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SUB-DOF (SUB-DEGREES OF FREEDOM) METHOD FOR APPLYING PERTURBATION PROCEDURE TO REDUCTION OF EIGENVALUE COMPUTATION IN STRUCTURAL ANALYSIS

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ABSTRACT

In this paper, a new way to reduce the eigenvalue computation in structural analysis is presented. The degrees of freedom of a structure may be classified into groups. The groups are termed as sub-degrees of freedom. The coupling stiffnesses between sub-degrees of freedom are selected small. The eigensolution from the computation for each sub-degrees of freedom is an approximation and a little computing time is required. As the weak coupling is considered to be a perturbation from sub-degrees of freedom to full degrees of freedom, the eigensolution is served as the zeroth-order result and the perturbation algorithm is used to obtain an accurate result. The application to a truss is shown and the numerical result is discussed. The accuracy of perturbation depends on the coupling between sub-degrees of freedom. The better result comes from the weaker coupling. The suggested procedure can be used to simplify a problem of three dimensions to that of two dimensions or from two dimensions to one diminution. It is efficient to determine the dynamic behavior. The method is able to be extended to analyze a structure under the seismic loading.

KEYWORDS

Structural analysis; dynamics; eigenvalue; reduction; sub-degrees of freedom; perturbation.

INTRODUCTION

The finite element method is widely used in the structural analysis nowadays. The degrees of freedom are chosen as variables to discretize the structure and hence the structure is analyzed. The accurate result is relied on the number of degrees of freedom. The more number of degrees of freedom is, the more accurate structural behavior is obtained. It is common that hundreds or thousands of degrees of freedom are used for the analysis of a structure. However, because the capacity of a computer and the efficiency of analysis, to reduce the number of degrees of freedom is desired. This is even more important for a structure to be analyzed under the earthquake loading.

Usually, the earthquake or dynamic analysis focuses on the eigenvalue problem to obtain the dynamic behavior of the structure and the mode shapes. The lower dynamic modes are excited under an earthquake and that are mostly

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concerned. It has been known that the eigenvalue computation takes much computing time. The various ways have been developed to reduce the computation. To reduce the size of stiffness matrix with fewer degrees of freedom is the most used way of reduction.

Substructuring is a normal technique to reduce the structural analysis (Bathe and Wilson, 1976). The structure is partitioned into substructures and into even multi-level substructures. The output degrees of freedom from a substructure are formed in the connectors between substructures and the rest of degrees of freedom are condensed. By this way, the number of degrees of freedom are reduced and the structural matrix becomes much small. The modal synthesis of substructures is used to reduce eigenvalue analysis (Hurty, 1965; Bathe and Wilson, 1976; Craig, 1977). The eigensolution of the structure is obtained from the dynamic modes of substructures, which may be from the eigenvalue computation or the modal testing for each substructure. For an accurate eigensolution, the weak connectors between the substructures must be found. However in many cases, it is rather difficult to find the weak connectors and the structure is unable to be partitioned into substructures.

The structural degrees of freedom may be classified into groups, for example, the degrees of freedom in any one of two horizontal directions for a frame structure. The group of degrees of freedom may be termed as sub-degrees of freedom. The weak coupling between the groups can be found. The weak coupling leads to the small structural stiffnesses and masses in sub-matrices. The structural matrix and mass matrix are divided into two or more parts related to sub-degrees of freedom, in which sub-matrices of the weak coupling are not considered. The structural analysis is carried on the each of sub-degrees of freedom.

The perturbation method is an approximate way to obtain the solution in the form of an asymptotic expansion from the zeroth-order result. It is useful for the structural analysis with small changes of properties and for the modal sensitive analysis. The reduction by the perturbation method has been studied. The perturbation method is used for substructuring of structures with weak connectors (Chen and Liu, 1993). A reduction algorithm is applied to the structures with large stiffnesses and small masses (Liu, 1995; Liu, 1996).

In this paper, a new perturbation way to reduce the eigenvalue computation in structural dynamics is presented. The term of sub-dof, sub-degrees of freedom, is introduced. The perturbation algorithm is employed to obtain an accurate eigensolution.

For a structure is partitioned into sub-dofs, the eigenvalue analysis is reduced to the separate computation for each sub-dof. The computing time takes only a small portion of the each computation and the total time is much shorter than that for the computation with full degrees of freedom. The eigensolution obtained is an approximation due to the small influence caused by the weak coupling. Taking the eigensolution to be the zeroth-order result, the perturbation procedure can be used to obtain an accurate result. Because the weak coupling matrices between sub-dofs, the problem of full degrees of freedom is only a perturbation upon the zeroth-order result. The perturbed result is assumed to be a sequence of small parameter and is convergent with the increase of the order of perturbation. Using the standard perturbation procedure, the simple algorithm is given in the paper. The perturbed eigenvector is obtained from the first-order or higher-order perturbation, while the perturbed eigenvalue is obtained from the second-order perturbation.

The application of the method is shown in the numerical example, a plane truss with 5 bars. Two sub-dofs are considered for the respective directions. The perturbed result is obtained from the second-order perturbation. The relative errors with the increase of the coupling between sub-dofs are examined.

SUB-DEGREES OF FREEDOM

For the structural dynamics, the eigenvalue equation with the symmetrical stiffness matrix K and mass matrix M is given by

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{u} = \mathbf{0} \tag{1}$$

where λ and u are eigenvalue and eigenvector respectively. The structure is discretized into N degrees of freedom through the finite element method.

Suppose eigenvector u may be classified into two or even more groups. The degrees of freedom for the groups could be termed as sub-degrees of freedom, *i.e.* sub-dof. Hence equation (1) is rewritten to

$$\left(\begin{bmatrix} \mathbf{K_{aa}} & \mathbf{K_{ab}} \\ \mathbf{K_{ba}} & \mathbf{K_{bb}} \end{bmatrix} - \lambda \begin{bmatrix} \mathbf{M_{aa}} & \mathbf{M_{ab}} \\ \mathbf{M_{ba}} & \mathbf{M_{bb}} \end{bmatrix} \right) \begin{Bmatrix} \mathbf{u_a} \\ \mathbf{u_b} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{2}$$

where a and b denote two sub-dofs.

For each of two sub-dofs a and b, an eigenvalue problem is set up and is solved separately. Therefore, yields the eigenvalue equations

$$(\mathbf{K_{aa}} - \lambda_a \mathbf{M_{aa}})\,\mathbf{u_a} = \mathbf{0} \tag{3}$$

$$(\mathbf{K_{bb}} - \lambda_b \mathbf{M_{bb}})\,\mathbf{u_b} = \mathbf{0} \tag{4}$$

Computation of the equations yields the eigensolution, which may be considered to be from equation (1) or equation (2) without the coupling matrices between two sub-dofs a and b. As the numbers of degrees of freedom in equation (3) and equation (4) are less than that for equation (2), the matrices K_{aa}, K_{bb}, M_{aa} and M_{bb} become much small and only little computing time is required for the eigenvalue computation. Eventually, the whole computing time is reduced greatly.

The similar reduction technique has widely used in the structural analysis. In most cases, to analyze a complex structure in three dimensions is not applicable. Only a section of the structure is considered and it is simplified into a plane structure of two dimensions. The third coordinate of the structure is neglected. By this way, the structure is analyzed and the computational result is obtained. The plane frame is a usual analytical model for the frame structure. Each frame in any one of two horizontal directions is separately computed. However, this is only a simple way and the error may be very big. The eigensolution obtained is only for the lateral and vertical vectors. The torsional mode is excluded and it can be very important in the structural dynamics. This makes the result lacks the character of three dimensions.

If the coupling between the sub-dofs is weak, the stiffnesses and masses in matrices K_{ab}, K_{ba}, M_{ab} and M_{ba} are small. Therefore, the eigensolutions obtained from the computation of each sub-dof are possible to be an approximation. However, to achieve a better result, the new method needs to be developed.

PERTURBATION EQUATIONS

If sub-dofs have been carefully selected, the coupling stiffness and mass matrices between sub-dofs, *i.e.* K_{ab}, K_{ba}, M_{ab} and M_{ba} in equation (2), are in the small order compared to the diagonal matrices K_{aa}, K_{bb}, M_{aa} and M_{bb}. These matrices may be expressed in first-order according to the perturbation theory to

$$\varepsilon \mathbf{K}^{(1)} = \begin{bmatrix} \mathbf{0} & \mathbf{K_{ab}} \\ \mathbf{K_{ba}} & \mathbf{0} \end{bmatrix} = \varepsilon \begin{bmatrix} \mathbf{0} & \mathbf{K_{ab}^{(1)}} \\ \mathbf{K_{ba}^{(1)}} & \mathbf{0} \end{bmatrix} \tag{5}$$

$$\varepsilon \mathbf{M}^{(1)} = \begin{bmatrix} \mathbf{0} & \mathbf{M_{ab}} \\ \mathbf{M_{ba}} & \mathbf{0} \end{bmatrix} = \varepsilon \begin{bmatrix} \mathbf{0} & \mathbf{M_{ab}^{(1)}} \\ \mathbf{M_{ba}^{(1)}} & \mathbf{0} \end{bmatrix} \tag{6}$$

ε is a small positive parameter, that is $0<\varepsilon<<1$. Then, the structural stiffness and mass matrices become

$$\mathbf{K} = \mathbf{K}^{(0)} + \varepsilon \mathbf{K}^{(1)} \tag{7}$$

$$\mathbf{M} = \mathbf{M}^{(0)} + \varepsilon \mathbf{M}^{(1)} \tag{8}$$

where superscript (0) denotes the zeroth-order. The zeroth-order stiffness matrix $K^{(0)}$ and the zeroth-order mass matrix $M^{(0)}$ are given by

$$\mathbf{K}^{(0)} = \begin{bmatrix} \mathbf{K}_{aa} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{bb} \end{bmatrix}$$

$$\mathbf{M}^{(0)} = \begin{bmatrix} \mathbf{M}_{aa} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{bb} \end{bmatrix}$$

Considering the zeroth-order eigenvalue problem from equation (1) for the perturbed problem, the eigenvalue equation is

$$(\mathbf{K}^{(0)} - \lambda^{(0)}\mathbf{M}^{(0)})\,\mathbf{u}^{(0)} = \mathbf{0} \tag{9}$$

This is just the eigenvalue equations (3) and (4) for sub-dofs a and b. So the solution is formed from the results of equations (3) and (4)

$$\lambda^{(0)} = \begin{pmatrix} \lambda_a & \lambda_b \end{pmatrix} \tag{10}$$

$$\mathbf{u}^{(0)} = \begin{bmatrix} \mathbf{u}_a & \mathbf{0} \\ \mathbf{0} & \mathbf{u}_b \end{bmatrix} \tag{11}$$

which are N×n matrices, and n is the number of eigenvalues. Suppose the numbers of eigenvalues obtained from equation (3) and equation (4) are n_a and n_b for two sub-dofs a and b respectively, then

$$\mathbf{n} = \mathbf{n}_a + \mathbf{n}_b \tag{12}$$

n may be smaller than the number of degrees of freedom N.

According the perturbation theory, the eigensolution of equation (1) is assumed to take an asymptotic sequence of the small parameter ε

$$\lambda_i = \lambda_i^{(0)} + \varepsilon\,\lambda_i^{(1)} + \varepsilon^2\lambda_i^{(2)} + \ldots \tag{13}$$

$$\mathbf{u}_i = \mathbf{u}_i^{(0)} + \varepsilon\,\mathbf{u}_i^{(1)} + \varepsilon^2\,\mathbf{u}_i^{(2)} + \ldots \tag{14}$$

where i=1, . . ., n.

Based on the zeroth-order result, the solution of equations (13) and (14) is obtained through the standard perturbation procedure. The simple formulas are as follows
the first-order perturbation

$$\lambda_i^{(1)} = 0 \tag{15}$$

for $i \subset [1, n_a]$

$$\mathbf{u}_i^{(1)} = \sum_{j=n_a+1}^{n} \frac{1}{\lambda_i^{(0)} - \lambda_j^{(0)}} [\; \mathbf{u}_j^{(0)T}(\mathbf{K}^{(1)} - \lambda_i^{(0)}\mathbf{M}^{(1)})\,\mathbf{u}_i^{(0)} \;]\; \mathbf{u}_j^{(0)} \tag{16}$$

and for $i \subset [\, n_a+1, n]$

$$\mathbf{u}_i^{(1)} = \sum_{j=1}^{n_a} \frac{1}{\lambda_i^{(0)} - \lambda_j^{(0)}} [\; \mathbf{u}_j^{(0)T}(\mathbf{K}^{(1)} - \lambda_i^{(0)}\mathbf{M}^{(1)})\,\mathbf{u}_i^{(0)} \;]\; \mathbf{u}_j^{(0)} \tag{17}$$

the second-order perturbation

$$\lambda_i^{(2)} = \mathbf{u}_i^{(0)T}(\;\mathbf{K}^{(1)} - \lambda_i^{(0)T}\mathbf{M}^{(1)}\;)\,\mathbf{u}_i^{(1)} \tag{18}$$

for $i \subset [1, n_a]$

$$\mathbf{u}_i^{(2)} = \sum_{\substack{j=1 \\ j\neq i}}^{n_a} \frac{1}{\lambda_i^{(0)} - \lambda_j^{(0)}} [\mathbf{u}_j^{(0)T}(\mathbf{K}^{(1)} - \lambda_i^{(0)}\mathbf{M}^{(1)})\,\mathbf{u}_i^{(1)}]\,\mathbf{u}_j^{(0)} - \frac{1}{2}(\mathbf{u}_i^{(1)T}\mathbf{M}^{(0)}\mathbf{u}_i^{(1)} + 2\,\mathbf{u}_i^{(0)T}\mathbf{M}^{(1)}\mathbf{u}_i^{(1)})\,\mathbf{u}_i^{(0)} \tag{19}$$

and for $i \subset [\, n_a+1, n]$

$$\mathbf{u}_i^{(2)} = \sum_{\substack{j=n_a+1 \\ j\neq i}}^{n} \frac{1}{\lambda_i^{(0)} - \lambda_j^{(0)}} [\mathbf{u}_j^{(0)T}(\mathbf{K}^{(1)} - \lambda_i^{(0)}\mathbf{M}^{(1)})\,\mathbf{u}_i^{(1)}]\,\mathbf{u}_j^{(0)} - \frac{1}{2}(\mathbf{u}_i^{(1)T}\mathbf{M}^{(0)}\mathbf{u}_i^{(1)} + 2\,\mathbf{u}_i^{(0)T}\mathbf{M}^{(1)}\mathbf{u}_i^{(1)})\,\mathbf{u}_i^{(0)} \tag{20}$$

Note that the first-order perturbation of eigenvalue is unable to be obtained. Though only the first-order perturbation and the second-order perturbation are given, the perturbation may be further computed in the straight forward way. However, the first two order perturbations are usually enough for a good perturbed result and the higher order perturbation is not needed. The equations are for two sub-dofs, while for more sub-dofs the computation is the same.

As sub-dofs are considered to make the stiffness matrix into small ones and the zeroth-order result is from the computation for sub-dofs, the computing time is shortened. It is demonstrated that the computing time for the perturbation takes only very small part (Liu, 1996). Consequently, the whole time can be reduced.

NUMERICAL EXAMPLE

For a plane structure, the degrees of freedom are considered in two directions of coordinates. There are degrees of freedom in direction X and Y for a plane truss. For a plane frame, additional rotational degrees of freedom are considered. Assume x and y are the eigenvectors in X and Y coordinates and may be taken to be two sub-dofs respectively, that is

$$u_a = x$$
$$u_b = y$$

Fig. 1 shows a 5-elements plane truss. The eigenvector is $u=\{ u_a\ u_b \}^T=\{ x_1\ x_2\ y_1 y_2 \}^T$

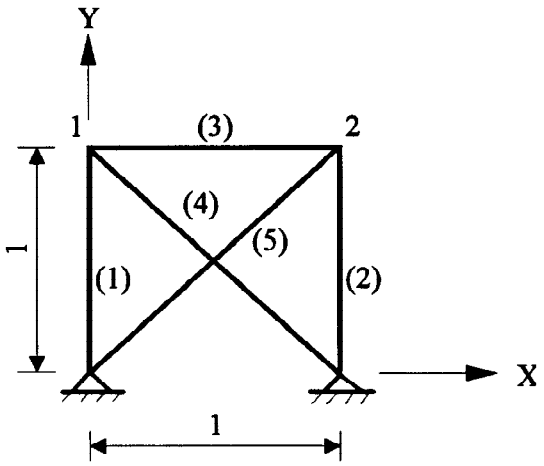

Fig. 1. A plane truss with 5 bars and 4 degrees of freedom

The sectional stiffnesses of bars and the nodal masses are given by

$$k_1 = k_2 = 1.5;\quad k_3 = 1.0;\quad k_4 = k_5 = 0.5.$$
$$m_1 = m_2 = 1.0$$

Thus, yields the stiffness matrix and the mass matrix

$$K = \begin{bmatrix} 1.176777 & -1.000000 & -0.176777 & 0.000000 \\ -1.000000 & 1.176777 & 0.000000 & 0.176777 \\ -0.176777 & 0.000000 & 1.676777 & 0.000000 \\ 0.000000 & 0.176777 & 0.000000 & 1.676777 \end{bmatrix}$$

$$M = \text{diag}[\ 1.0\ \ 1.0\ \ 1.0\ \ 1.0\ ]$$

The zeroth-order stiffness and mass matrices are taken as

$$\mathbf{K}^{(0)} = \begin{bmatrix} \mathbf{K}_{aa} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{bb} \end{bmatrix} = \begin{bmatrix} 1.176777 & -1.000000 & 0.000000 & 0.000000 \\ -1.000000 & 1.176777 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 1.676777 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 1.676777 \end{bmatrix}$$

$$\mathbf{M}^{(0)} = \mathbf{diag}[\ 1.0\ \ 1.0\ \ 1.0\ \ 1.0\ ]$$

then the first-order matrices are

$$\varepsilon\mathbf{K}^{(1)} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_{ab} \\ \mathbf{K}_{ba} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0.000000 & 0.000000 & -0.176777 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.176777 \\ -0.176777 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.176777 & 0.000000 & 0.000000 \end{bmatrix}$$

$$\varepsilon\mathbf{M}^{(1)} = \mathbf{0}$$

Because the lumped nodal mass matrix is used, there is no coupling between sub-dofs. Therefore, the first-order mass matrix is the zero matrix.

The eigensolutions are compared in Table 1. The exact, zeroth-order and perturbed eigenvalues and eigenvectors are given. The relative errors in percentage for the zeroth-order eigenvalues and the perturbed eigenvalues are computed. The perturbed eigenvalues are in the good agreement with the exact result. The errors become smaller with the increase of dynamic modes. The big change is found in the first mode, with the relative error from 13.1554% to -0.1805%.

Table 1. Comparison of exact, zeroth-order and perturbed eigensolutions

Mode		exact	zeroth	perturbed
1	Eigenvalue	0.156225	0.176777	0.155943
	error %		13.1554	-0.1805
	Eigenvector	0.702376	0.707107	0.702196
		0.702376	0.707107	0.702196
		0.081657	0.000000	0.083333
		0.081660	0.000000	0.083333
2	Eigenvalue	1.620591	1.676777	1.614277
	error %		3.4670	-0.3896
	Eigenvector	0.214186	0.000000	0.250000
		0.214186	0.000000	0.250000
		0.673887	0.707107	0.662913
		0.673888	0.707107	0.662913
3	Eigenvalue	1.697329	1.676777	1.697610
	error %		-1.2108	0.0166
	Eigenvector	0.081656	0.000000	0.083333
		0.081658	0.000000	0.083333
		0.702376	0.707107	0.702196
		0.702376	0.707107	0.702196
4	Eigenvalue	2.232963	2.176777	2.239277
	error %		-2.5162	0.2828
	Eigenvector	0.673887	0.707107	0.662913
		0.673887	0.707107	0.662913
		0.214186	0.000000	0.250000
		0.214186	0.000000	0.250000

It is noted that the coupling matrices K_{ab} and K_{ba} depend on the sectional stiffnesses k_4 and k_5 of bar 4 and 5. To compare the eigensolution with the change of the coupling between sub-dofs, the sectional stiffnesses k_4 and k_5 are considered to be a variable and take changes from 0.1 to 1.0. The different eigensolutions are obtained. The relative errors of the perturbed eigenvalue for the first dynamic mode together with the relative errors of the zeroth-order eigenvalue are plotted in Fig 2.

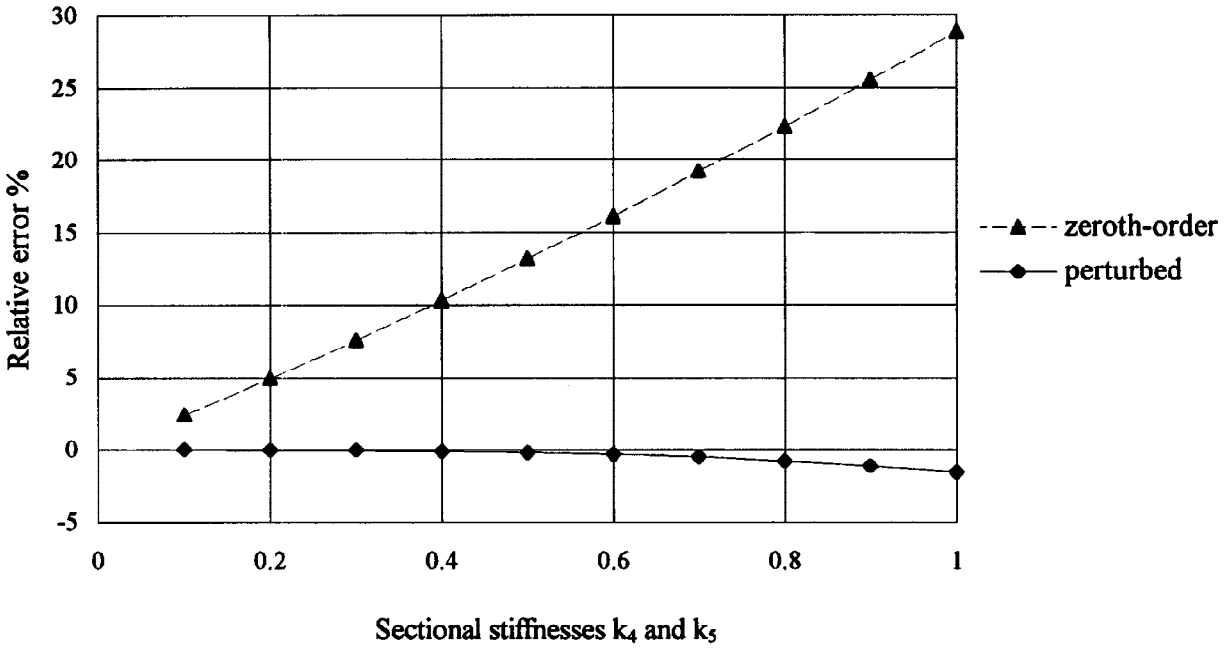

Fig. 2. Relative error of the perturbed eigenvalue for the first dynamic mode vs. sectional stiffness of bar 4 and 5.

From Fig. 2, it is noticed that the relative errors for the eigenvalue of first mode increase with changes of the sectional stiffnesses k_4 and k_5 from 0.1 to 1.0. The relative error for the perturbed eigenvalue goes from -0.0008% to -1.5222%, although it is from 2.4131% to 28.8473% for the zeroth-order. The perturbed eigenvalue is smaller than the exact one, while the zeroth-order result is larger. The error is so small that result of the perturbed eigenvalue becomes acceptable.

CONCLUSION

The sub-dof is introduced in the paper and is used to divided the stiffness matrix as well as the mass matrix into several parts. The eigenvalue computation is performed with the reduced number of degrees of freedom. The perturbation algorithm is useful to obtain the eigensolution with the full degrees of freedom. Therefore, the computing time is reduced. The method is effective in the reduction and is a new way for the efficient computation of eigenvalue problem in the structural analysis.

The numerical example of a simple plane truss shows the effectiveness of the method. The eigensolution obtained from the second-order perturbation is greatly improved from the result for sub-dofs and is in the good agreement with the result for full dof. The error depends on the coupling matrices of structural stiffness and mass between sub-dofs. The weaker coupling yields the better perturbed result. Therefore, the sub-dofs, between which the weak coupling stiffness matrices are found, are preferable to be selected.

A structure of three dimensions may be divided into sub-dofs with two dimensions and hence is analyzed. However, the torsional dynamic mode must be obtained from the perturbation, as the result of sub-dofs is only for

the lateral and vertical modes. If the earthquake loading is considered, the idea of sub-dof may be extended to obtain the structural behavior.

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