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A QUICK METHOD FOR EIGENVALUE ESTIMATION OF MULTISTORY BUILDINGS

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SUMMARY

A simplified method is presented to estimate desired modal frequency of a *chain structure* independently from any other modes' calculations. It is based on the developed mapping between the chain structure and an equivalent beam model thus leading to a dimensionless frequency equation. For the considered case of fixed-free boundary condition an estimation technique is used to find the desired root of the dimensionless frequency equation. The *missed mass effect* during the proposed mapping is taken into account to improve estimated results. Finally the effectiveness and accuracy of the presented method is discussed via illustrative examples of regular and irregular chain structures even with the non-uniform distribution of mass and stiffness among their height.

Key words: Regular chain structure, Frequency equation, Missed mass effect

INTRODUCTION

Estimation of dynamic structural characteristics is of interest in many applications such as time-consuming iterations of non-linear analyses or in the case of optimization problems where structural model is highly changed during the process. Most of the recent methods require exhaustive computational efforts (Clough [1], Hosseini [2]) among them Holzer method [3] may

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be concerned. It estimates desired modal frequency independently from other modes, however dependently to the first guess of the corresponding mode-shape. A modified version of that based on energy considerations, has already been developed known as Holzer-Hosseini method [2].

This paper aims to develop a quick simplified method to achieve an estimation of a desired mode's frequency for a fixed-ended chain structure independently from other modes' calculations. It is also expected to find out the corresponding mode-shape simultaneously. Such an approximate result may be then used as a proper starting point for some other known exact-iterative methods aiming to achieve more efficiency and accuracy in the final results.

PROBLEM DEFINITION AND THEORY

A chain structure is defined as a system including elements of mass and stiffness which are alternatively arranged in a sequence along a main axis. Thus, many practical structures fall in this category such as pipelines, railways, tall process towers and so on. In this work, multistory building frames are approximately treated as chain structures with fixed-free boundary conditions. Such an analogy will be more accurate after condensation of rotational degrees of freedom within each story, leading to a lumped mass model (Fig.1). Deriving chain structure' properties may be done regarding approximate methods such as Hosseini [4].

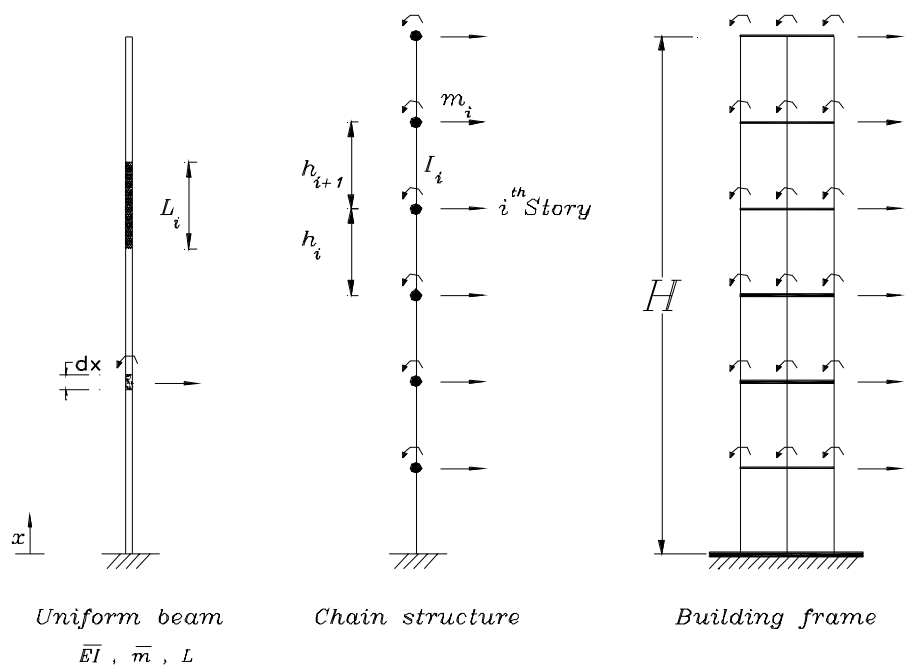

Figure 1. Chain structure model of a building frame and equivalent uniform beam

Consider a beam with uniform mass of $\overline{m}$ per length and moment inertia of $\overline{I}$ over the whole length L [1]. Its free-vibration equation regarding flexural behavior is given by:

$$(1)\quad \frac{\partial^4 v(x,t)}{\partial x^4} + \frac{\overline{m}L}{\overline{EI}}\frac{\partial^4 v(x,t)}{\partial t^4} = 0$$

Applying common assumption in variables' separation lemma, i.e. $v(x,t) = \varphi(x).Y(t)$ leads to:

(2) $\varphi^{iv}(x) - a^4\varphi(x) = 0$

(3) $\ddot{Y}(t) + \omega^2 Y(t) = 0$

Whereas:

(4) $\varphi(x) = A_1 \sin(ax) + A_2 \cos(ax) + A_3 \sinh(ax) + A_4 \cosh(ax)$

(5) $\omega = a^2 \sqrt{\dfrac{\overline{EI}}{\overline{mL}}} = {(aL)^2}\Big/{L^2} \sqrt{\dfrac{\overline{EI}}{\overline{mL}}}$

It is desired to find vibration frequency (or dimensionless frequency: q=aL). The fixed-free boundary conditions are;

(6) $\varphi(0) = 0$

(7) $\varphi'(L) = 0$

(8) $\overline{EI}\varphi''(L) = 0$

(9) $\overline{EI}\varphi'''(L) = 0$

In order to have non-zero solution for the above set of homogenous equations, its determinant should be assigned to zero. It consequently leads to the following <u>dimensionless frequency equation</u> for such a uniform property beam with Fixed-Free boundary conditions:

(10) $1 + Cos(q)Cosh(q) = 0$, $q = aL$

Omitting the remained dependent coefficient between (Eq.8) and (Eq.9), the mode-shape function is found in terms of one unknown coefficient, i.e. amplification factor A_1:

(11) $\varphi(x) = A_1\{\sin(ax) - \sinh(ax) + \dfrac{\sin(aL) + \sinh(aL)}{\cos(aL) + \cosh(aL)}[\cosh(ax) - \cos(ax)]\}$

As (Eq.10) and (Eq.11) are dimensionless, the effect of geometrical and mechanical properties of the corresponding beam will be considered by (Eq.5).

In order to develop the relation between parameters of chain structure and corresponding flexural beam a reverse mapping scheme is used.

Consider a flexural beam with fixed-free boundary conditions having parameters of uniform mass $\overline{m}$, moment of inertia $\overline{EI}$ and length L, be discretized to a lumped mass model in n equal-height story levels of the target chain structure (Fig.2). During such a mapping, mass of each discrete element is lumped to its center location, i.e. corresponding story level in the chain structure. Thus the length of each element; so-called *height portion* of the corresponding story, will be the same for all elements except the top and bottom ones. As a result, the mass of ground and top level will be half the value of all other similar lumped masses. This fact is considered the basis of definition of regular chain structure.

(12) $M = \overline{m}L = \sum_{i=0}^{n} m_i = (n-1)m_1 + 2\dfrac{m_1}{2}$, $L = H$

(13) $m_i = m_1 = \overline{m}\dfrac{L}{n} = \overline{m}L_i \quad i = 1,2,...,n-1 \Rightarrow \quad \overline{m} = \dfrac{m_i}{L_i} = {m_n}\Big/{L_n/2}$

Despite the parent beam, in the discretized model the mass lumped to ground elevation does not experience any displacements either and so its participation in modal analysis is zero. This will cause an error, so-called the *missed mass effect.*

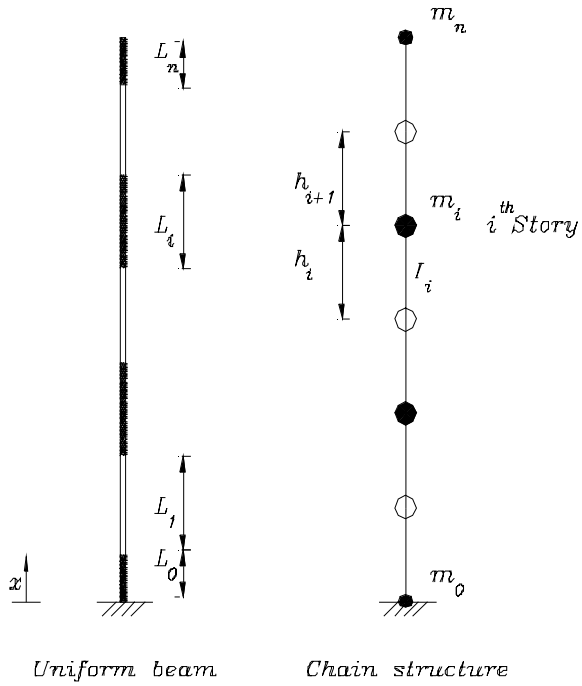

Figure 2. Discretizing a uniform beam and mapping it to a lumped mass model

Regarding reverse mapping, the similarity of mass, stiffness or height of stories in chain structure model may not always be the case in practice. So an equivalent unit mass and height portion for every story is defined based on the above analogy:

(14) $M = \sum_{i=1}^{n-1} m_i + m_n \quad , \quad \overline{m}_0 = \dfrac{M}{H - \frac{1}{2} h_1}$

(15) $L_i = \dfrac{h_i}{2} + \dfrac{h_{i+1}}{2}$

(16) $\overline{m_i} = \dfrac{m_i}{L_i}, i = 1,..,n$

As the value of ground level mass is not determined, the height portion of this level is subtracted from the total height in (Eq.14) in order to consider the missed mass effect. Overall unit mass, $\overline{m}_0$, is then calculated using this modified height.

Regarding mode-shapes of uniform beam vibration, it is investigated that the difference between the displacements of the first height portion with respect to the others is reduced in higher modes. As a result, the higher the mode number, the more the missed mass modal participation will be. So the following approximate relation is used to consider the missed mass effect for the jth mode:

(17) $\overline{m}_j = \overline{m}.(1 + \frac{\frac{1}{2}\overline{m}h_1}{M})^j$

Effect of relative changes in story mass and stiffness

Similarly, the mean stiffness of story-i and average stiffness over the whole height are defined as:

(18) $\overline{EI}_i = \frac{EI_i}{L_i}$

(19) $\overline{EI}_0 = \frac{1}{H}\sum_{i=1}^{n}\overline{EI}_i$

Based on recently developed mapping of the continuous uniform beam to the lumped mass model, *regular chain structure* can be defined under the following conditions:

(a) Story mean (unit) mass $\overline{m}_i$ (Eq.16) be equal to overall average mass $\overline{m}_0$ (Eq.14)

(b) Story mean (unit) stiffness $\overline{EI}_i$ (Eq.18) be equal to overall average mass, $\overline{EI}_0$ (Eq.19)

For such a regular case, the effective length of equivalent beam, L_e, is taken as structure's height, *H*. Otherwise, it is altered so that the equivalent beam frequency is as close as possible to that of the chain structure. The overall average mass and stiffness are uniformly distributed along the equivalent beam length to take advantage of its simple formulation. So the effect of mass/stiffness irregularity is entered through modified story height portion, L'_i. It is increased if the mass of i th story is greater than the total mean mass and vice versa. For the case of stiffness, however, the relation is reversed as below:

(20) $L'_i = L_i\alpha_i$, $\alpha_i = (\frac{\overline{m}_i}{\overline{m}_0})^{(\frac{x_i}{H})s_1}$

(21) $h'_i = h_i\beta_i$, $\beta_i = (\frac{\overline{EI}_i}{\overline{EI}_0})^{(\frac{x_i}{H})s_2}$

Here, the parameter s_1 is taken as 0.5 and s_2 as -0.5 due to reverse relation of mass to frequency with respect to stiffness as in Eq.5. The term (x_i /H) is employed due to the fact that the higher the story (number) is, the more its mass participation in modal frequency will be. Consequently, the total length of equivalent beam is given as sum of the modified height portions of story levels added to half modified height of the first level:

(22) $L_e = \sum_{i=1}^{n} L'_i + \frac{1}{2}h'_1$

QUICK SOLUTION OF THE FREQUENCY EQUATION

The dimensionless frequency equation of uniform beam with flexural behavior (Eq.10) is of hyperbolic type and needs considerable numerical effort to be solved. However, a simplifying scheme is employed here to find its approximate roots. Consider both sides of Eq.10 be multiplied by non-zero term e^{-q}, so its left side changes to :

(23) $g(q) = e^{-q}f(q) = e^{-q} + \cos(q)\frac{1+e^{-2q}}{2}$

As the dimensionless frequency, *q,* increases, the term e^{-q} will rapidly tend to zero. So the equation will be simplified to:

(24) $g(q) \approx \frac{1}{2}\cos(q) = 0$

Its well-known roots are given by the following binomial relation:

(25) $q_j = (j - 0.5) * \pi$

While j stands for mode number and $q = al$ is the dimensionless frequency. Fig.3 shows good satisfactory between roots of cos(q) and f(q).

This approximation error is reduced with j increase, so that it will be of the order 10^{-5} after the three first modes. Thus, it is considered acceptable for estimation purposes. Since the frequency equation (Eq.10) is dimensionless, exact solution of the first three modes can be saved once for ever as:

(26) $q_1 = 1.875104$, $q_2 = 4.694091$, $q_3 = 7.854757$

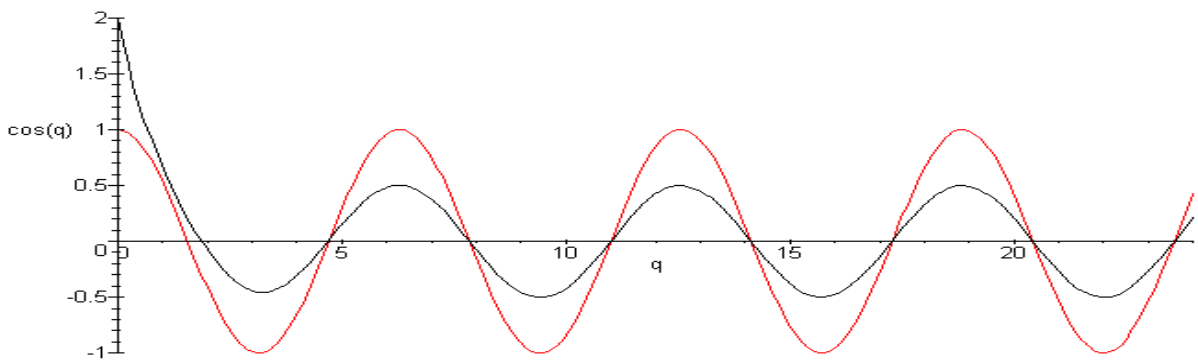

Figure 3. Comparison between roots of frequency equations g(q)=0 and cos(q)=0

ESTIMATION ALGORITHM

Based on the presented theory, the jth -mode frequency and shape of a tall multistory building can be directly estimated through the following steps:

1. Determine the jth dimensionless frequency using (Eq.25) or (Eq.26).
2. Generate the chain structure model and determine its parameters of mass, stiffness and story height.
3. Through mapping to uniform equivalent beam determine:
   3.1. Story height portion, L_i, by (Eq.15)
   3.2. Mean mass, $\overline{m_i}$, for every story (Eq.16) and also total average mass, $\overline{m_0}$ (Eq.14).
   3.3. Mean stiffness, $\overline{EI_i}$, for every story (Eq.18) and total average stiffness, $\overline{EI_0}$ (Eq.19).
   3.4. If $\overline{m_i} = \overline{m_0}$ and $\overline{EI_i} = \overline{EI_0}$ then chain structure is *regular* so $L_e = H$, else select the value of $s_2 = -s_1$ and compute the effective beam length, L_e .
4. Recover the missed mass effect as in (Eq.17) to obtain modified total mean mass in mode-j:

   (27) $\overline{m}_j = \dfrac{\overline{m}_0 H}{L_e}(1 + \dfrac{\frac{1}{2}\overline{m}h_1}{M})^j$

5. Modify the total mean stiffness in mode-*j* for the effective length by:

   (28) $\overline{EI}_j = \dfrac{L_e}{H}\overline{EI}_0$

6. Estimate the mode-*j* frequency using (Eq.5).

7. To obtain the desired mode-shape, $\varphi_j(x_i)$, substitute the mapped coordinates x_i in (Eq.11) from the following relation:

$$(29)\ x_i = \frac{h_1'}{2} + \sum_{m=1}^{i} L_m' - \tfrac{1}{2} L_i'$$

ILLUSTRATIVE EXAMPLES

Here, the suitability of the proposed method is examined through some samples of regular and irregular chain structures. In each example, the results have been compared with those of accelerated iterative subspace method as a benchmark. The ($KN-m$) unit system is employed for all the cases.

Regular ten story building

A 10-story building with the following specifications is modeled:

$m_i = 600$ for $i = 1,..,9$ while $m_{10} = 300$.

$E = 2*10^9, I_i = 0.2$, $h_i = 3$ for $i = 1,..,10$.

$L_i = 3$ for $i = 1,..,9$ and $L_{10} = 3/2$.

$$\forall i \rightarrow \overline{m}_i = m_i / L_i = 200 \Rightarrow \overline{m}_0 = \overline{m}_i \quad \text{and} \quad \overline{EI}_0 = \overline{EI}_i = EI_i / h_i = \frac{4*10^8}{3}.$$

Thus, such a model satisfies both conditions: (a) and (b), so it is considered a regular chain structure. Each node in this model has a rotational as well as a transversal degree of freedom. Total mass and the missed mass are computed as below:

$$M = \sum_{i=1}^{10} m_i = 9*600 + 300 = 5700\ ,\ m_{lost} = \overline{m}_0 h / 2 = 200*3/2 = 300$$

The results of the current approximate method have been compared with the exact one through computed relative errors for each mode, in table 1.

Table 1. Comparison of exact vs. approximate periods obtained for regular 10-story building regarding missed mass recovery (Average relative error = 5.5%)

Mode No.	1	2	3	4	5	6	7	8	9	10
Exact period T *(s)*	1.3519	.2084	.0728	.0366	.0219	.0146	.0106	.0082	.0067	.0060
Estimated T *(s)*	1.1668	.1908	.0697	.0364	.0224	.0154	.0112	.0086	.0068	.0055
RelativeError %	-13.7	-8.4	-4.3	-0.5	-2.3	5.5	5.7	4.9	1.5	-8.3

Considering the case of ignoring the missed mass effect as in table 2, it is found that for such a case the maximum and mean relative error have been increased from 13.7% to 25% and from 5.5% to 12.9% correspondingly. This notifies the necessity of regarding missed mass recovery through estimation.

Table 2. The effect of missed mass on the accuracy of estimated periods (Average relative error = 12.9%)

Mode No.	1	2	3	4	5	6	7	8	9	10
Exact period T *(s)*	1.3519	.2084	.0728	.0366	.0219	.0146	.0106	.0082	.0067	.0060
Estimated T *(s)*	1.1373	.1815	.0645	.0331	.0200	.0134	.0096	.0072	.0056	.0045
RelativeError %	-15.9	-12.9	-11.0	-9.6	-8.7	-8.2	-9.4	-12.2	-16.4	-25.0

In addition, a conformation can be found between the estimated and exact mode shapes. For example, shape of mode-7 is illustrated in Fig.4.

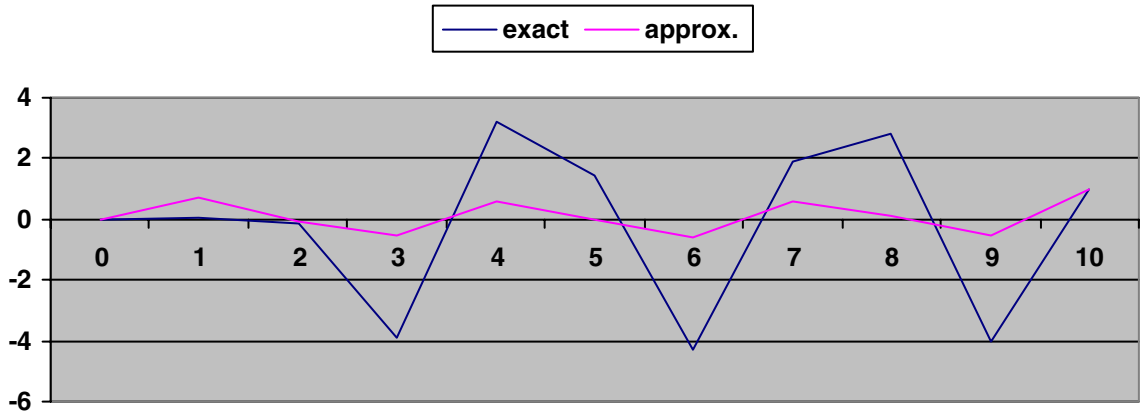

Figure 4. Example of estimated vs. exact 7th mode shape

Irregular 10-story building

This example shows the ability of the method in modeling highly irregular chain structures. To this end, some characteristics of the previous regular model are changed as:

$h_1 = 5$,

$m_2 = m_3 = 1400$, $m_{10} = 600$, $m_6 = 1000$

$I_6 = 0.08, I_7 = I_8 = I_9 = I_{10} = 0.10$

Decreasing stiffness or increasing mass in some levels model the soft story effect. The accuracy of approximate results with respect to exact solution is given in Table 3.

Table 3. Comparison of exact vs. approximate periods obtained for irregular 10-story building (Average relative error = 9.0%)

Mode No.	1	2	3	4	5	6	7	8	9	10
Exact period T *(s)*	1.3519	.2084	.0728	.0366	.0219	.0146	.0106	.0082	.0067	.0060
Estimated T *(s)*	1.1668	.1908	.0697	.0364	.0224	.0154	.0112	.0086	.0068	.0055
RelativeError %	-13.7	-8.4	-4.3	-0.5	-2.3	5.5	5.7	4.9	1.5	-8.3

Regular 30-story building

The effect of more distributed models has been studied in this example of a 30-story regular building model with the same characteristics of the 1st example. As a comparison parameter, the mean relative error has decreased from 5.5% for 10-story model to 3.9% in this case (Table 4) while the maximum error has reduced to 13%.

Table 4. Comparison of exact vs. approximate periods obtained for regular 30-story building (Average relative error = 3.9%)

Mode No.	1	2	10	20	28	30
Exact period T *(s)*	10.904	1.731	.0418	.0100	0.0060	0.0058
Estimate T *(s)*	10.322	1.661	.0439	.0331	0.0061	0.0054
RelativeError %	-5.3	-4.0	5.0	13.0	1.7	-6.9

CONCLUDING REMARKS

The proposed method has developed mapping of the chain structure model of a multistory building to the uniform flexural beam. The frequency equation of corresponding equivalent beam is dimensionless, thus independent of building structural properties except the boundary conditions and dynamic behavior. As an outstanding point, high computational effort in solving such a hyperbolic equation is reduced to simply evaluating a binomial for each mode number.

The missed mass effect during the mapping has been declared and treated by a mass recovery technique. Results obtained through examples of multi-story buildings notify the importance of regarding such an effect. The proposed mapping introduces new definition of regular structure in height, especially for the mass/stiffness of the last story with respect to the others.

The effect of mass and stiffness irregularity was treated by modifying equivalent beam parameters, specially its effective length. The method was then applied to high irregular examples in the case of mass, stiffness and story height (up to 200, 80 and 60 percent correspondingly). According to treated examples, the estimated results were in acceptable range of accuracy even when assessing such high irregular cases including soft story effect. The mean relative error varies from 5.5% for 10-story and less than 4% in 30-story example which introduces better mean accuracy for taller buildings. Reasonable satisfactory was also found between exact and estimated mode shapes.

Another advantage of the proposed method is its capability to predict both desired mode shape and eigenvalue simultaneously but independent from other modes' calculations. The results can be considered acceptable for some applications such as analysis quick-checks; however, they can be further improved being used as initial point for exact iterative methods in order to achieve even higher accuracy and efficiency. In addition, such a direct modal estimation may be of interest for steps of highly model-changing iterative methods such as optimization or nonlinear analyses.

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