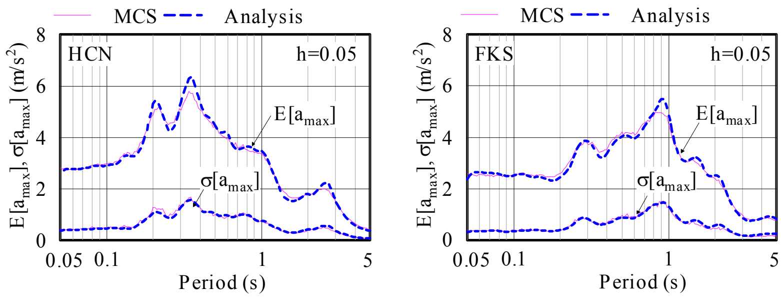
a) HCN b) FKS

Fig.10 PDF of maximum displacement (T=1s)



a) HCN b) FKS

Fig.11 PDF of maximum displacement (T=0.1s)



a) HCN b) FKS

Fig.12 Mean and standard deviation of acceleration response spectrum

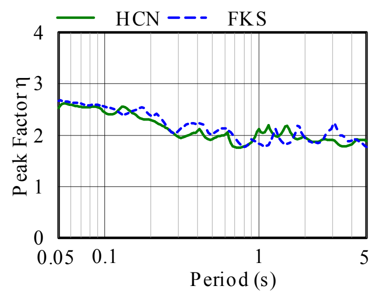


Fig.13 Peak factor

3.2 Numerical Results for Input Motion synthesized from Modeling Faults

The above procedure applies to the input motion case which is synthesized to use modeling faults by the Statistical Green's Function method (SGF) tends to recognize its efficiency by simulation analysis for observed records. Input motion $f_m(\omega)$ is obtained as below with the SGF method[7].

$$f_m(\omega) = H(\omega) * f_e(\omega) = [F_1(\omega) * F_2(\omega)] * f_e(\omega) \tag{15}$$

$f_e(\omega)$ is a small event motion expressed as a ω^{-2} model. $F_1(\omega)$ is a function referring to compounded sub-faults. $F_2(\omega)$ is a function to correct slip time between faults and sub-faults. $*$ denotes products in the frequency domain. When small event motion is expressed as power spectral density function $S_{0e}(\omega)$ and envelope function $e(t)$, EPS function $S_{0m}(\omega)$ for input motion $f_m(\omega)$ is as follows[8].

$$S_{0m}(\omega) = [\Phi(\omega, t) * \Phi^*(\omega, t)] * S_{0e}(\omega) \tag{16}$$

$$\Phi(\omega, t) = \int_0^t e(t) \cdot \tilde{H}(t-u) \cdot e^{i\omega u} du, \qquad H(t) = \Im^{-1}[H(\omega)] \tag{17}$$

$\Im^{-1}[\ ]$ is the inverse fourier transformation. Subscript * denotes complex conjugations. Input motion is synthesized using this procedure and modeling faults are shown in Fig.14. EPS function and RMS acceleration derived from its integral across period axis is shown in Fig.15. The parameters of modeling faults follows in the reference 9.

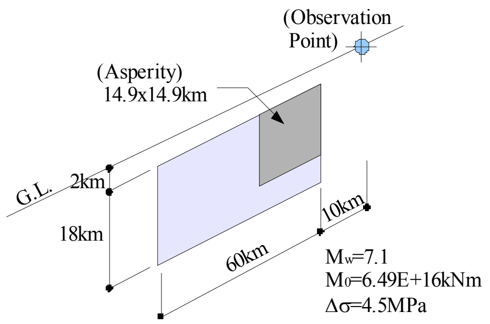


Fig.14 Modeling of Faults

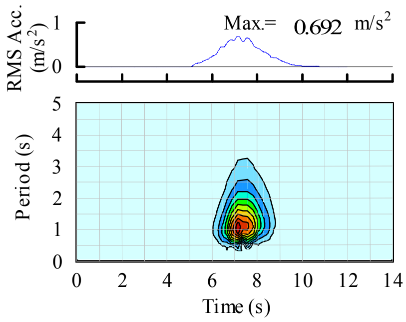
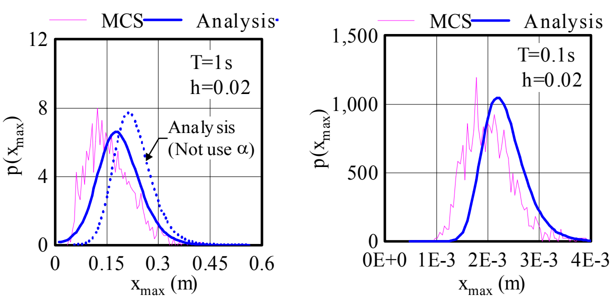


Fig.15 EPS Function (SGF)



a) T=1s b) T=0.1s

Fig.16 PDF of maximum displacement (SGF)

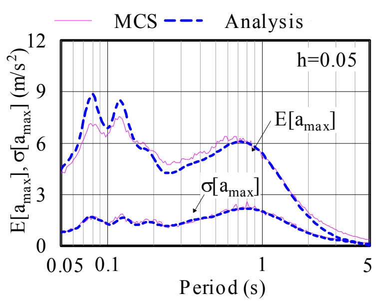


Fig.17 Mean and standard deviation of acceleration response spectrum

The PDF of maximum displacement converted from PMF for the SDOF system is shown in Fig.16. The results of analysis are slightly higher than MCS in all cases and the shape of the PDF fits well in any period. The mean and variance of the acceleration response spectrum obtained from the above procedure is shown in Fig.17 compared with the results of MCS. Both results fit well in any period but slight errors occur at the top of peaks.

4. Conclusion

The proposed method to estimate the mean and variance of maximum displacement excited by input motion expressed as an arbitrary shape power spectrum density function under non-stationary conditions has been presented. Its accuracy is examined by comparing with the results of MCS. The conclusions are shown below.

- The number of across over mean maximum displacement $N_c(E[X_{max}])$ obtained from MCS is about 0.6 corresponding to the value derived from the Poisson process over short periods. However, it has time-dependency and becomes generously large as periods increase. When using the proposed coefficient α based on Cartwright's method to reflect this time-dependency, statistics for maximum displacement are able to be accurately estimated.
- When input motion is expressed as an arbitrary shape power spectrum function and an envelope function, the method to obtain the RMS response of the linear SDOF System is proposed. When using the above procedure, the mean and standard deviation of analysis fit well with the results of MCS.
- When the non-stationary spectrum property of input motion is considered as an evolutionary power spectrum function, statistics for maximum displacement examined by the above procedure also fit well with the results of MCS. However, the mean of maximum displacement may be different at the peak of the input motion spectrum due to coefficient α being smoothly set as an average of MCS for various input motion spectrums.

The proposed method will apply to cases in which the system has strong non-linearity unless the PDF of response displacement or velocity are severely different to normal distribution. It is necessary to find out whether the proposed method is possible or not for non-linearity progress. In this paper, coefficient α that modifies non-across probability when the Poisson process is not available is obtained approximately from an average of MCS. However, this coefficient should be derived from the analytical method to clarify relationships between the number of across and the across mean maximum displacement with input motion spectrum.

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