THE SHALLOW FOUNDATION BEHAVIOR WITH ELASTIC BED AFFECTED BY RANDOM VERTICAL LOADING

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ABSTRACT

In many cases, the loading on the foundation is random vertical, such as foundation of mechanical machines and loading on the foundation while earthquake and vertical acceleration. The most important index of random loading is that it's not periodic, so we can't understand their exact analytical amount based on the previous data of system input and output. We can just guess through probability of their amount in the future by use of random vibration theory. This article aims at studying the shallow foundation behavior with elastic bed affected by random vertical loading in order to determine the most probable resulted amount of deflection and the stress. Regarding to the dynamic balance of an element, differential equation of motion of plate with damping, inertia and spring force, was obtained. Then regarding to boundary Conditions and choosing appropriate functions, this equation was solved. Having movement equation, Self–correlation function of deflection and stress were obtained based on input intersecting spectral density, supposing permanent loading process. Then we obtained the square average of maximum deflection and stress, supposing whitening of input density and then we show the results in graphs. Finally, the different parameters such as reaction coefficient of elastic bed, foundation dimensional ratio and the damping coefficient were studied. We consider here a Shallow foundation on the elastic bed affected by a random loading that is a function of time with uniform distribution. It aims at finding out the most probable output response of system, like deflection and stress in foundation and determining the effect of different parameters on that. Based on white spectral density and the supposing permanent process, and considering continuum system for a concrete foundation with an elastic bed affected by the random vertical loading, we obtained the output autocorrelation function (deflection) and also stress from bending in foundation, as well as the most probable amounts of vertical deflection and bending stress. To survey the parameters effecting, we studied damping coefficient, bed soil hardening, and the ratio of width to length and thickness, and in every cases based on the obtained functions they were solved by MATLAB numerically and we offered the effect of parameters amounts of most probable vertical deflection and bending stress in foundation in the third part of paper. Using probability and random vibration theory, we can find the maximum of deflection and stress and by extending this idea we can put other structures under random loading, survey the spectral density and its special conditions in order to guess their responses based on probability and random vibrations of structure against earthquake or any random loading kind.

KEYWORDS: Random Vibration, Elastic Bed, Auto–Correlation Function, Spectral Density

Figure 22 INTRODUCTION

In many cases, the loading on the foundation is random vertical, such as foundation of mechanical machines and loading on the foundation due to earthquake and vertical acceleration. To be no periodic is the most important index of random loading, so we can't understand their exact analytical amount based on the previous data of system input and output. We can just guess through probability of their amount in the future by use of random vibration theory. In this paper, we consider a Shallow foundation on the elastic bed due to random loading that is a function of time with uniform distribution P_0. It aims at finding out the most probable output response of system, like deflection and stress in foundation and determining the effect of different parameters on that.

2- THE CONSTITUTIVE RELATIONS OF THE VIBRATION OF SHOLLOW FOUNDATION WITH ELASTIC BED DUE TO RANDOM LOADING

Figure 1 shows schematic of the foundation on the elastic bed figure. Using figure 2 and balance equation in z axis, we have the following relation for an element in dy, dx:

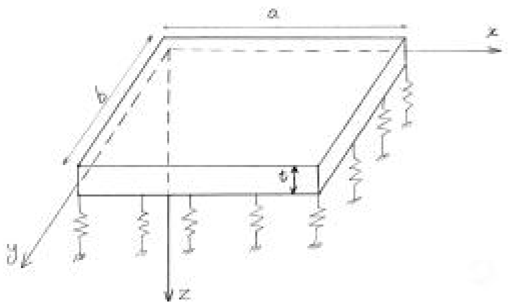

Fig. 1- Schematic of the foundation on the elastic bed figure

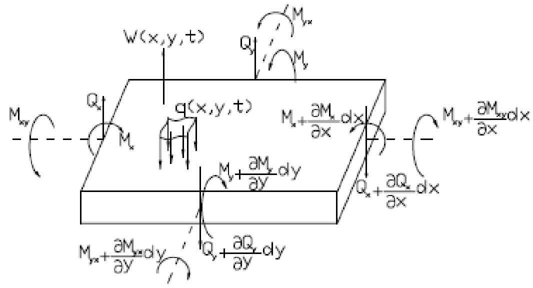

Fig. 2- The element of a foundation in dimension of dx, dy under random loading of external load and internal forces

$$\frac{\partial Q_x}{\partial x}\delta x\delta y+\frac{\partial Q_y}{\partial y}\delta x\delta y+q\delta x\delta y-\rho t\ddot{w}\delta x\delta y-c\dot{w}\delta x\delta y-kw\delta x\delta y=0 \tag{1}$$

Or:

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q-\rho t\ddot{w}-C\dot{w}-Kw=0 \tag{2}$$

Where, w is the vertical deflection of foundation, K is the soil stiffness coefficient in unit area, ρ is the foundation density , t is foundation thickness and Qx , Qy are the internal shear force of foundation along x, y, respectively. Using the moments balance, we have:

$$\frac{\partial Mx}{\partial x}\delta x\delta y-\frac{\partial Mxy}{\partial y}\delta x\delta y-Qx\delta x\delta y=0 \tag{3}$$

$$\frac{\partial Mx}{\partial x}-\frac{\partial Mxy}{\partial y}=Qx \tag{4}$$

$$\frac{\partial My}{\partial y}-\frac{\partial Mxy}{\partial x}=Qy \tag{5}$$

Using the relation between moments and deflection (w) and based on plate theories, we have:

$$Mx=-D(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}) \tag{6}$$

$$My=-D(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}) \tag{7}$$

$$Mxy=D(1-\nu)\frac{\partial^2 w}{\partial x\partial y} \tag{8}$$

Where in above relations, $D=\frac{Et3}{1-\nu^2}$ and ν is the Poisson coefficient of the foundation. Substituting the relations (6-8) in (4-5), we have:

$$Qx = -D(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}) \tag{9}$$

$$Qy = -D(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial y \partial x^2}) \tag{10}$$

Substituting the relations (9-10) in (2), we have:

$$D\nabla^4 w + \rho t \ddot{w} + C\dot{w} + Kw = q(x, y, t) \tag{11}$$

In which the random loading of q(x, y, t) has been considered to be in the form of P_0f(t) that f(t) is a permanent process. So we can have the relation (11) in the following form:

$$\nabla^4 w + \frac{\rho t}{D}\ddot{w} + \frac{C}{D}\dot{w} + \frac{K}{D}w = \frac{1}{D}P_o f(t) \tag{12}$$

Or:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{\rho t}{D}\ddot{w} + \frac{C}{D}\dot{w} + \frac{K}{D}w = \frac{1}{D}P_o f(t) \tag{13}$$

Using the separation of variable method and modal analysis relations, we can consider the answer in the following form:

$$W(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \psi_{mn}(x, y) y_k(t) \tag{14}$$

In which regarding to the boundary conditions the function ψ_{mn} (x, y), will be:

$$\psi_{mn}(x, y) = \sin\frac{m\pi x}{a} . \sin\frac{n\pi y}{b} \tag{15}$$

With substituting of answer form of (Eq. 15) in Eq.(14), we obtain:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[D\nabla^4 w\psi_{mn} y_k(t) + \rho t\psi_{mn}\ddot{y}_k(t) + C\psi_{mn}\dot{y}_k(t) + k\psi_{mn} y_k(t)\right] = q(x,y,t) = P_0 f(t) \tag{16}$$

In case of homogeny equation: $C = 0$ and $q(x, y, t) = 0$, so:

$$D\nabla^4 \psi_{mn} y_k(t) + \rho t\psi_{mn}\ddot{y}_k(t) + k\psi_{mn} y_k(t) = 0 \tag{17}$$

$$[D\nabla^4 \psi_{mn} + k\psi_{mn}] y_k(t) + \rho t\psi_{mn}\ddot{y}_k(t) = 0 \tag{18}$$

$$D[\frac{\partial^4 \psi_{mn}}{\partial x^4} + 2\frac{\partial^4 \psi_{mn}}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi_{mn}}{\partial y^4} + k\psi_{mn}] y_k(t) + \rho t\psi_{mn}\ddot{y}_k(t) = 0 \tag{19}$$

$$[\frac{\partial^4\psi_{mn}}{\partial x^4}+2\frac{\partial^4\psi_{mn}}{\partial x^2\partial y^2}+\frac{\partial^4\psi_{mn}}{\partial y^4}+\frac{k}{D}\psi_{mn}]y_k(t)=-\frac{\rho t}{D}\psi_{mn}\ddot{y}_k(t) \tag{20}$$

Where: $\overline{C}^2=\frac{\rho t}{D}$

Where a and b are the foundation dimension along x and along y, respectively.
Substituting (14) in relation (13) and solving the homogenous equation, we can obtain the natural frequency of the foundation:

$$\omega^2{}_{mn}=\frac{D\pi^4}{\rho t}\left[(\frac{m}{a})^4+2(\frac{m}{a})^2(\frac{n}{b})^2+(\frac{n}{b})^4\right]+\frac{k}{\rho t} \tag{21}$$

With substituting of $\omega^2{}_{nm}$ in Eq. (16), we have:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[D[\,(\frac{n\pi}{a})^4+2(\frac{n\pi}{a})^2(\frac{m\pi}{b})^2+(\frac{m\pi}{b})^4\,]\psi_{nm}y_k(t)+k\psi_{nm}y_k(t)+\rho t\psi_{mn}\ddot{y}_k(t)+C\psi_{mn}\dot{y}_k(t)\right]=q(x,y,t)=p_0f(t) \tag{22}$$

So:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[D[\,(\frac{n\pi}{a})^4+2(\frac{n\pi}{a})^2(\frac{m\pi}{b})^2+(\frac{m\pi}{b})^4+k\,]y_k(t)+\rho t\,\ddot{y}_k(t)+C\,\dot{y}_k(t)\right]\psi_{mn}=q(x,y,t)=p_0f(t) \tag{23}$$

Or:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[\,\rho t\,\ddot{y}_k(t)+C\,\dot{y}_k(t)+\omega^2{}_{nm}.\rho t.y_k(t)\,]\psi_{mn}=q(x,y,t)=p_0f(t) \tag{24}$$

Using the modal perpendicularity characteristics and substituting ω_{mn} in the motion equation, we have:

$$\ddot{y}_{mn}(t)+2\zeta_{mn}\omega_{mn}\dot{y}(t)+\omega^2_{mn}y_{mn}(t)=\frac{4P_\circ f(t)}{ab\rho t}\int_0^a\int_0^b\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy \tag{25}$$

Where the modal coefficient is defined like the following:

$$\zeta_{mn}=\frac{C}{2\rho t\omega_{mn}} \tag{26}$$

We can reach answer y_{mn}(t) by solving differential equation. The response function (deflection) will be as the following form:

$$w(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}y_{mn}(t)\ \sin\frac{m\pi x}{a}.\sin\frac{n\pi y}{b} \tag{27}$$

Obtaining response function (deflection) of w(x, y, t), output autocorrelation function (deflection) is obtained, supposing the f(t) to be permanent process:

$$E\left[w(x_1,y_1,t),w(x_2,y_2,t+\tau)\right]=\int_{-\infty}^{+\infty} S^w(x_1,x_2,y_1,y_2,\omega)e^{i\omega\tau}d\omega \tag{28}$$

Where:

$$S^w(x_1,x_2,y_1,y_2)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty}\psi_{mn}(x_1,y_1)\,\psi_{m'n'}(x_2,y_2)S^f_{mnm'n'}(\omega)H^*_{mn}(\omega)H_{m'n'}(\omega) \tag{29}$$

In this relation:

$$S^f_{mnm'n'}(\omega)=\alpha_{mn}\alpha_{m'n'}S_f(\omega) \tag{30}$$

$$\alpha_{mn}=\frac{4p_\circ}{\rho t\,mn\pi^2}(1-\cos m\pi)(1-\cos n\pi) \tag{31}$$

$$\alpha_{m'n'}=\frac{4p_\circ}{\rho t\,m'n'\pi^2}(1-\cos m\pi)(1-\cos n\pi) \tag{32}$$

$$S_f(w)=S_\circ \tag{33}$$

Where, S_0 is the density amount of white spectral.

$$H^*_{mn}(\omega)=\frac{1}{\omega_{mn}^2-\omega^2-2i\zeta_{mn}\omega\omega_{mn}} \tag{34}$$

$$H_{mn}(\omega)=\frac{1}{\omega_{mn}^2-\omega^2+2i\zeta_{mn}\omega\omega_{mn}} \tag{35}$$

Regarding to (29), the square average of deflection for each point of the plate will be equals to:

$$E\left[w^2(x,y,t)\right]=S_\circ\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty}\alpha_{mn}\alpha_{m'n'}\ ,\ \psi_{mn}(x,y)\psi_{m'n'}(x,y)\int_{-\infty}^{+\infty}H^*_{mn}(\omega)H_{m'n'}(\omega)d\omega \tag{36}$$

In other hand, we have:

$$\int H^*_{mn}(\omega)H_{m'n'}(\omega)d\omega=\frac{4\pi}{r_{mn}r_{m'n'}}\frac{2\eta}{(\omega_{m'n'}^2-\omega_{mn}^2)^2+8\eta^2(\omega_{m'n'}^2+\omega_{mn}^2)} \tag{37}$$

Where $y=\dfrac{c}{2pt}$ and $r_{mn}=\dfrac{ptab}{4}$.

Substituting (37) in (36), the square average of deflection in each point of the foundation is equals to:

$$E\left[w^2(x,y,t)\right]=\Big(\frac{64\,S_\circ P_\circ^2}{\rho^2h^2\pi^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty}\frac{(1-\cos m\pi)\sin\dfrac{m\pi x}{a}}{m}\frac{(1-\cos n\pi)\sin\dfrac{n\pi y}{b}}{n}$$
$$\times\frac{(1-\cos m'\pi)\sin\dfrac{m'\pi x}{a}}{m'}\frac{(1-\cos n'\pi)\sin\dfrac{n'\pi y}{b}}{n'}\frac{1}{r_{mn}r_{m'n'}}\times\frac{2\eta}{(\omega_{m'n'}^2-\omega_{mn}^2)^2+8\eta^2(\omega_{m'n'}^2+\omega_{mn}^2)}\Big) \tag{38}$$

Using the stress-moments and stress-deflection relations, we can obtain square average of stress in each point, similar to deflection:

$$I_x = \frac{1}{12}(1)t^3 = \frac{t^3}{12} \quad , \quad z = \frac{t}{2} \quad , \quad \sigma_{x(\max)} = \frac{M_x.z}{I_x} \tag{39}$$

$$I_y = \frac{1}{12}(1)t^3 = \frac{t^3}{12} \quad , \quad z = \frac{t}{2} \quad , \quad \sigma_{y(\max)} = \frac{M_y.z}{I_y} \tag{40}$$

$$E\left[\sigma^2(x,y,t)\right] = \frac{16E^2 S_\circ P_\circ^2 \pi}{\rho^2(1-v^2)^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty}\Bigg(\frac{(1-\cos m\pi)\sin\frac{m\pi x}{a}}{m}\times\frac{(1-\cos n\pi)\sin\frac{n\pi y}{b}}{n}$$

$$\times\frac{(1-\cos m'\pi)\sin\frac{m'\pi x}{a}}{m'}\,\frac{(1-\cos n'\pi)\sin\frac{n'\pi y}{b}}{n'}\times\left(\frac{m^2}{a^2}+v\frac{n^2}{b^2}\right)\times\left(\frac{m'^2}{a^2}+v\frac{n'^2}{b^2}\right)\frac{1}{r_{mn}r_{m'n'}}$$

$$\times\left(\frac{m^2}{a^2}+v\frac{n^2}{b^2}\right)\times\left(\frac{m'^2}{a^2}+v\frac{n'^2}{b^2}\right)\frac{1}{r_{mn}r_{m'n'}}\times\frac{2\eta}{(\omega_{m'n'}^2-\omega_{mn}^2)^2+8\eta^2(\omega_{m'n'}^2+\omega_{mn}^2)}\Bigg) \tag{41}$$

3- NUMERICAL RESULT SURVEY

To study the effect of different parameters in foundation deflection and stress, it is necessary to solve the square average of maximum deflection (Eq. 38), square average of maximum stress (Eq. 41) which has a linear relation with white spectral density and a non-linear relation with damping coefficient (c) or damping ratio (ξ) or other parameters, by a numerical solving software. Based on obtained relations and using MATLAB software, we provided numerical solving programs. By using of MAPLE software we obtained same results. The input data with elastic modulus of foundation concrete are:

The elastic modulus is considered to be $E = 2\times 10^{10}\ N/m^2$, Poisson coefficient ν=0.2, concrete density ρ =2400 kg/m^3, and the length of foundation is considered to be a=3m. The other parameters are considered by the following amounts:

1-damping ratio: ξ=0.02, 0.05, 0.1, 0.2

2- Bed soil stiffness: K=5E^7, 5E^8, 5E^9 N/m^2

3- Width to length ratio: b/a=0.2,0.35,0.5,0.7,1.0,1.5

4- Foundation thickness: t=30, 40, 50 cm

In different conditions, the maximum of probable deflection and stress are plotted along $y = \frac{b}{2}$.

Figure (3) shows the effects of foundation width to length ratio on the most probable stress amount. We can see the stress increases by the plate ratio increasing. Figure (4) shows the effects of foundation width to length ratio on the most probable deflection amount. The increasing of the ratio results the increasing in deflection. Figure (5) shows the effect of foundation thickness on the most probable stress. In this case, it is considered that the plate width is 2m and damping ratio is 5%. We can see the stress decreases by the foundation thickness increasing, and also by increasing the thickness, the stress difference decreases in the one fifth of middle length with the other points. Figure (6) shows the effect of foundation thickness changes on the deflection amount. With increasing the foundation thickness, deflection decreases. Figure (7) shows the damping ratio change effects on the most probable deflection. As we can see, with increasing this amount, deflection amount decreases. The effect of changing of damping ratio on most probable amount of stress is shown in the figure (8). With increasing damping ratio, the amount of the most probable stress decreases and in middle of span, the most probable stress is greater than other point. Figures (9) and (10) show the effect of bed stiffness on the most probable amount of deflection and stress. With increasing of stiffness of the bed, amount of deflection and stress is decreased.

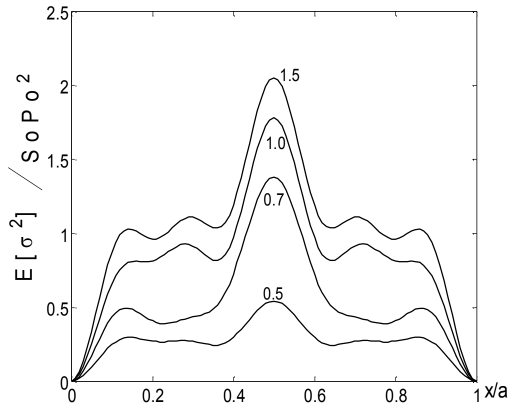

Figure (3) - The most probable stress for the different ratios of width to length

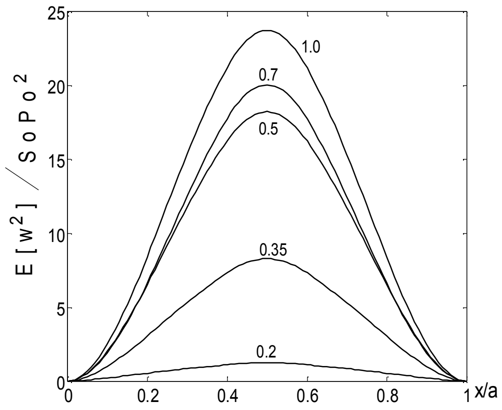

Figure (4) - The most probable deflection for the different ratios of width to length

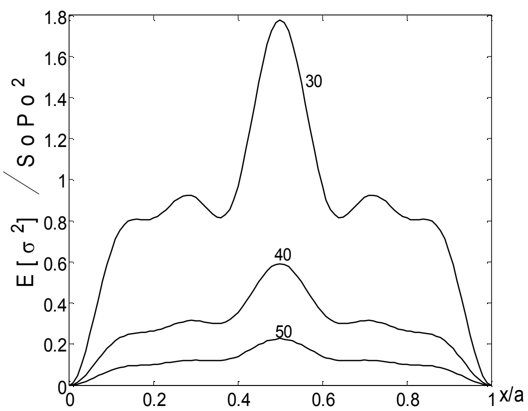

Figure (5) - The most probable stress for the different foundation thickness (cm)

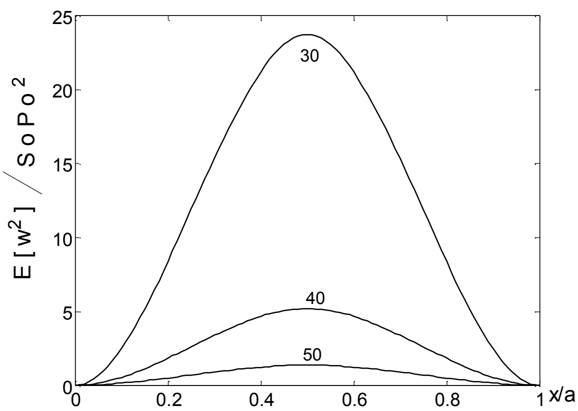

Figure (6) - The most probable deflection for the different foundation thickness (cm)

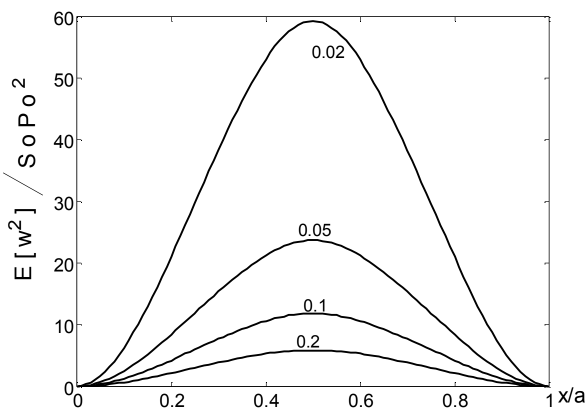

Figure (7) - The most probable deflection for the different amount of damping ratio

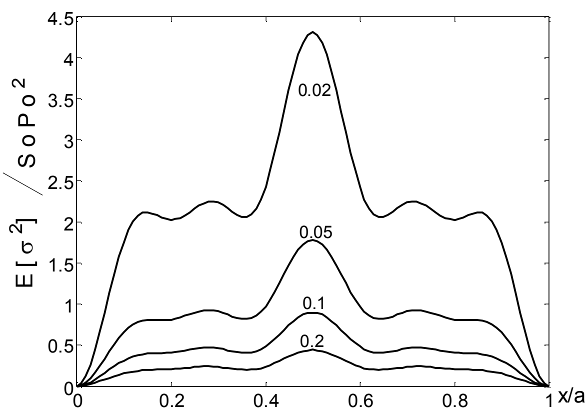

Figure (8) The most probable stress for the different amount of damping ratio

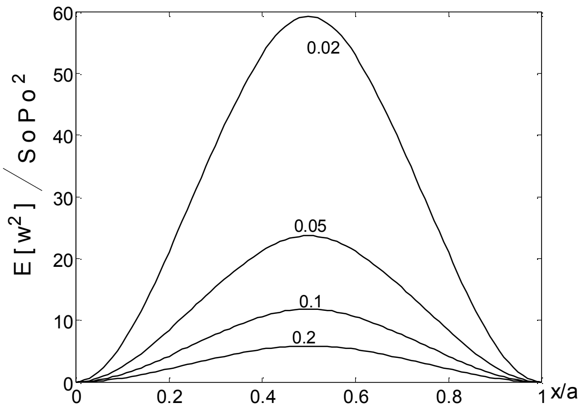

Figure (9) - The most probable deflection for the different amount of bed hardening $N/m/m^2$

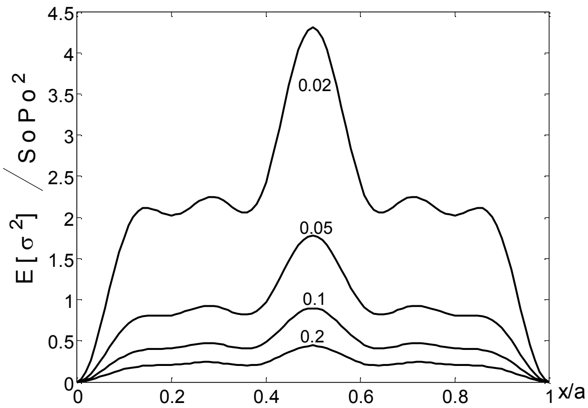

Figure (10)- The most probable stress for the different amount of bed hardening $N/m/m^2$

4- CONCLUSION

1- Based on white spectral density and the supposing stationary process, and considering continuum system for a concrete foundation with an elastic bed due to the random vertical loading, we obtained the output autocorrelation function (deflection) and also stress from bending in foundation, as well as the most probable amounts of vertical deflection and bending stress. 2- To survey the parameters effecting, we studied damping coefficient, bed soil stiffness, and the ratio of width to length and thickness, and in every cases based on the obtained functions they were solved numerically by MATLAB software. We present the effect of parameters amounts of most probable vertical deflection and bending stress in foundation in the third part of paper. 3- Using probability and random vibration theory, we can find the maximum of deflection and stress. By extending this idea, we can put other structures under random loading, survey the spectral density and its special conditions in order to guess their responses based on probability and random vibrations of structure against earthquake or any random loading kind.

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