THREE-DIMENSIONAL SEISMIC ANALYSIS OF MASONRY COMBINED SYSTEMS

N. Augenti[1] and F. Parisi[2]

[1] *Associate Professor, Dept. of Structural Engineering, University of Naples Federico II, Naples, Italy*
[2] *Ph.D. Student, Dept. of Structural Engineering, University of Naples Federico II, Naples, Italy*
Email: augenti@unina.it, fulvio.parisi@unina.it

ABSTRACT:

Recent numerical analyses on some types of planar Masonry Combined Systems (MCSs) showed a great variability of the lateral stiffness along the height, following to the mutual interaction between different parallel structural elements. This may induce significant eccentricities between centre of mass and centre of rigidity, in addition to the ones potentially due to U-shaped walls placed at the perimeter of the building or to irregular in-plan frame-wall dual systems. The mixed structure of the building becomes dramatically prone to damage due to twist-induced displacements, so a 3D seismic analysis is required to account for its torsional response.
In this paper a matrix algorithm for seismic analysis of structural systems formed by masonry or Reinforced Concrete (RC) shear walls (with or without openings), frames and structural cores, is proposed. Such lateral load-resisting elements are considered to be arbitrarily arranged in plan and subjected to a generic pattern of horizontal forces. The torsional stiffness matrix of the cores is built up by means of the non-uniform torsion theory and the stiffness matrix of the whole mixed structure is subsequently defined.
The presented analytical formulation provides a good prediction of the seismic behaviour of torsionally eccentric buildings with MCSs, and allow to evaluate strength and displacement demands.

KEYWORDS: Masonry Combined Systems, 3D seismic analysis, non-uniform torsion theory.

1. INTRODUCTION

The seismic behaviour of the Masonry Combined Systems (MCSs) under lateral forces is characterised by a significant mutual interaction between the lateral load-resisting elements due to a great variability of the stiffness over the height. In this case the seismic codes, at both national and international levels, require a multi-modal dynamic analysis of the building structure. Since the lateral stiffnesses change along the height, the centre of rigidity may drastically vary from one to another storey. On the contrary, the centre of mass change significantly only in particular conditions (i.e., eccentric mass concentrations), so these additional in-plan eccentricities may increase the ones potentially due to cores placed at the perimeter of the building or to non-orthogonal bracing systems. In such cases the seismic response of the structure may have significant torsional components, so 2D analyses on separate planar models along two different directions of the building plan should result in large errors and lack of safety. For this reasons, a 3D seismic analysis is required.
The present work deals with this topic looking to provide an analytical tool, accurate and simple at the same time, that is able to describe the seismic behaviour of torsionally eccentric buildings with MCSs. In particular, the distribution of the horizontal forces between frames, masonry or Reinforced Concrete (RC) shear walls, and structural cores is studied. The 3D analysis is carried out by constructing the stiffness matrix of the whole structure, which includes both the Saint-Venant and warping torsional rigidities of the cores. The torsional stiffness matrix is defined through the non-uniform torsion theory.

2. THE PROPOSED ALGORITHM

The 3D seismic analysis of buildings formed by frames, shear walls and structural cores is performed by applying the direct stiffness method (also called displacement or deformation method), that is particularly suited for computer-automated analyses of statically indeterminate structures.

The algorithm is based on the following hypotheses:
- lateral load-resisting elements mutually connected by rigid diaphragms;
- horizontal forces applied to the centre of mass at each level;
- frames with inextensible beams and columns;
- prismatic solid (masonry or RC) shear walls with both flexural and shear flexibilities;
- opened masonry shear walls discretised in macro-elements by using the RAN method (Augenti 2004);
- prismatic RC cores having thin-walled, open or multi-celled, cross-section;
- equal interstorey heights.

Both the out-of-plane lateral stiffness and the torsional rigidity can be neglected for shear walls and frames; the structural cores can be modelled as cantilever beams with both lateral stiffness along any direction and torsional rigidity. P-Delta effects and axial deformations are also neglected in this study, since they are significant for metallic and tall structures.

In the following treatment $E_{k,p}$ is the generic static or kinematic parameter referred to the p-th storey of k-th wall. Storeys and lateral load-resisting elements are progressively numbered from the top to the base and from the left to the right, respectively. Let us consider the structural system showed in Figure 1, composed by (1) an opened masonry wall, (2) a solid shear wall, (3) a moment-resisting frame, and (4) a thin-walled open cross-section core. We define a Global Coordinate System (GCS), O(x, y, z), having xy-plane overlapped to the p-th floor and z-axis with right-handed positive orientation. Given the generic element k, whose cross-section has the shear centre C_k at the p-th storey of the building, we introduce a Local Coordinate System (LCS), C_k $(\bar{x}_k, \bar{y}_k, \bar{z}_k)$, having $\bar{x}_k, \bar{y}_k$ parallels to the principal inertia axes and $\bar{z}_k$ oriented as the z-axis.

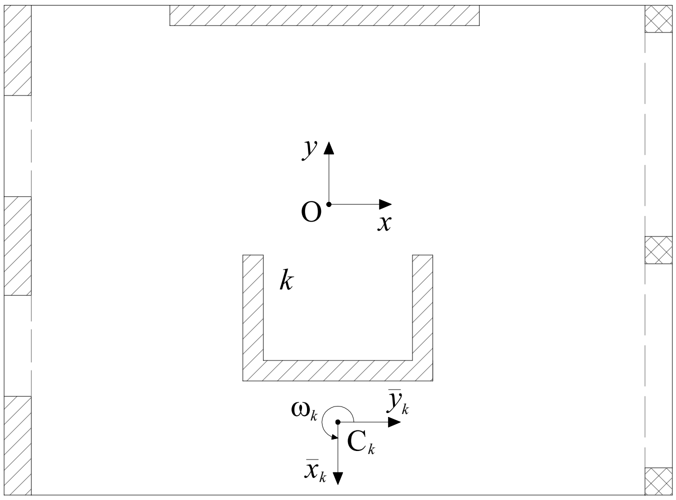

Figure 1. Arrangement of lateral load-resisting elements

In general, $\bar{x}_k, \bar{y}_k$ are rotated of an angle ω_k about the respective x, y axes of the GCS, so a rotation matrix of the structural element have to be defined to transform the coordinates from an orthonormal basis to another one. Assuming the diaphragms to be rigid in their own plane, the displacement of a point lying on the p-th floor is univocally defined by: (1) the horizontal translation along the x-axis, u_p; (2) the horizontal translation along the y-axis, v_p; and (3) the torsional rotation about the z-axis, θ_p.

If the centres of rigidity are not coincident at each level or not lined up in the z-direction, the seismic response of the structure has non-zero torsional component and the lateral-torsional coupling of the bracing elements have to be evaluated by means of a 3D analysis. In this second case, denoting by s the number of storeys, it follows that: (1) the structure has $3s$ Degrees of Freedom (DOFs); (2) external load vector, **Q**, and displacement vector, **S**, have dimensions $3s\times1$; the stiffness matrix, **K**, has size equal to $3s$. In the GCS the distribution of the horizontal forces between the lateral load-resisting elements is governed by the equilibrium matrix equation:

$$\mathbf{K}\,\mathbf{S} = \mathbf{Q} \tag{2.1}$$

which can be specialised to the k-th element and written in the LCS. Both the load and displacement vectors may be decomposed along the three axes, so the problem reduces to the construction of the stiffness matrix of the structure. The $3s\times3s$ rotation matrix of the k-th element may be expressed as follows:

$$\mathbf{\Omega}_k = \left[\begin{array}{c:c:c} \mathbf{cos\omega}_k & -\mathbf{sen\omega}_k & \mathbf{0} \\ \hdashline \mathbf{sen\omega}_k & \mathbf{cos\omega}_k & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{I} \end{array}\right] \tag{2.2}$$

where every $s\times s$ sub-matrix is diagonal. The vector $\mathbf{S}_k$ is related, in turn, to $\mathbf{S}$ by the compatibility matrix:

$$\mathbf{C}_k = \left[\begin{array}{c:c:c:c} \mathbf{c}_{k,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{c}_{k,2} & \cdots & \mathbf{0} \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline \mathbf{0} & \mathbf{0} & \cdots & \mathbf{c}_{k,s} \end{array}\right] \tag{2.3}$$

composed by a number of s^2 square sub-matrices related to each storey p of the element under consideration. The global coordinates of the shear centre appear in the latter matrix and allow the representation of the compatibility matrix through nine $s\times s$ diagonal sub-matrices, by grouping together the coordinates of all the floor cross-sections of the k-th element in the vectors $\mathbf{X}_k$ and $\mathbf{Y}_k$. Finally, the transfer matrix $\mathbf{T}_k$ of such element can then be defined, so the displacement vector in the LCS is (Pozzati 1977):

$$\overline{\mathbf{S}}_k = \mathbf{\Omega}_k\,\mathbf{C}_k\,\mathbf{S} = \mathbf{T}_k\,\mathbf{S} \tag{2.4}$$

Since $\mathbf{T}_k$ is orthogonal, the stiffness matrix of the k-th element is given by:

$$\mathbf{K}_k = \mathbf{T}_k^{\mathrm{T}}\,\overline{\mathbf{K}}_k\,\mathbf{T}_k \tag{2.5}$$

and can be written also in terms of sub-matrices as:

$$\mathbf{K}_k = \left[\begin{array}{c:c:c} \mathbf{K}_{k,uu} & \mathbf{K}_{k,uv} & \mathbf{K}_{k,u\theta} \\ \hdashline \mathbf{K}_{k,vu} & \mathbf{K}_{k,vv} & \mathbf{K}_{k,v\theta} \\ \hdashline \mathbf{K}_{k,\theta u} & \mathbf{K}_{k,\theta v} & \mathbf{K}_{k,\theta\theta} \end{array}\right] \tag{2.6}$$

being (2.7):

$$\begin{aligned}
\mathbf{K}_{k,uu} &= \mathbf{cos}^2\boldsymbol{\omega}_k\,\overline{\mathbf{k}}_{k,u} + \mathbf{sen}^2\boldsymbol{\omega}_k\,\overline{\mathbf{k}}_{k,v} \\
\mathbf{K}_{k,vu} &= \mathbf{K}_{k,uv} = \mathbf{cos\omega}_k\mathbf{sen\omega}_k\left(-\overline{\mathbf{k}}_{k,u} + \overline{\mathbf{k}}_{k,v}\right) \\
\mathbf{K}_{k,vv} &= \mathbf{sen}^2\boldsymbol{\omega}_k\,\overline{\mathbf{k}}_{k,u} + \mathbf{cos}^2\boldsymbol{\omega}_k\,\overline{\mathbf{k}}_{k,v} \\
\mathbf{K}_{k,\theta u} &= \mathbf{K}_{k,u\theta} = \mathbf{cos\omega}_k\left(-\mathbf{X}_k\mathbf{sen\omega}_k - \mathbf{Y}_k\mathbf{cos\omega}_k\right)\overline{\mathbf{k}}_{k,u} + \mathbf{sen\omega}_k\left(\mathbf{X}_k\mathbf{cos\omega}_k - \mathbf{Y}_k\mathbf{sen\omega}_k\right)\overline{\mathbf{k}}_{k,v} \\
\mathbf{K}_{k,\theta v} &= \mathbf{K}_{k,v\theta} = \mathbf{sen\omega}_k\left(\mathbf{X}_k\mathbf{sen\omega}_k + \mathbf{Y}_k\mathbf{cos\omega}_k\right)\overline{\mathbf{k}}_{k,u} + \mathbf{cos\omega}_k\left(\mathbf{X}_k\mathbf{cos\omega}_k - \mathbf{Y}_k\mathbf{sen\omega}_k\right)\overline{\mathbf{k}}_{k,v} \\
\mathbf{K}_{k,\theta\theta} &= \left(\mathbf{X}_k\mathbf{sen\omega}_k + \mathbf{Y}_k\mathbf{cos\omega}_k\right)^2\overline{\mathbf{k}}_{k,u} + \left(\mathbf{X}_k\mathbf{cos\omega}_k - \mathbf{Y}_k\mathbf{sen\omega}_k\right)^2\overline{\mathbf{k}}_{k,v} + \overline{\mathbf{k}}_{k,\theta}
\end{aligned}$$

Such equations contain the lateral stiffness matrices, $\overline{\mathbf{k}}_{k,u}$ and $\overline{\mathbf{k}}_{k,v}$, as well as the torsional stiffness matrix, $\overline{\mathbf{k}}_{k,\theta}$, and allow to construct the following stiffness matrix for the *k*-th element in the LCS:

$$\overline{\mathbf{K}}_k = \left[\begin{array}{c:c:c} \overline{\mathbf{k}}_{k,u} & \mathbf{0} & \mathbf{0} \\ \hdashline \mathbf{0} & \overline{\mathbf{k}}_{k,v} & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} & \overline{\mathbf{k}}_{k,\theta} \end{array}\right] \tag{2.8}$$

If this lateral load-bearing element is a shear wall or a moment-resisting frame, both the out-of-plane lateral stiffnesses and the torsional stiffnesses can be neglected, so the computational work associated with 3D analysis of the mixed structure is drastically reduced. Augenti and Parisi (2008) proposed some analytical expressions of the lateral stiffness matrices for both opened masonry shear walls and solid (masonry or RC) shear walls.
The stiffness matrix of the whole mixed structure is obtained by assembling the matrices $\mathbf{K}_k$. Obviously, in the case of mutually orthogonal planar load-bearing elements, $\mathbf{K}_{uv}$ and $\mathbf{K}_{vu}$ are zero matrices, so the translational components of motion along the x and y axes are uncoupled. In such conditions, the coupling of the lateral load-resisting elements is only due to the torsional response. Therefore, the global displacements at each level may be derived by the inversion of Eqn. 2.1. Subsequently, displacements and load vectors can be estimated for every load-bearing element at each storey, and one can then assess their seismic performance.
To achieve a better physical understanding of the 3D seismic distribution of the horizontal actions between different lateral load-resisting elements, distribution matrices are usefully defined. Given that:

$$\mathbf{S} = \mathbf{T}_k^{\mathrm{T}}\,\overline{\mathbf{S}}_k = \mathbf{K}^{-1}\mathbf{Q} \qquad \text{and} \qquad \overline{\mathbf{S}}_k = \overline{\mathbf{K}}_k^{-1}\,\overline{\mathbf{Q}}_k \tag{2.9}$$

one can get:

$$\overline{\mathbf{Q}}_k = \left(\overline{\mathbf{K}}_k\,\mathbf{T}_k\,\mathbf{K}^{-1}\right)\mathbf{Q} = \mathbf{\Gamma}_k\,\mathbf{Q} \tag{2.10}$$

The distribution matrix of the *k*-th element, $\mathbf{\Gamma}_k$, is squared as well, but whilst its size is equal to s for planar models, it becomes $3s$ for 3D models. Also the distribution matrix can be written in the following form:

$$\mathbf{\Gamma}_k = \left[\begin{array}{c:c:c} \mathbf{\Gamma}_{k,uu} & \mathbf{\Gamma}_{k,uv} & \mathbf{\Gamma}_{k,u\theta} \\ \hdashline \mathbf{\Gamma}_{k,vu} & \mathbf{\Gamma}_{k,vv} & \mathbf{\Gamma}_{k,v\theta} \\ \hdashline \mathbf{\Gamma}_{k,\theta u} & \mathbf{\Gamma}_{k,\theta v} & \mathbf{\Gamma}_{k,\theta\theta} \end{array}\right] \tag{2.11}$$

where the sub-matrices are expressed as follows (2.12):

$$\mathbf{\Gamma}_{k,uu} = \overline{\mathbf{K}}_{k,u}\left[\cos\omega_k\mathbf{D}_{uu} - \mathrm{sen}\omega_k\mathbf{D}_{vu} - \left(\mathbf{X}_k\mathrm{sen}\omega_k + \mathbf{Y}_k\cos\omega_k\right)\mathbf{D}_{\theta u}\right]$$
$$\mathbf{\Gamma}_{k,uv} = \overline{\mathbf{K}}_{k,u}\left[\cos\omega_k\mathbf{D}_{uv} - \mathrm{sen}\omega_k\mathbf{D}_{vv} - \left(\mathbf{X}_k\mathrm{sen}\omega_k + \mathbf{Y}_k\cos\omega_k\right)\mathbf{D}_{\theta v}\right]$$
$$\mathbf{\Gamma}_{k,u\theta} = \overline{\mathbf{K}}_{k,u}\left[\cos\omega_k\mathbf{D}_{u\theta} - \mathrm{sen}\omega_k\mathbf{D}_{v\theta} - \left(\mathbf{X}_k\mathrm{sen}\omega_k + \mathbf{Y}_k\cos\omega_k\right)\mathbf{D}_{\theta\theta}\right]$$
$$\mathbf{\Gamma}_{k,vu} = \overline{\mathbf{K}}_{k,v}\left[\mathrm{sen}\omega_k\mathbf{D}_{uu} + \cos\omega_k\mathbf{D}_{vu} + \left(\mathbf{X}_k\cos\omega_k - \mathbf{Y}_k\mathrm{sen}\omega_k\right)\mathbf{D}_{\theta u}\right]$$
$$\mathbf{\Gamma}_{k,vv} = \overline{\mathbf{K}}_{k,v}\left[\mathrm{sen}\omega_k\mathbf{D}_{uv} + \cos\omega_k\mathbf{D}_{vv} + \left(\mathbf{X}_k\cos\omega_k - \mathbf{Y}_k\mathrm{sen}\omega_k\right)\mathbf{D}_{\theta v}\right]$$
$$\mathbf{\Gamma}_{k,v\theta} = \overline{\mathbf{K}}_{k,v}\left[\mathrm{sen}\omega_k\mathbf{D}_{u\theta} + \cos\omega_k\mathbf{D}_{v\theta} + \left(\mathbf{X}_k\cos\omega_k - \mathbf{Y}_k\mathrm{sen}\omega_k\right)\mathbf{D}_{\theta\theta}\right]$$

$$\mathbf{\Gamma}_{k,\theta u} = \overline{\mathbf{K}}_{k,\theta}\,\mathbf{D}_{\theta u}$$
$$\mathbf{\Gamma}_{k,\theta v} = \overline{\mathbf{K}}_{k,v}\,\mathbf{D}_{\theta v}$$
$$\mathbf{\Gamma}_{k,\theta\theta} = \overline{\mathbf{K}}_{k,v}\,\mathbf{D}_{\theta\theta}$$

Flexibility sub-matrices of the structure appear in such formulas and may be clearly derived by the inversion of the stiffness matrix of the whole mixed structure. Eqn. 2.10 allows the direct distribution of the applied actions between the lateral load-resisting elements and can be expressed also by a set of matrix equations, so each load vector of the element is a linear combination of the external load ones. When there are non-orthogonal lateral load-resisting elements the first two equations are coupled themselves. This occurs also if centre of mass and centre of stiffness are coincident at every level: in such case, the algebraic system is composed just by two matrix equations, but the sub-matrices $\mathbf{\Gamma}_{k,uv}$ and $\mathbf{\Gamma}_{k,vu}$ have nonzero entries. These last ones become zero matrices only when the external horizontal actions are applied along the principal directions of the floor plan, which still change along the height even if the lateral load-resisting elements have constant cross-sections.

3. MODELLING OF THE STOREY TORQUES

The proposed algorithm provide good results only if the storey torques are accurately modelled. Such operation can be uniquely performed after eccentricity matrices are built up. These lower triangular matrices can be defined, in turn, only after the coordinates of the centres of rigidity are determined at each level. In general, such algebraic problem can be solved via direct or iterative procedures, but in this case "trial and error" algorithms may result in high numerical approximations on the final solutions, because (1) the stiffness matrices have large condition numbers (ill-conditioned problems) and (2) an unacceptable step-by-step propagation of errors may occur (Ferziger *et al*. 1998, Higham 2002, Quarteroni *et al*. 2007).
The centres of rigidity can be estimated through a simple direct procedure based on the definition of centre of vector field. After the stiffness matrices are constructed in the GCS, the position of the centres can be derived by solving the following equation:

$$\sum_{k=1}^{w}\left[\mathbf{K}_k\cdot\left(\mathbf{r}_k-\mathbf{r}_c\right)\right]=0 \tag{3.1}$$

where $\mathbf{r}_c$ is the unknown radius vector of the centres of rigidity of the whole structure and $\mathbf{r}_k$ is the radius vector of the k-the lateral force-resisting element. Such $3s\times1$ vectors may be divided into the sub-vectors $\mathbf{X}_c$, $\mathbf{Y}_c$, $\mathbf{Z}_c$ and $\mathbf{X}_k$, $\mathbf{Y}_k$, $\mathbf{Z}_k$, respectively, which contain the global coordinates x, y, z, so it turns out to be:

$$\mathbf{r}_c=\mathbf{D}\left(\sum_{k=1}^{w}\mathbf{K}_k\,\mathbf{r}_k\right) \tag{3.2}$$

where $\mathbf{D}$ is the flexibility matrix of the whole structure. Obviously, Eqn. 3.2 may be written also as:

$$\begin{bmatrix}\mathbf{K}_{uu} & \mathbf{K}_{uv} & \mathbf{K}_{u\theta}\\ \mathbf{K}_{vu} & \mathbf{K}_{vv} & \mathbf{K}_{v\theta}\\ \mathbf{K}_{\theta u} & \mathbf{K}_{\theta v} & \mathbf{K}_{\theta\theta}\end{bmatrix}\begin{Bmatrix}\mathbf{X}_c\\ \mathbf{Y}_c\\ \mathbf{Z}_c\end{Bmatrix}=\sum_{k=1}^{w}\begin{bmatrix}\mathbf{K}_{k,uu} & \mathbf{K}_{k,uv} & \mathbf{K}_{k,u\theta}\\ \mathbf{K}_{k,vu} & \mathbf{K}_{k,vv} & \mathbf{K}_{k,v\theta}\\ \mathbf{K}_{k,\theta u} & \mathbf{K}_{k,\theta v} & \mathbf{K}_{k,\theta\theta}\end{bmatrix}\begin{Bmatrix}\mathbf{X}_k\\ \mathbf{Y}_k\\ \mathbf{Z}_k\end{Bmatrix} \tag{3.3}$$

or also as a system of three matrix equations in the unknown vectors $\mathbf{X}_c$, $\mathbf{Y}_c$, $\mathbf{Z}_c$. The exact solution of the problem is formed also by the vector $\mathbf{Z}_c$, which does not contain the z coordinates of the diaphragms because each centre of rigidity does not actually lie in the middle plane of the floor, but it is generically placed in the 3D space. One could then assume $\mathbf{Z}_c$ equal to the vector $\mathbf{Z}_p$ containing the z coordinates of the floors, but such constraint equation should reduce the number of unknowns and the algebraic problem should become indeterminate. Thus, to estimate directly the positions of the centres of rigidity, this equality relation have to be imposed only after Eqn. 3.2 is solved. This modus operandi does not result in significant errors, since storey torques depend just on the eccentricities along the x and y axes.

4. TORSIONAL STIFFNESS MATRIX OF A STRUCTURAL CORE

The last formula of Eqns. 2.7 allows the definition of the torsional stiffness matrix of a lateral force-resisting element in the GCS: it includes both the Saint-Venant torsional stiffness matrices, as well as the warping rigidity matrix. The first ones are derived from the translational stiffness matrices written in the LCS; the latter is presented below for a generic structural core by applying the non-uniform torsion theory (Vlasov 1961, Kolbrunner and Basler 1969).
For a prismatic thin-walled beam, the fundamental equation of torsion is the following linear, non-homogeneous Ordinary Differential Equation (ODE) of order 4 with constant coefficients:

$$k^{(2)}\theta''''(z)-k^{(1)}\theta''(z)=m_z(z) \qquad (4.1)$$

being: m_z the applied torque per unit length; θ the angle of twist; $k^{(1)}$ the Saint-Venant torsional stiffness, defined as product of the shear modulus G by the torsion constant I^*; and $k^{(2)}$ the warping rigidity, defined as product of the Young's modulus E by the warping constant $I_{\lambda\lambda}$. Note that λ is the generalised warping function defined by Augenti (1992) for multi-celled cross-sections. The general solution of Eqn. 4.1 is:

$$\theta(z)=A\cdot sh\,\alpha z+B\cdot ch\,\alpha z+C\cdot z+D+\bar{\theta}(z) \qquad (4.2)$$

where: A, B, C, D are the constants of integration; α is the inverse characteristic length; and $\bar{\theta}(z)$ is the particular solution of the ODE.
Let us consider a prismatic cantilever thin-walled beam, having torsional restraint at the base and subjected to a torque $\mathfrak{M}_z$ at the top. In such case Eqn. 4.1 becomes homogeneous and has a particular solution set to zero, so the following expression of the angle of twist is derived by imposing the boundary conditions:

$$\theta(z)=\frac{\mathfrak{M}_z}{\alpha\cdot k^{(1)}}\cdot\left[-sh\,\alpha z+th\,\alpha z\cdot\left(ch\,\alpha z-1\right)+\alpha z\right] \qquad (4.3)$$

Since at $z=H$ one can get:

$$\theta(H)=\frac{\mathfrak{M}_z}{\alpha\cdot k^{(1)}}\cdot\left(-th\,\alpha H+\alpha H\right)=\frac{\mathfrak{M}_z\cdot H}{k^{(1)}}\cdot\left(1-\frac{th\,\alpha H}{\alpha H}\right) \qquad (4.4)$$

the torsional stiffness at this height is given by:

$$k_{k,\theta}=\frac{\alpha\cdot k^{(1)}}{-th\,\alpha H+\alpha H} \qquad (4.5)$$

To obtain a generalised analytical expression of the torsional stiffness matrix of a core with height sH, consider a cantilever beam of height $3H$ subjected to the torques $\mathfrak{M}_{z,1}$, $\mathfrak{M}_{z,2}$, $\mathfrak{M}_{z,3}$ at H, $2H$, $3H$, and let us analyse three separate schemes on which we apply the principle of effects superposition (see Figure 2).

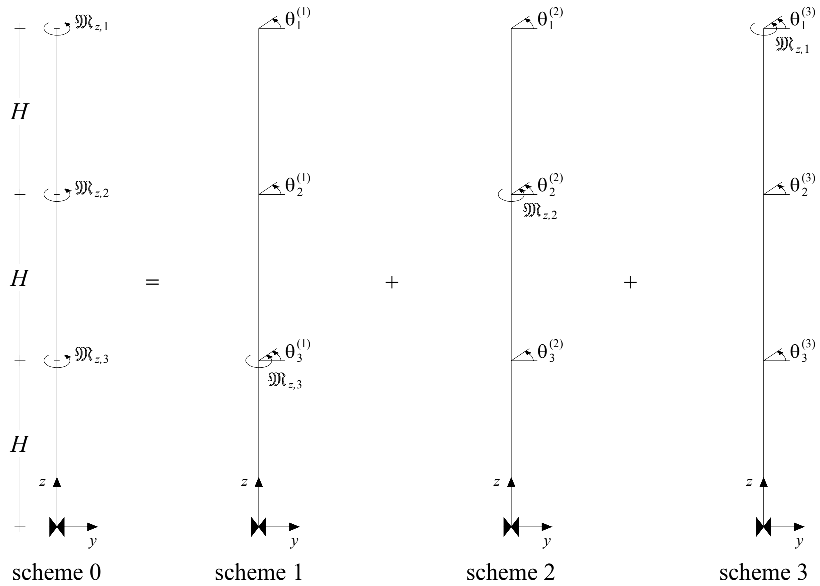

Figure 2. Effects superposition for a structural core of height 3*H*

The angle of twist θ_p at the generic level of the cantilever beam under the applied torques may be expressed as sum of three rates: the first one caused by the actions applied to the levels below the one held in consideration; the second one induced by the action applied to the level taken in consideration; the last one due to the actions applied to the levels above the one held in consideration. Thus, one can write:

$$\theta_1 = \theta_1^{(1)} + \theta_1^{(2)} + \theta_1^{(3)} \qquad \theta_2 = \theta_2^{(1)} + \theta_2^{(2)} + \theta_2^{(3)} \qquad \theta_3 = \theta_3^{(1)} + \theta_3^{(2)} + \theta_3^{(3)} \tag{4.6}$$

where each superscript is referred to the generic scheme to be superposed. Taking into account that deformed configurations of schemes (1) and (2) are composed by an elastic part and a rigid part, the twist rotation-torque relationships can be simply obtained, so the torsional flexibility matrix, $\bar{\mathbf{d}}_{k,\theta}$, may be constructed.

This approach may be generalised to a cantilever beam of height *sH*, reaching to an *s*×*s* torsional flexibility matrix whose entries are:

$$\begin{aligned}
d_{k,\theta,11} &= \frac{-th\,\alpha(sH) + \alpha(sH)}{\alpha \cdot k^{(1)}} \\
d_{k,\theta,12} &= \frac{-th\,\alpha\left[(s-1)H\right] + \alpha\left[(s-1)H\right]}{\alpha \cdot k^{(1)}} \\
&\vdots \\
d_{k,\theta,1s} &= \frac{-th\,\alpha H + \alpha H}{\alpha \cdot k^{(1)}}
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
d_{k,\theta,21} &= \frac{-sh\,\alpha\left[(s-1)H\right] + th\,\alpha(sH)\cdot\left\{ch\,\alpha\left[(s-1)H\right] - 1\right\} + \alpha\left[(s-1)H\right]}{\alpha \cdot k^{(1)}} \\
d_{k,\theta,22} &= \frac{-th\,\alpha\left[(s-1)H\right] + \alpha\left[(s-1)H\right]}{\alpha \cdot k^{(1)}} \\
&\vdots \\
d_{k,\theta,2s} &= \frac{-th\,\alpha H + \alpha H}{\alpha \cdot k^{(1)}}
\end{aligned} \tag{4.8}$$

$$d_{k,\theta,s1} = \frac{-sh\alpha H + th\alpha\left(sH\right)\cdot\left(ch\alpha H - 1\right) + \alpha H}{\alpha \cdot k^{(1)}}$$

$$d_{k,\theta,s2} = \frac{-sh\alpha H + th\alpha\left[\left(s-1\right)H\right]\cdot\left(ch\alpha H - 1\right) + \alpha H}{\alpha \cdot k^{(1)}} \tag{4.9}$$

$$\vdots$$

$$d_{k,\theta,ss} = \frac{-th\alpha H + \alpha H}{\alpha \cdot k^{(1)}}$$

Finally, the torsional stiffness matrix, $\overline{\mathbf{k}}_{k,\theta}$, is derived by the inversion of $\overline{\mathbf{d}}_{k,\theta}$, so one can get the matrix $\mathbf{K}_{k,\theta\theta}$.

CONCLUSIONS

The matrix algorithm proposed in this work allows to carry out 3D seismic analyses of torsionally eccentric masonry-RC mixed structures formed by frames, (masonry or RC) shear walls, and cores, connected themselves by rigid diaphragms. The analytical formulation is based on the direct stiffness method, so simple equations have been presented to define both the translational and torsional stiffness matrices for each lateral load-resisting element. Either the lateral force vectors or the storey torque vectors may be estimated for each structural element, by determining the stiffness-based distribution of the external horizontal actions applied to the centres of mass. To model the storey torques and to run the seismic analysis, the centres of rigidity have to be determined first. This problem may be solved by means of a direct method which provide little approximations on the final numerical solutions.
Also the distribution matrices may be defined, so the lateral load-resisting elements that mainly resist the external horizontal actions can be identified. These analytical tools are very important, especially when the seismic upgrading or retrofit of an existing building is needed.
An essential step of the proposed method is the construction of the torsional stiffness matrices of the lateral load-resisting elements. They were defined for RC cores by analysing the behaviour of a prismatic cantilever thin-walled beams, torsionally fixed at the base and subjected to a generic pattern of torques. A generalised analytical expression of such matrices was found by using the non-uniform torsion theory, taken into account that the distribution of the torsional moment in the Saint-Venant and warping parts depends not only on the ratio between the respective rigidities, but also on the height of the structural element.
The proposed algorithm is applicable not only for linear equivalent static analyses, but also for pushover analyses in which a step-by-step update of the stiffness matrix may be simply performed.

REFERENCES

Augenti, N. (1992). *Introduzione al calcolo delle strutture in parete sottile*, Liguori, Naples, Italy (in Italian).
Augenti, N. (2004). *Il calcolo sismico degli edifici in muratura*, UTET, Turin, Italy (in Italian).
Augenti, N. and Parisi, F. (2008). Preliminary analysis on masonry–RC combined systems: structural assessment. *Proceedings of 14th International Brick & Block Masonry Conference*, Sydney, 58-67.
Ferziger, J.H. (1998). *Numerical Methods for Engineering Applications*, John Wiley & Sons, New York, U.S.A.
Higham, N.J. (2002). *Accuracy and Stability of Numerical Algorithms*, Society of Industrial and Applied Mathematics, Philadelphia, U.S.A.
Kolbrunner, C.F. and Basler, K. (1969). *Torsion in structures*, Springer-Verlag, Berlin, Germany.
Pozzati, P. (1977). *Teoria e tecnica delle strutture*, UTET, Turin, Italy (in Italian).
Quarteroni, A., Sacco, R. and Saleri, F. (2007). *Numerical Mathematics*, Springer-Verlag, New York, U.S.A.
Vlasov, V.Z. (1961) *Thin-walled elastic beams*, National Science Foundation, Washington D.C., U.S.A., Oldbourne Press, London, U.K.