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AN APPROACH FOR DETERMINING THE EIGENFREQUENCIES OF RANDOM SOIL WITH THE FOKKER-PLANCK EQUATION

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ABSTRACT

The eigenfrequencies of the soil are mainly determined by the local soil conditions. They can be analytically calculated by solving the wave equation in inhomogeneous or layered media. However a stochastic approach seems to be necessary, because the parameters of the soil system can only be determined in a stochastic sense. Thus the solution of the stochastic wave equation is sought. In this paper the wave equation is transformed in such a way that the transition probability is governed by the Fokker-Planck equation. This is an interesting alternative method to the usual perturbation or stochastic Green's function method, because not only the first moments but the complete stochastic behaviour of the solution process can be achieved. As a case example the Fokker-Planck equation for the one-dimensional harmonic scalar Helmholtz equation is derived and numerically solved with the Finite Differences method.

KEYWORDS

stochastic wave propagation, Fokker-Planck equation, finite differences, eigenfrequencies, random soil, SH-wave, soil response

INTRODUCTION

Predicting the eigenfrequencies of the soil is an important task in the framework of earthquake engineering. The concentration of energy of an earthquake in the vicinity of the natural frequencies of man-made structures can lead to severe damages. Empirical observations show that often power spectral densities during different earthquakes or aftershocks at the same stations show significant stable peaks. In fig. 1 two normalised PSD of strong motion records (NS-component) recorded at INCERC station in Romania during the Vrancea 1977 and 1986 earthquake are shown. The first two peaks of the spectrum nearly coincide showing the dominant site effects (Lungu et al., 1994). Theoretical investigations (Zsohar and Scherer, 1995) show that the eigenfrequencies of the soil are mainly determined by the local soil system and that the location of the source is of minor importance.

Because the parameters of the local soil system, i.e. wave velocities and damping cannot be determined for any given location in a deterministic sense, uncertainties of the local soil conditions have to be taken into

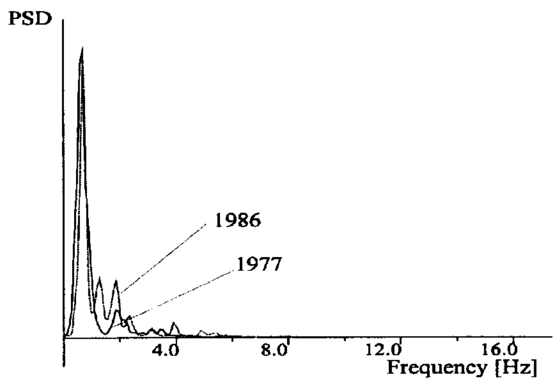

Fig. 1 Power Spectral Densities (NS) of INCERC-station during 1977 and 1986 Vrancea earthquake in Romania

account. Thus it is a reasonable attempt to model the soil as a random medium, i.e. to model the parameters of the soil as stochastic processes or random variables. In a first approach to this problem (Zsohar and Scherer, 1995) Monte Carlo Simulations for the damping, the wave velocities, the layer thickness' and the angle of inclination of the wave trace into the bedrock have been carried out.

The interest of this research is focused on a generally applicable model, which is able to predict the eigenfrequencies of uncertain soil conditions. Therefore the stochastic wave equation has to be solved. The methods available in the literature can be classified into analytical and numerical methods. Among the first ones are approximate methods like the perturbation method (Sobczyk, 1985), the smoothing method (Karal and Keller, 1964; Hryniewicz, 1989) or the stochastic Green's function method (Manolis and Bagtzogluo, 1992). Among the latter ones are stochastic Finite Elements (Waubke and Grundmann, 1994), stochastic Finite Differences (Frankel and Clayton, 1984) or Monte Carlo Simulations. A major drawback of all these methods is that they allow to determine the mean wave of an ensemble of waves only. Some of them can calculate the second moments, e.g. the Born approximation (Sobczyk, 1985), but neither of them the probability density function.

In this paper an alternative method of solving the stochastic wave equation will be presented, which allows to calculate the probability density function of the amplitude of the wave. It is based upon a work of Frisch (1968). The basic idea is to give a formulation of the scalar wave equation in such a way that the transition probability is governed by the Fokker-Planck Equation (FPE). Therefore all stochastic processes have to be assumed Markovian. In a first step the one-dimensional harmonic scalar wave equation (Helmholtz equation) in one layer, in which the S-wave velocity is randomly disturbed, is considered. The corresponding four-dimensional FPE is solved numerically with the Finite Differences Method. The results for different frequencies are presented and compared. This method can be easily extended to the case of layered soil by the usual transfer matrices. For the case of randomly layered soil the necessary tools can be found in (Kotulski, 1990).

SCALAR STOCHASTIC WAVE PROPAGATION

Dealing with scalar seismic waves leads to a wave equation with a forcing function representing the excitating source.

$$\nabla^2 \mathbf{G}(\mathbf{r},\mathbf{r}_S) + k_0^2 n^2(\mathbf{r}) \mathbf{G}(\mathbf{r},\mathbf{r}_S) = -\delta(\mathbf{r}-\mathbf{r}_S) \quad (1)$$

In this Helmholtz equation $\mathbf{r}$ is the position vector, $\mathbf{r}_S$ the source vector, $\mathbf{G}$ the Green's function, k_0 the wave number, δ the Dirac delta function and n the index of refraction of the medium. An invariable index, i.e. $n(r) \equiv 1$ leads to the case of homogeneous wave propagation. In an unbounded homogeneous three-dimensional medium with a single impulse as source function (Dirac delta function) this problem can be analytically solved. The solution for homogeneous boundary conditions is the Green's function

$$G(\mathbf{r}-\mathbf{r_S}) = \frac{e^{ik_0|\mathbf{r}-\mathbf{r_S}|}}{4\pi|\mathbf{r}-\mathbf{r_S}|} \tag{2}$$

In the case of a variable index of refraction only few closed-form solutions exist. Many recent work (Li 1990) has been done for the case $n(\mathbf{r}) = \sqrt{1+Az}$, where z is the vertical component and A is a small negative factor.

Wave propagation in a random medium is characterised by a stochastic index of refraction $n(r) = n(r,\gamma)$. Karal and Keller (1964) solved this problem for the mean wave of an ensemble of waves by a second order perturbation method (smoothing method). Manolis (1992) gave a perturbation expansion for the stochastic Green's function. Chu et al. (1981) calculated a solution with the Laplace transform method under the assumption that the shear coefficient is an 'Uhlenbeck-Ornstein Process'. A common result of these researches is that the mean motion of a stochastic wave is different from the solution of the wave equation with the mean properties, although the waves are linear. An energy decay emerges by the scattering of the waves through the inhomogeneities of the medium. This is manifested by a non-empty imaginary part of the amplitude.

The one-dimensional scalar (SH) wave equation of interest reads

$$\Psi_{xx}(x,t) = \frac{1}{\alpha^2}\Psi_{tt}(x,t) \tag{3}$$

with wave velocity α. When looking at the eigenfrequencies of random soil it is sufficient to consider harmonically excitated layers. Considering harmonic waves with frequency ω

$$\Psi(x,\omega,t) = \Psi(x,\omega)\cdot e^{i\omega t} \tag{4}$$

and omitting the time factor on both sides the wave equation simplifies to

$$\Psi''(x) = -\frac{\omega^2}{\alpha^2}\Psi(x) \tag{5}$$

The wave can now be analysed separately for each frequency ω_i because of the principle of superposition.

Stochastic Case

The approach chosen follows the work of (Frisch, 1968). With wave number $k^2 = \frac{\omega^2}{\alpha^2}$ the Stochastic Helmholtz Equation (SHE) in a homogeneous, isotropic, stationary random medium is

$$\frac{d^2}{dx^2}\Psi(x) + k_0^2 n^2(x)\Psi(x) = g(x) \tag{6}$$

- n is the index of refraction. As the interest of this research is focused on stochastic wave propagation, $n^2(x) = 1+\mu(x)$ is a real homogeneous, isotropic random function with unit mean square value. It represents a random disturbation in the medium, i.e. a fluctuation in the wave velocity.
- g can be chosen as the Dirac delta function representing a point source or taken to be zero. In the latter case the excitation of the medium is to be fulfilled by some boundary conditions.

The Helmholtz equation describes the propagation of a wave Ψ with a special frequency ω_i from the source (x=0) to the receiver (x=L). The solution of the Helmholtz equation at x=L is the amplitude $A(\omega_i)$ of the wave at the receiver. Thus the bigger the amount of the amplitude of the special ω_i is the larger is the amplification function of the layer at this frequency. The maximum amplitude corresponds to the dominant eigenfrequency of the layer. The solution of the stochastic Helmholtz equation is the probability density $P(x;\mu(x),\Psi(x),\Psi'(x))$ at x=L. Having a solution of the SHE means having the probability that the corresponding amplitude is less than a certain value. Considering all interesting frequencies ω_i the one with the maximum probability can be found.

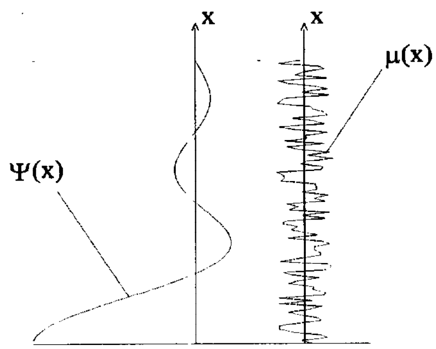

Fig. 2 A scalar wave $\Psi(x)$ propagates along the x-axis with wave velocity α, which is randomly disturbed by a stochastic process $\mu(x)$

Procedure. Taking the index of refraction as a stochastic process the SHE can be splitted into a constant and fluctuating part

$$\frac{d^2}{dx^2}\Psi(x) + k_0^2(1+\mu(x))\Psi(x) = 0 \tag{7}$$

With given initial conditions for $\Psi(0) = \Psi_0$ and $\Psi'(0) = \Psi_0'$ this equation is a linear random differential equation. The stochastic process $\mu(x)$ is assumed as an Uhlenbeck-Ornstein process, i.e. a centred, stationary, Gaussian and Markovian random function with spatial correlation function

$$E\{\mu(x)\mu(x')\} = \varepsilon^2 e^{-\frac{|x-x'|}{L}} \tag{8}$$

leading to a linear differential equation with Markov coefficients. The simplifications

- $L \equiv 1$
- $\mu(x) \mapsto \varepsilon\mu(x)$

lead to the simpler correlation function

$$E\{\mu(x)\mu(x')\} = e^{-|x-x'|}$$

and Helmholtz equation

$$\frac{d^2}{dx^2}\Psi(x) + k_0^2(1+\varepsilon\mu(x))\Psi(x) = 0 \tag{9}$$

Frisch showed that such a linear (Markovian) random differential equation possesses an associated transition probability density function satisfying a FPE. In order to apply this equation, two new variables

$$y_1 = \Psi(x) \qquad \text{and} \qquad y_2 = \frac{1}{k_0}\Psi'(x)$$

are introduced into the second order wave equation leading to the system of first order equations

$$\frac{d}{dx}y_1(x) = k_0 y_2(x) \tag{10}$$

$$\frac{d}{dx}y_2(x) = -k_0^2(1+\varepsilon\mu(x))y_1(x) = 0 \tag{11}$$

or in state vector form

$$\frac{d}{dx}\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 0 & k_0 \\ -k_0(1+\varepsilon\mu(x)) & 0 \end{bmatrix}\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} \tag{12}$$

$\mu(x)$ is centred, stationary and Gaussian, thus the corresponding probability density is

$$W(\mu) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\mu^2\right) \tag{13}$$

a stationary normalised solution of the FPE

$$\frac{\partial}{\partial x} W(x;\mu) = \frac{\partial}{\partial \mu}(\mu W) + \frac{\partial^2}{\partial \mu^2} W \qquad (14)$$

with drift $m_W=\mu$ and diffusion $b_W=1$.

The drift-vector for the wave equation is

$$\begin{bmatrix} m_1(x) \\ m_2(x) \end{bmatrix} = \begin{bmatrix} k_0 y_2(x) \\ -k_0(1+\varepsilon\mu(x))y_1(x) \end{bmatrix} \qquad (15)$$

The diffusion coefficients b_{11} , ..., b_{22} equal zero, for there is no excitating white noise process. Combining this there exists a FPE for the process μ, y_1, y_2

$$\frac{\partial}{\partial x} P(x;\mu,y_1,y_2) = \frac{\partial}{\partial \mu}(\mu P) + \frac{\partial^2}{\partial \mu^2} P - \frac{\partial}{\partial y_1}(k_0 y_2 P) + \frac{\partial}{\partial y_2}\left(k_0(1+\varepsilon\mu)y_1 P\right) \qquad (16)$$

with initial condition

$$\frac{\partial}{\partial x} P(0;\mu,y_1,y_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mu^2} \delta\left(y_1 - y_1(0)\right)\delta\left(y_2 - y_2(0)\right) \qquad (17)$$

No solution of this FPE is known. Frisch gave a solution for the first moment of the displacement and the velocity only. But it is clear that the usefulness of the FPE lies in the fact that it determines with given initial conditions the complete probabilistic behaviour of the solution process, namely the probability density function. Deriving the FPE and then solving only for the mean value is some kind of a contradiction. Thus in this paper an attempt is made to solve the FPE numerically.

NUMERICAL SOLUTION OF THE FOKKER-PLANCK EQUATION

Carrying out the differentiation of eq. (15) the FPE can be written as

$$\frac{\partial}{\partial x} P(x;\mu,y_1,y_2) = P + \mu\frac{\partial}{\partial \mu} P + \frac{\partial^2}{\partial \mu^2} P - k_0 y_2 \frac{\partial}{\partial y_1} P + k_0(1+\varepsilon\mu)y_1 \frac{\partial}{\partial y_2} P \qquad (18)$$

<u>Discretization</u>. The different derivatives are approximated by their forward differences, e.g.

$$\frac{\partial P(x;\mu,y_1,y_2)}{x} = \frac{P(x+h;\mu,y_1,y_2) - P(x;\mu,y_1,y_2)}{h} \qquad (19)$$

leading to a Finite Differences scheme which propagates into x-direction

$$\begin{aligned} P(x+h;\mu,y_1,y_2) &= P(x;\mu,y_1,y_2) \\ &+ h\Bigg\{ P(x;\mu,y_1,y_2) + \mu\frac{P(x;\mu+k,y_1,y_2) - P(x;\mu,y_1,y_2)}{k} \\ &+ \frac{P(x;\mu+k,y_1,y_2) - 2P(x;\mu,y_1,y_2) + P(x;\mu-k,y_1,y_2)}{k^2} \\ &- k_0 y_2 \frac{P(x;\mu,y_1+l,y_2) - P(x;\mu,y_1,y_2)}{l} + k_0(1+\varepsilon\mu)y_1 \frac{P(x;\mu,y_1,y_2+m) - P(x;\mu,y_1,y_2)}{m} \Bigg\} \end{aligned} \qquad (20)$$

h,k,l,m are the steps in the x, μ, y_1 and y_2 directions respectively. The following points need special care:

1. Range of definition of the four variables
2. Initial conditions (delta functions)
3. Boundary conditions
4. Stability

1. In order to solve this equation the ranges of the different variables have to be determined.

- As x is the spatial coordinate of the wave equation, it determines the travel path of the wave through one layer. Thus it starts at 0 and ends at L, e.g. L=10 m.
- y_1 represents the displacement amplitude, which is normalised to be in range of [-1, 1]
- y_2 represents the velocity, thus the same range [-1, 1] as for the displacement is assumed..
- μ is a stochastic process (SP) with mean 0 and standard deviation 1. Thus a range of [-5,5] suffices.

2. The initial conditions are the normal distribution for μ and the delta functions for y_1 and y_2. The delta function has to be approximated with a discretised function keeping the basic properties. It must pick the values of the normal distribution at the initial values for the displacement and velocity and must set all other values to zero. The second property is that the integral over the delta function must be unity. Therefore a normalising factor for the density values has to be introduced. The stochastic interpretation of this initial conditions says that the probability that the wave gets in a state, which is not the initial state, is zero.

3. The discretization of the FPE means a transformation from an infinite region into a limited grid. Therefore at the maximum and minimum values of the grid values for the transition probability have to be prescribed. The natural conditions are to set these values to zero, because the probability that the wave gets into a state, where e.g. the amplitude is bigger than the maximum amplitude clearly is 0. As the discretization steps forward into the x-direction, for x only the initial conditions has to be prescribed.

4. An extensive theoretical stability investigation has not been carried out. Testing various grid steps a step size of 10^{-3} to 10^{-4} for the spatial coordinate has been chosen. Due to stability necessities the other three grid steps have to be much larger, i.e. greater than 0.1. If a finer discretization in the other three variables is needed, the spatial grid step size has to be decreased.

Results

Four different frequencies have been chosen, i.e. 1, 2.5, 5 and 10 Hz representing the low to intermediate and high frequency range concerning buildings. In fig. 3 the probabilities for the displacement to be less than 0.5 and 0.2 are plotted. The initial conditions are $y_1(0) = 1.0$ and $y_2(0) = 0$, i.e. $\Psi(0) = 1.0$ and $\Psi'(0) = 0$. The SH-wave velocity is assumed as 200 m/s. For the numerical calculations the following grid steps have been chosen: Δx=0.0001m, $\Delta\mu$=0.25, $\Delta y_1=\Delta y_2$=0.1. The CPU time needed on a HP 715-100 workstation was approximately 40 h for 10^6 iterations for each frequency.

The amplitudes of the higher frequency wave clearly are decreasing faster. The reason for this is that the one-dimensional Green's function for the deterministic homogeneous problem follows the cosine function without a geometrical 1/r damping as in the three-dimensional case. Thus the stochastic case shows the same effects as the homogeneous one. In fig. 4 the probability densities $P(x;\Psi(x))$ for a frequency of 1 and 10 Hz are plotted against the displacement and the spatial coordinate. The natural boundary conditions at $\Psi(x) = \pm 1$ can be seen. The common peaks at the beginning (x=0) are caused by the delta function at $\Psi(0) = 1$. For the 1 Hz component a second peak arises at 80m.

Comparison of low and high frequency case

The influence of the stochastic variation of the wave velocity is now analysed in greater detail for the 1 Hz and 10 Hz cases. The Green's function for the one-dimensional homogeneous case is known (fig.5). The homogeneous case says that the displacement is less than 0.5 at 33.33m. The analogous probability $P_{1Hz}(\Psi(33) \le 0.5)$ equals 0.392. For the 10Hz component the displacement will be less than 0.5 at 3.3 m. $P_{10Hz}(\Psi(3.3) \le 0.5)$ is slightly smaller, i.e. 0.390. The same is now done for the probabilities of the displacement to be less than 0.2, i.e. $P_{1Hz}(\Psi(43.0) \le 0.2)$=0.346 against $P_{10Hz}(\Psi(4.3) \le 0.2) = 0.344$ and for the probabilities to be less than zero, i.e. $P_{1Hz}(\Psi(50) \le 0)$=0.392 against $P_{10Hz}(\Psi(5) \le 0) = 0.390$. The probabilities of the corresponding 2.5 and 5Hz cases always are in between the extreme cases. The corresponding mean values show this effect (fig.5) more clearly: $E(\Psi_{1Hz}(33))$= 0.577 against $E(\Psi_{10Hz}(3.3))$= 0.584 and $E(\Psi_{1Hz}(50))$= 0.113 against $E(\Psi_{10Hz}(5))$= 0.132. Two consequences can be drawn: First, the random disturbation of the medium acts slightly different on the different frequency parts of the wave, i.e. the lower frequency part amplitude is more decreased than the higher frequency part. Second, the mean value of the amplitude is not the solution of the wave equation with mean properties (fig. 5), because in this case e.g. $E(\Psi_{1Hz}(50.0))$ and $E(\Psi_{10Hz}(50.0))$ had to equal zero. The first effect seems negligible and can also be caused by numerical errors because of the different length of the travel paths. The second effect is much more important and can be explained by the scattering of the waves through the inhomogeneities in the medium (Chu, 1981).

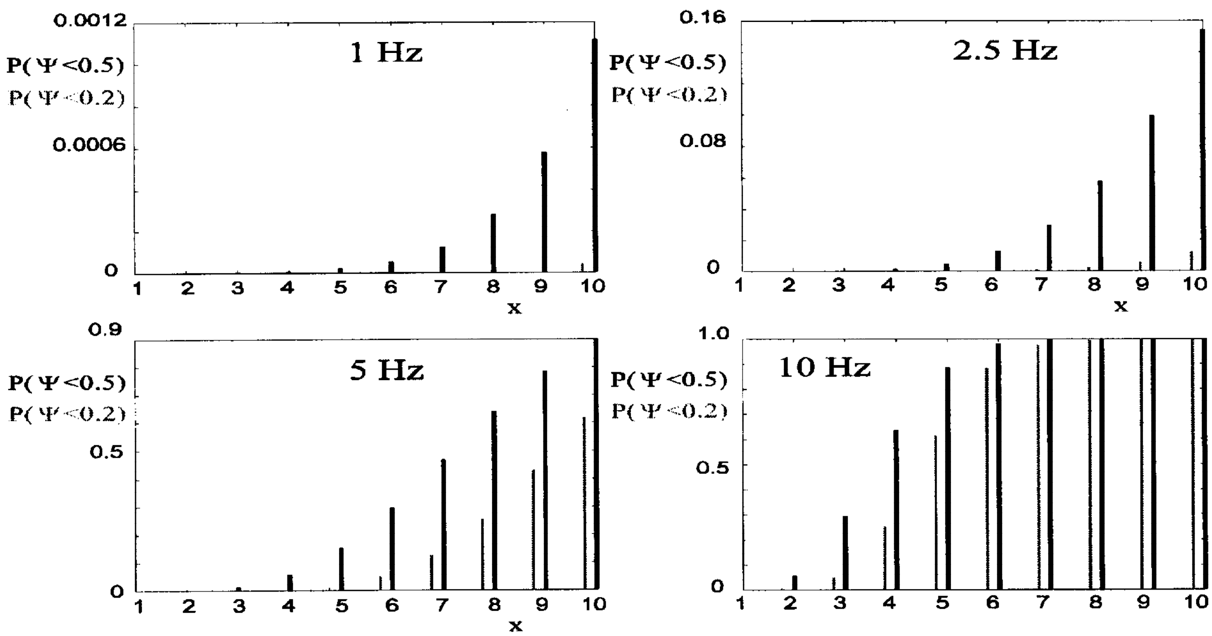

Fig. 3 Probability of the displacement $\Psi(x)$ to be less than 0.5 and 0.2 (initial condition $\Psi(0)=1.0$ and $\Psi'(0)=0$) for four different frequencies (1,2.5,5 and 10 Hz) over the travel distance x [m]. The wave velocity α=200 m/s is disturbed by the SP $\mu(x)$.

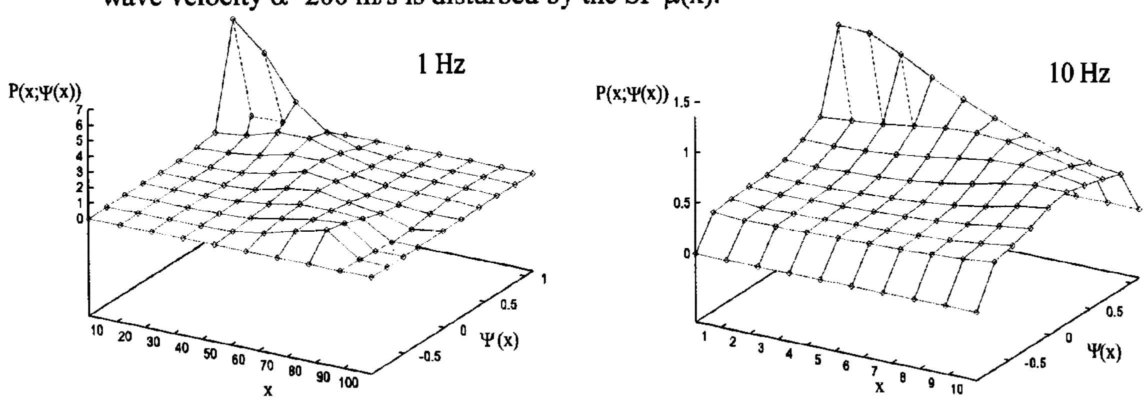

Fig. 4 Probability density functions of the displacement $P(x;\Psi(x))$ against x [m] and $\Psi(x)$ (initial condition $\Psi(0)=1.0$ and $\Psi'(0)=0$) for two different frequencies (left: 1 Hz for $x\in[0,100m]$, right: 10 Hz for $x\in[0,10m]$). The wave velocity α=200 m/s is disturbed by the SP $\mu(x)$.

CONCLUSIONS

In this paper the problem of wave propagation in a random medium with the aim of determining the probabilistic behaviour of the eigenfrequencies of random soil was investigated. The approach chosen shows the need for a stochastic treatment of the propagation of earthquake waves, because the effects of the random disturbations of the medium act in a non-linear way. In contrast to the most methods available in the literature the formulation with the aid of the FPE allows to determine the probability density function of the solution process. A minor shortcoming of this method is the fact that the stochastic process disturbing the medium has to be assumed Markovian. The major drawback is the extreme high computational efforts for solving the four-dimensional partial differential equation (FPE). Describing realistic models one has to resort to at least the two-dimensional problem. Therefore the approach solving the FPE seems to be not as attractive because in this case the CPU-time is approximately the second power of the one-dimensional case, i.e. 40^2 hours for the HP715-100. Fast supercomputers and optimised algorithms will be favourable. However, it is still an open question whether the solution of the FPE is faster than sophisticated simulating.

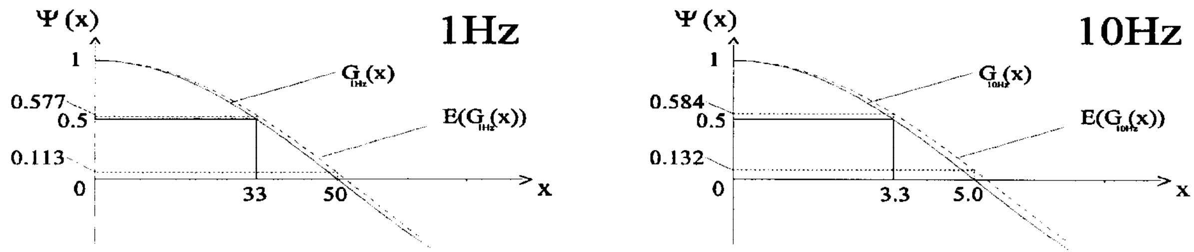

Fig. 5 Homogeneous solution G(x) (solid line) and mean solution E(G(x)) (dashed line) for the 1 Hz (left) and 10Hz (right) components

The results achieved for one layer will be extended in an ongoing research to layered soil. The source will be implemented as a band-limited white noise process as proposed in (Zsohar and Scherer, 1995). This leads to an Itô differential equation (Soong, 1993), whose transition probability again is governed by the FPE.

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