Paper No. 773. (quote when citing this article)
Eleventh World Conference on Earthquake Engineering
ISBN: 0 08 042822 3

FLEXIBILITY-BASED FRAME MODELS FOR NONLINEAR DYNAMIC ANALYSIS

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ABSTRACT

This paper presents a family of frame models based on the flexibility method of structural analysis. The models are used for the nonlinear dynamic analysis of both reinforced concrete and steel frames. The increased computational cost of the element state determination is compensated by the element robustness that leads to the reduction of both total elements and degrees of freedom of the nonlinear frame analysis. Two nonlinear moment-curvature relations and a fiber section model are presented. Computational aspects related to the number of integration points and the section refinement are discussed. Finally, a consistent method for the application of distributed loads is presented in which section axial force and bending moments are always in equilibrium with both nodal and distributed loads.

KEYWORDS

Beam element; finite element; flexibility method; fiber model; endochronic model; hysteretic model; reinforced concrete; element loads.

ELEMENT FORMULATION

Most frame elements implemented in present finite element codes are based on the classical stiffness method of structural analysis. It is however recognized that such elements are inaccurate when strong ground motions induce highly nonlinear responses in the structural members and fine meshes are needed to obtain satisfactory results. In recent years there has been a growing interest in the development of frame elements based on the flexibility method of structural analysis. The advantage of this method derives from the assumption of the exact force distributions in the elements. The main challenge presented by this approach lies in the element implementation in a general purpose finite element program, because the section resisting forces are not readily related to the element resisting forces, as is the case of elements based on the stiffness formulation.

The element formulation proposed herein can be derived using either a flexibility approach or a mixed method. In the case of a beam element it is possible to show that the two formulations yield the same matrix relation between element forces and element deformations. Because of its generality, the mixed approach is followed. The beam element is shown in Figure 1. The element is represented in the local reference system without rigid-body modes, thus element generalized forces and deformations ($\boldsymbol{Q}$ and $\boldsymbol{q}$, respectively), are measured with respect to the cord connecting the two end nodes.

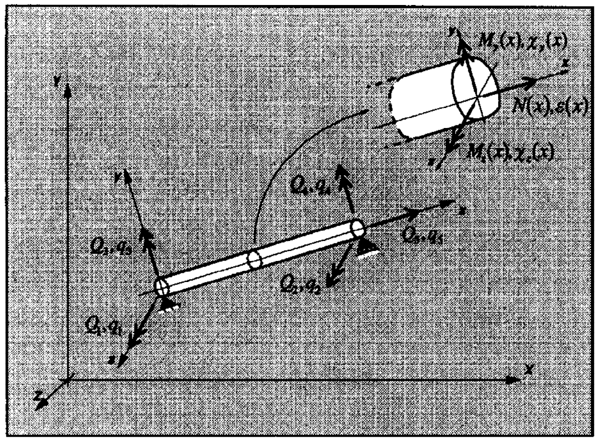


Fig. 1. Beam element forces and deformations without rigid body modes in local reference system.

Denoting with Δ increments of the corresponding quantities, the deformation and force fields ($\boldsymbol{d}(x)$ and $\boldsymbol{D}(\boldsymbol{x})$, respectively) are written

$$\begin{aligned} \Delta \boldsymbol{d}(x) &= \boldsymbol{a}(x)\,\Delta \boldsymbol{q} \\ \Delta \boldsymbol{D}(x) &= \boldsymbol{b}(x)\,\Delta \boldsymbol{Q} \end{aligned} \tag{1}$$

where matrices $\boldsymbol{a}(x)$ and $\boldsymbol{b}(x)$ denote the deformation and force interpolation functions, respectively. The incremental section constitutive relation is written in the form

$$\Delta \boldsymbol{d}(x) \;=\; \boldsymbol{f}(x)\,\Delta \boldsymbol{D}(x) \;+\; \boldsymbol{r}(x) \tag{2}$$

where $\boldsymbol{f}(x)$ and $\boldsymbol{r}(x)$ are the section flexibility and residual deformations, respectively. The residual deformations can be seen as the linear approximation to the deformation error made in the linearization of the section force-deformation relation. Two integral equations are needed in the two-field mixed method. One is the weighted integral form of Eq. (2):

$$\int_0^L \delta \boldsymbol{D}^T(x)\left[\Delta \boldsymbol{d}(x) - \boldsymbol{f}(x)\,\Delta \boldsymbol{D}(x) - \boldsymbol{r}(x)\right] dx = 0 \tag{3}$$

The other is the equilibrium equation of the beam element, which can be obtained from the virtual displacement principle:

$$\int_0^L \delta \boldsymbol{d}^T(x)\left[\boldsymbol{D}(x) + \Delta \boldsymbol{D}(x)\right] dx \;=\; \delta \boldsymbol{q}^T \boldsymbol{Q} \tag{4}$$

Upon substitution of Eq. (1) in Eqs. (3) and (4) and after rearrangement of the resulting expressions, the final element matrix equation is

$$\boldsymbol{T}^T \left[\boldsymbol{F}\right]^{-1} \left(\boldsymbol{T}\,\Delta \boldsymbol{q} - \boldsymbol{s}\right) \;=\; \boldsymbol{Q} - \boldsymbol{T}^T \boldsymbol{Q} \tag{5}$$

where

$$\boldsymbol{T} \;=\; \int_0^L \boldsymbol{b}^T(x)\,\boldsymbol{a}(x)\;dx \tag{6}$$

$$\boldsymbol{F} \;=\; \int_0^L \boldsymbol{b}^T(x)\,\boldsymbol{f}(x)\,\boldsymbol{b}(x)\;dx \tag{7}$$

$$s = \int_0^L b^T(x)\, r(x)\ dx \tag{8}$$

$\boldsymbol{T}$ is a matrix that depends only on the interpolation functions, $\boldsymbol{F}$ is the element flexibility matrix and $\boldsymbol{s}$ is the element residual deformation vector.

The selection of the interpolation functions $\boldsymbol{b}(x)$ and $\boldsymbol{a}(x)$ for the beam element greatly simplifies the above expression. Matrix $\boldsymbol{b}(x)$ is computed from the assumption of constant axial force and linear bending distributions within the element. Spacone *et al.* (1996) show that the selection of $\boldsymbol{a}(x)$ in the present application of the mixed approach does not affect the element formulation because $\boldsymbol{q}$ and $\boldsymbol{Q}$ are conjugate resultants from a work viewpoint. This fact, peculiar to the proposed Bernoulli beam, leads to $\boldsymbol{T}=\boldsymbol{I}$, where $\boldsymbol{I}$ is the 3x3 identity matrix, irrespective of the selection of $\boldsymbol{a}(x)$. Eq. (5) thus becomes

$$[\boldsymbol{F}]^{-1}(\Delta \boldsymbol{q} - \boldsymbol{s}) = \Delta \boldsymbol{Q} \tag{9}$$

The element state determination is based on the element residual deformations (8). These deformations cannot be applied at the element nodes because they violate node compatibility, thus end forces are applied to the element to impose end deformations $-\boldsymbol{s}$ using the current tangent element stiffness matrix. These forces change the element force field and yield new section deformations that cause new section residuals $\boldsymbol{r}(x)$. The iterations continue until the element residual deformations become sufficiently small. During the iterations the element force and deformation fields are adjusted until the section constitutive relations are satisfied, while always satisfying equilibrium along the element. Details on the element formulation and state determination can be found in Spacone *et al.* (1996). The element is presently implemented in the general purpose finite element program FEAP developed by Professor R.L. Taylor at the University of California at Berkeley and documented in Zienkiewicz and Taylor (1989 and 1991).

SECTION BEHAVIOR

The computation of the section resisting forces $\boldsymbol{D}_R(x)$ and flexibility matrix $\boldsymbol{f}(x)$ is an important step of the element state determination. In the present version of the element, three different section models are implemented (Figure 2): a section model based on the endochronic theory of plasticity, a piecewise linear model and a model in which the section is subdivided into longitudinal fibers. The first two models assume that axial and flexural responses are decoupled, while the fiber section naturally accounts for the interaction between axial and flexural behaviors.

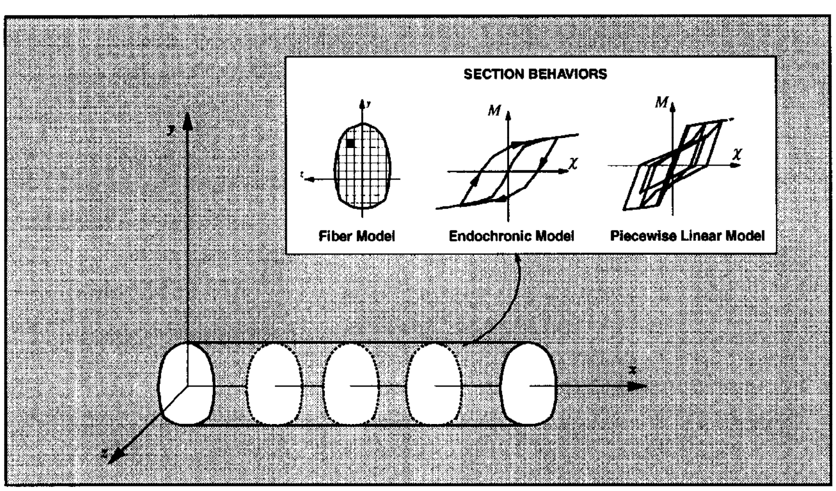


Fig. 2. Beam element integration points and section behavior.

The first section model to be implemented in the nonlinear element was the moment-curvature relation derived from the endochronic theory. The endochronic model is based on the differential equation of a simple viscoelastic model, modified by the introduction of an "intrinsic time" that yields a differential moment-curvature relation $dM/d\chi = g(x)$. Appropriate modification of $g(x)$ permits the introduction of pinching and damage in the section response. The application of the endochronic model to the study of a reinforced concrete cantilever beam tested at the Earthquake Engineering Research Center of the University of California at Berkeley is shown in Figure 3. Details on the element endochronic model are found in Spacone *et al.* (1992).

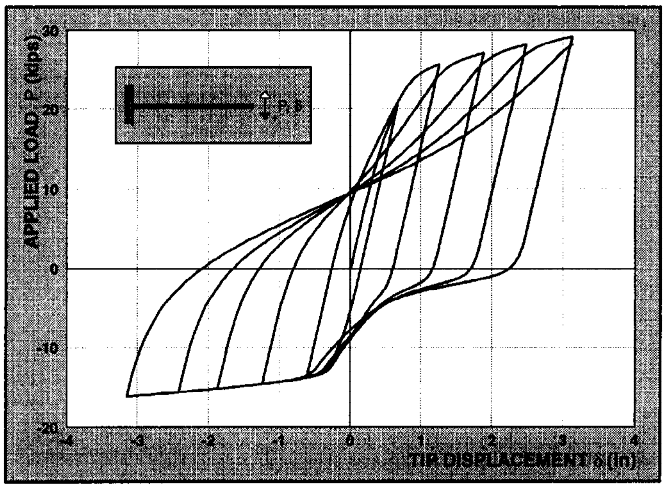


Fig. 3. Analytical load-tip displacement relation of specimen R-3 (test data from Ma *et al.*, 1976).

The second hysteretic moment-curvature model is a piecewise linear law with a bilinear or trilinear envelope. The model with the trilinear envelope is shown Figure 4 with a negative (softening) second slope in the negative moment-curvature quadrant. Details on this model can be found in Filippou (1996).

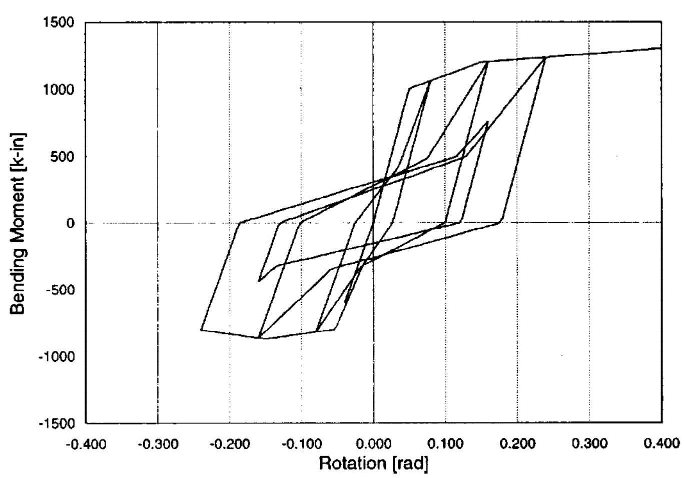


Fig. 4. Piecewise linear moment-curvature law.

In the third section model the element is subdivided into longitudinal fibers and the integral of the response of the single fibers yields the section response. From the hypothesis that plane sections remain plane and normal to the longitudinal axis the strain at point (y,z) of cross section x is $\varepsilon(x,y,z) = \boldsymbol{l}(y,z)\,\boldsymbol{d}(x)$, where $\boldsymbol{l}(y,z)$ is the simple geometric vector $\boldsymbol{l}(y,z) = \{-y \quad z \quad 1\}$. The strain distribution $\varepsilon(x,y,z)$ with the constitutive relations of the constituent materials yields Young's modulus $E(x,y,z)$ and stress $\sigma(x,y,z)$. The section stiffness matrix $\boldsymbol{k}(x)$ and the resisting forces $\boldsymbol{D}_R(x)$ are then determined with the virtual force principle. The evaluation of the integrals requires the selection of a numerical integration scheme. In the present work the section x is subdivided into $nfib(x)$ fibers and the midpoint integration rule is used for the integrals. The resulting section stiffness and forces are

$$\boldsymbol{k}(x) = \int_{A(x)} \boldsymbol{l}^T(y,z)\, E(x,y,z)\, \boldsymbol{l}(y,z)\, dA \cong \sum_{ifib=1}^{nfib(x)} \boldsymbol{l}^T\left(x, y_{ifib}, z_{ifib}\right)(EA)_{ifib}\, \boldsymbol{l}\left(x, y_{ifib}, z_{ifib}\right) \tag{10}$$

$$\boldsymbol{D}_R(x) = \int_{A(x)} \boldsymbol{l}^T(y,z)\, \sigma(x,y,z)\, dA \cong \sum_{ifib=1}^{nfib(x)} \boldsymbol{l}^T\left(x, y_{ifib}, z_{ifib}\right)(\sigma A)_{ifib} \tag{11}$$

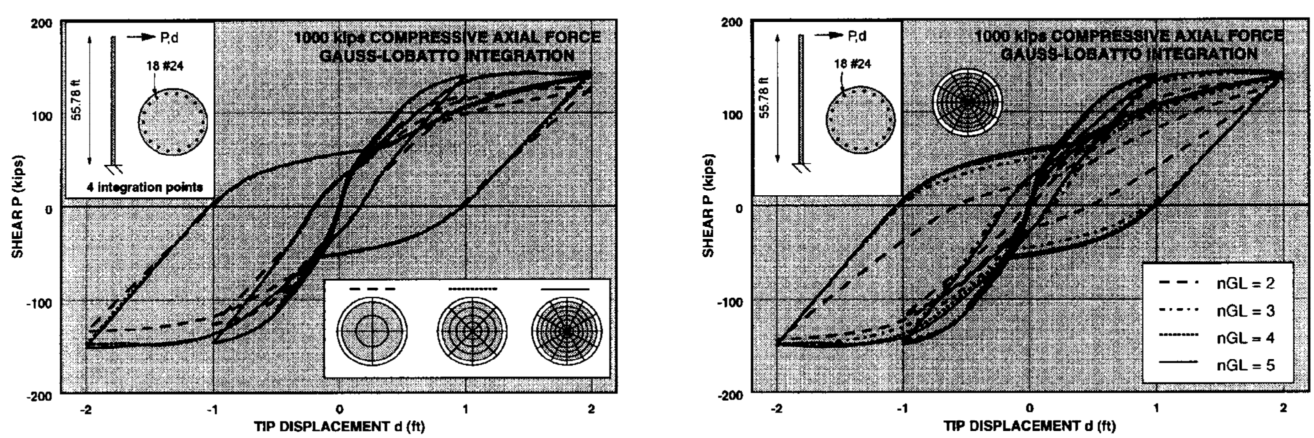
The above integrals depend on the computation of the response of the single fibers. Presently, a library of uniaxial constitutive laws is implemented and their modular implementation permits an easy interchange of different model. Among others, the concrete uniaxial constitutive model proposed by Kent and Park (1971), as extended later by Scott *et al.* (1982), and the steel model proposed by Menegotto and Pinto (1973), as modified by Filippou *et al.* (1983), are implemented.

Because of the interaction between axial and flexural responses, the fiber section is a more precise model for studying columns or, in general, members with high axial forces, such as those in prestressed concrete structures. The fiber model is in general a more rational tool, while some feel that the other moment curvature laws may save computational time. More details on the fiber model can be found in Spacone *et al.* (1995a and b). The computational costs of different models and the robustness of the element iteration scheme with respect to different section models will be addressed in future studies.

COMPUTATIONAL ISSUES

Several computational issues are involved in the implementation and performance of the proposed element and a thorough discussion of all these issues is beyond the scope of this article. A general overview is given in Spacone (1994) and further studies are needed. In this section, two of the main issues involved in the response of the element with fiber sections are discussed; a) the element sensitivity to the number of section fibers and b) the element sensitivity to the number of integration points.

A 55.78 ft high reinforced concrete bridge column with a 36 in. diameter circular cross section is used to illustrate the problem. Imposed displacement cycles are applied at the free end of the column. In the first

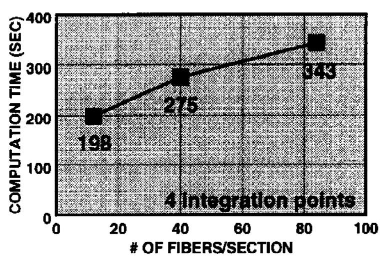


a) b)

Fig. 5. Response of a cantilever beam with circular cross section:
a) with three different section mesh refinements;
b) with different numbers of integration points.

analysis, illustrated in Figure 5a, the response of the column is studied for three different mesh refinements (twelve, forty and eighty-four fibers). The darker fibers in the section mesh indicate confined concrete. A constant 1000 kips compressive axial force was applied and the analysis was conducted modeling the column with a single beam element with four Gauss-Lobatto integration points. The section with twelve fibers does not yield good results and underestimates the column bending capacity, while the more refined meshes converge to the correct response. The same analysis was then repeated with the eighty-four fiber section to test the response sensitivity to the number of integration points. The corresponding results are presented in Figure 5b and they show that the response converges for more than three integration points. The column stiffness is underestimated with two and three integration points.

The selection of the appropriate section mesh and of the optimal number of integration points should be dictated by the optimization of both numerical efficiency and computational time. Computational times for the analysis of Figure 5 are illustrated in Figure 6 and seem to increase linearly both with the number of integration points and the number of fibers per section. From this study and from other analyses it was observed that selecting less than four integration points and adopting a very coarse fiber mesh do not yield satisfactory result. At the same time, section meshes with too many fibers and elements with too many integration points (more than five) do not improve the response accuracy but stretch the computational time considerably. Furthermore, it has been observed that if the section response starts softening, as is the case in reinforced concrete columns with high axial forces and large lateral sways, increasing the number of integration points leads to a strain localization in the end sections and to unrealistic strain predictions. This is a well known phenomenon that occurs in the analysis of strain-softening materials.



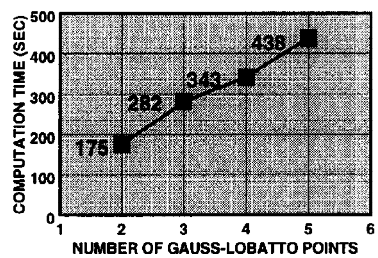


Fig. 6. Comparison of computational time for different section mesh refinements and for different numbers of integration points.

DISTRIBUTED LOADS

One of the most interesting and promising aspects of the flexibility approach is the application of distributed loads. Since the basic element is formulated without rigid body modes (Figure 1), thus reducing it to a simply supported beam, the internal forces due to any distributed load are found from simple equilibrium conditions. Calling $\Delta \boldsymbol{W}$ the element incremental load vector and $\boldsymbol{b}_g(x)$ the matrix relating section forces and element loads, the internal force distribution is computed using

$$\Delta \boldsymbol{D}(x) = \boldsymbol{b}(x)\, \Delta \boldsymbol{Q} + \boldsymbol{b}_g(x)\, \Delta \boldsymbol{W} \qquad (12)$$

instead of the second equation in (1). While the element formulation does not change, the element state determination is affected by the distributed loads. In the element nonlinear algorithm the so called fixed-end forces are computed. If the discussion is limited to the case of a uniformly distributed load and the element loads are grouped in a vector consisting of the loads per unit length $\boldsymbol{W} = \begin{bmatrix} w_x & w_y & w_z \end{bmatrix}^T$, the force transformation matrix $\boldsymbol{b}_g(x)$ in (12) is easily computed from equilibrium:

$$b_g(x) = \begin{bmatrix} 0 & \frac{L}{2}x(x-L) & 0 \\ 0 & 0 & \frac{L}{2}x(L-x) \\ L-x & 0 & 0 \end{bmatrix} \tag{13}$$

Different versions of the implementation of element loads have been proposed. The following procedure has been recently suggested and is presented here because of its clarity. In the general case, at Newton-Raphson step i element forces and stiffness matrix must be found corresponding to the new deformation increments $\Delta \boldsymbol{q}^i$ and the new distributed load increment $\Delta \boldsymbol{D}_g^i(x) = \boldsymbol{b}_g(x)\Delta \boldsymbol{W}_g^i$. This implies that in the first element iteration j=1 the element force increments are

$$\Delta \boldsymbol{Q}^{j=1} = \boldsymbol{K}^{j=0}\,\Delta \boldsymbol{q}^i + \Delta \boldsymbol{Q}_g^i \tag{14}$$

where $\Delta \boldsymbol{Q}_g^i$ is the element force vector necessary to maintain the imposed nodal deformations $\Delta \boldsymbol{q}^i$. The incremental vector $\Delta \boldsymbol{Q}_g^i$ can also be seen as the element fixed end forces due to the distributed load increment $\Delta \boldsymbol{W}_g^i$. This implies that the section deformation increments in the first iteration are

$$\Delta \boldsymbol{d}(x)^{j=1} = \boldsymbol{f}^{j=0}\left[\boldsymbol{b}(x)\left(\boldsymbol{K}^0\,\Delta \boldsymbol{q}^i + \Delta \boldsymbol{Q}_g^i\right) + \boldsymbol{b}_g(x)\,\Delta \boldsymbol{W}_g^i\right] \tag{15}$$

The expression for $\Delta \boldsymbol{Q}_g^i$ is found using the principle of virtual forces to enforce element compatibility. The results is :

$$\Delta \boldsymbol{Q}_g^i = -\boldsymbol{K}^{j=0}\left[\int_0^L \boldsymbol{b}^T(x)\,\boldsymbol{f}^{j=0}\,\boldsymbol{b}_g(x)dx\right]\Delta \boldsymbol{W}_g^i \tag{16}$$

The application of element loads has a wide range of applications, from considering dead and live loads in structural design to considering the effect of tendons in prestressed concrete elements or to the study of any composite member in which bond effects between different material components affects the structural response. These topics are the focus of ongoing research.

CONCLUSIONS

This paper presents a flexibility approach to the formulation of frame elements. The formulation is based on the assumption of the exact internal force distributions for both nodal and element loads. The element presents three major advantages over classical stiffness-based elements: a) the assumption of the exact force distribution leads to a very stable numerical response of the elements; b) because of the numerical accuracy of the element, only one finite element per frame member is used, thus introducing a significant decrease in the global number of degrees of freedom; c) distributed loads are treated in an exact way, by introducing the corresponding internal force distributions in the force interpolation functions. The approach has been applied to the formulation of a new family of finite elements for the nonlinear static and dynamic analysis of RC and steel buildings and bridges. Any nonlinear section constitutive law can be implemented in the element. Among those presently implemented, the fiber section, which is based on the section subdivision into longitudinal fibers, is the best solution if one wishes to consider the interaction between axial load and bending moments.

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