



## **FREQUENCY DOMAIN AND TIME DOMAIN IDENTIFICATION ON NONLINEAR STRUCTURAL SYSTEM**

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### **ABSTRACT**

The purpose of this paper is to develop a systematic method of time-domain nonlinear system identification using NARMA model. In the beginning the Hilbert transform was used to test the system nonlinearity, then the orthogonal parameter estimation algorithm together with the forward regression algorithm are applied to a NARMA model which represents the nonlinear system. Finally the nonlinear frequency response functions were computed. Application of the methodology to a Mathieu nonlinear equation and the seismic response data of Van Nuys building to both Whittier and Northridge earthquakes are studied.

### **KEYWORDS**

System Identification, NARMA Model, Signal Processing, Spectral Analysis

### **INTRODUCTION**

The fact that many structural systems exhibit linear behavior at low levels of excitation allows estimation of modal properties from test data, modelling of structures using constant parameter systems, and correlation of test and analysis results. Unfortunately, as the level of excitation increases, the system may not continue to respond in a linear fashion. Modal properties may differ greatly between low and high level excitation. Many attempts are made to develop a systematic procedures to identify the nonlinear system from measurements. One of the approach from frequency domain identification of nonlinear system is the reverse dynamic system of spectral analysis (Rice, 1988; Fitzpatrick, 1992; Bendat, 1992). To apply this method it is necessary to know the exact form of nonlinear feedback input. As a matter of fact the nonlinear damping-restoring force function of a nonlinear system is generally not known, and the application of this method is limited. The successful development of identification procedures for a nonlinear system depends upon the model which is used to represent the system under investigation. Traditionally the functional series descriptions of Volterra and Wiener have been used (Schetzen, 1980). Unfortunately, functional series models require an excessive kernel values to describe even simple nonlinear systems. However, by expanding the system output in terms of past input and output using a nonlinear auto-regressive moving average model with exogenous inputs (NARMA) model, a very concise representation for a wide class of nonlinear systems can be obtained (Korenberg, 1987; Hunter, 1990).

The purpose of this paper is to develop a systematic method of time domain nonlinear system identification using NARMA model. The orthogonal estimation algorithm will be used to estimate the modal parameters. Interpretation and properties of the nonlinear frequency response functions are discussed. A simulated nonlinear example (a Mathieu equation) is included to demonstrate the effectiveness of the algorithm. Finally, application of the identification algorithm to the building seismic response data is examined.

## SYSTEM DETECTION USING HILBERT TRANSFORM

The Hilbert Transform is an integral transform that relates the real and imaginary parts by any complex function. The Hilbert Transform of the complex function  $f(x)$  of real variable  $x$  is defined as (Gendat, 1986):

$$h(f(x)) = H(x) = \frac{1}{i \cdot \pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy \quad (1)$$

It is an integral transform that relates the real and imaginary part by any complex function. In the case of a general system, the system "frequency response function" will be defined as the ratio of the Fourier transforms of the output and the input signals:

$$G(\omega) = \mathcal{F}\{y(t)\} / \mathcal{F}\{x(t)\} = \text{Re}[G(\omega)] + i \text{Im}[G(\omega)] \quad (2)$$

where  $\mathcal{F}\{\cdot\}$  denotes the Fourier transform,  $y(t)$  and  $x(t)$  are system output and input signal, respectively. If the system is nonlinear,  $G(\omega)$  will depend on the input  $x(t)$ . The Hilbert Transform of the complex function  $G(\omega)$  is defined as:

$$H\{G(\omega)\} = H(\omega) = \text{Re}[H(\omega)] + i \text{Im}[H(\omega)] \quad (3)$$

where  $\text{Re}[H(\omega)]$  and  $\text{Im}[H(\omega)]$  represent the real and the imaginary parts of the Hilbert Transform defined by the following discrete form (Simon, 1984):

$$\text{Re}[H(\omega_j)] = -\frac{2}{\pi} \sum_{k=1}^n \frac{\text{Im}[G(\omega_k)] \omega_k \Delta\omega}{\omega_k^2 - \omega_j^2}, \quad \text{Im}[H(\omega_j)] = -\frac{2\omega_j}{\pi} \sum_{k=1}^n \frac{\text{Re}[G(\omega_k)] \omega_k \Delta\omega}{\omega_k^2 - \omega_j^2} \quad (4)$$

For a linear system, the response at any time does not depend on the future of the input, and the Hilbert Transform of a linear system is equal to the frequency response function of that system's  $G(\omega)$ . If the Hilbert Transform does not equal the frequency response function, then the system is nonlinear. Consider a broad class of SDOF nonlinear dynamic system. It can be described by the constant coefficient differential equation:

$$m \ddot{u} + c \dot{u} + k u + p(u, \dot{u}, t) = F(t) \quad (5)$$

where  $m$ ,  $c$ ,  $k$  denote the system mass, linear viscous damping and linear elastic stiffness, and  $p(u, \dot{u}, t)$  represents a general nonlinear damping-restoring force function components. For a Mathieu equation  $p(u, \dot{u}, t)$  is defined as  $k_2 \cos(2\pi f_m t)u$ . Use the fact that the Hilbert Transform of a nonlinear system will create changes in the Hermitian symmetry property of the original frequency response function. Figure 1 shows the comparison between the amplitude of the frequency response function and the modulus of the Hilbert Transform of the Mathieu equation. An index is suggest to compare the result of the Hilbert Transformed data to the original frequency response data from the use of statistical moments:

$$\text{ID}^n = 100 \left\{ \frac{H_v^n - G_v^n}{G_v^n} \right\} \quad (6)$$

where  $\text{ID}^0$  ( $n = 0$ ) represents the energy ratio and  $\text{ID}^1$  ( $n = 1$ ) represents the frequency ratio, and the moment integral about the vertical axis are defined as:

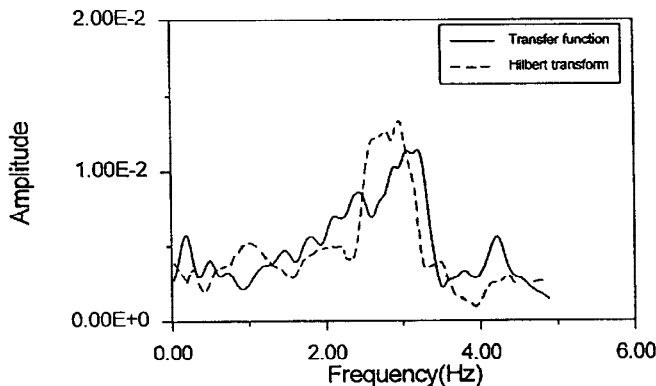


Fig. 1: Comparison between the system transfer function and its Hilbert transform for the nonlinear Mathieu equation ( $k = 170$ ,  $f = 1$ ).

$$H_v^n = \int_{\omega_1}^{\omega_n} \omega^n H(\omega) d\omega \quad \text{and} \quad G_v^n = \int_{\omega_1}^{\omega_n} \omega^n G(\omega) d\omega \quad (7)$$

Strong nonlinearity in the model will have a significant difference in the comparison on the energy ratio between linear and nonlinear model. It is concluded that this method provides a good index for the detection of system nonlinearity.

### THE NARMA METHODOLOGY

The theory associated with each of the stages in the NARMA methodology is well documented in the literature (Simon, 1984; Billings 1989; Billings, 1988). A single input single output system takes the form

$$y(t) = F^l[y(t-1), \dots, y(t-n_y), u(t), \dots, u(t-n_n), \varepsilon(t-1), \dots, \varepsilon(t-n_\varepsilon)] + \varepsilon(t) \quad (8)$$

where  $y(t)$  and  $u(t)$  represent the measured output and input respectively,  $\varepsilon(t)$  is the prediction error, and  $F^l[\cdot]$  is some nonlinear function. Several nonlinear models may include nonlinear damping force,  $F_D$ , and  $F_D$  can be a function of relative displacement and relative velocity. In the model equation of Eq. (8), one can incorporate the velocity term ( $\dot{y}(t)$ ) in the model to represent the nonlinear damping. The NARMA model equation may be represented by the regressional equation

$$y(k) = \sum_i^{n_\theta} p_i(k) \theta_i + \varepsilon(k) \quad (9)$$

where  $p_i(t)$  represents a term in the NARMA and no  $p_i(t)$ 's are identical. Rather than estimating the parameter  $\theta_i$  directly from Eq. (9) the orthogonal algorithm operates on an equivalent auxiliary model

$$y(k) = \sum_{i=1}^{n_\theta} g_i w_i(k) + \varepsilon(k) \quad (10)$$

where  $w_i(k)$  is constructed to be orthogonal over the data record. The parameters  $g_i$  in Eq. (10) can be estimated by implementing the orthogonal estimator.<sup>[8]</sup> The algorithm is described as follows:

1. Set  $w_1(t) = p_1(t)$  (generally  $p_1(t) = y(t-1)$ ) and

$$\hat{g}_1 = \frac{\sum_{k=1}^N w_1(k) y(k)}{\sum_{k=1}^N w_1^2(k)} \quad (11)$$

2. Set  $j = 2$  and compute

$$\alpha_{ij} = \frac{\sum_{k=1}^N w_i(k) p_j(k)}{\sum_{k=1}^N w_i^2(k)} \quad \text{for } i = 1, 2, \dots, j-1 \quad (12)$$

$$w_j(k) = p_j(k) - \sum_{i=1}^{j-1} \alpha_{ij} w_i(k) \quad \text{and} \quad \hat{g}_j = \frac{\sum_{k=1}^N w_j(k) y(k)}{\sum_{k=1}^N w_j^2(k)} \quad (13, 14)$$

increment  $j$  and compute Eqs. (12), (13) and (14).

3. Compute NARMA parameter  $\theta_i$  backwards using  $\hat{\theta}_{n_\theta} = \hat{g}_{n_\theta}$

$$\hat{\theta}_i = g_i - \sum_{j=i+1}^{n_\theta} \alpha_{ij} \hat{\theta}_j \quad \text{for } i = n_\theta - 1, n_\theta - 2, \dots, 1 \quad (15)$$

Besides the estimation of model parameter  $\theta_i$  from Eq. (15), it is necessary to detect which terms should be included within the NARMA model. It can be achieved by computing the error reduction ratio for the  $i$ -th terms as

$$\text{ERR}_i = \frac{\hat{g}_i^2 \sum_{k=1}^N w_i^2(k)}{\sum_{k=1}^N y^2(k)} \times 100 \quad (16)$$

$\text{ERR}_i$  provides a measure of the reduction in mean square error. The procedure is terminated when at any step  $q$  say:  $[\text{ERR}_i]_q < C_d$  (prescribed value  $C_d = 0.0001$ ).

Table 1: Linear model log loss function.

Log Loss function table for Mathieu model				
d (time delay)				
n (order)	1	2	3	4
1	-0.52506	-0.56045	-0.47292	-0.45995
2	-1.66426	-1.57760	-1.29974	-1.20868
3	-2.06782	-1.83202	-1.60499	-1.61857
4	-2.53661	-2.14432	-2.08040	-1.96514

Table 2: Identified NARMA model.

Mathieu model
$\ddot{x} + 3.77\dot{x} + 355.3x + 170 \cos(2\pi f_m t)x = -\ddot{u}_g$
Input: EL centro (PGA=0.2g)
$y(t) = (2.1792352)y(t-1)$ $- (1.9818847)y(t-2)$ $+ (7.3760353e-1)y(t-3)$ $+ (8.6106837e-4)u(t-1)$ $- (4.3139012e-4)u(t-3)$ $- (4.0843676e-6)u(t-1)^2y(t-3)$ $- (6.2759548e-2)y(t-3)^3$ $+ (4.5408963e-4)u(t-3)y(t-3)^2$ $- (4.3974071e-4)u(t-2)y(t-1)y(t-2)$

Taking an example of the Mathieu equation, the initial analysis involved fitting linear models of various order ( $n_y = n_u = 1, 2, 3, 4$ ) and delays ( $d = 0, 1, 2, 3$ ) and computing the loss function (sum of square errors) of the fitted models using the forward inclusion algorithm with  $l = 1$ . The result is summarized in Table 1. It shows that the time delay is  $n_d = 1$  and that an appropriate model order would be  $n_u = n_y = 3$  because the loss function will monotonically decrease with increasing  $n_u$  and  $n_y$ . Inputting these values into the forward regression estimator (Billings, 1989) and setting the degree of nonlinearity  $l = 3$ , one can select the terms which are significant from the possible terms. Table 2 shows the final process model of the nonlinear system. Model validation test of the system can be performed by calculating the autocorrelation function of residuals  $\xi$  (difference between the recorded output and the output from the model) and the cross-correlation function between input and residuals. Figure 2 shows the modal validation test from  $\phi_{\xi\xi}(\tau)$  and  $\phi_{u\xi}(\tau)$  and the result are within 95% confidence interval.

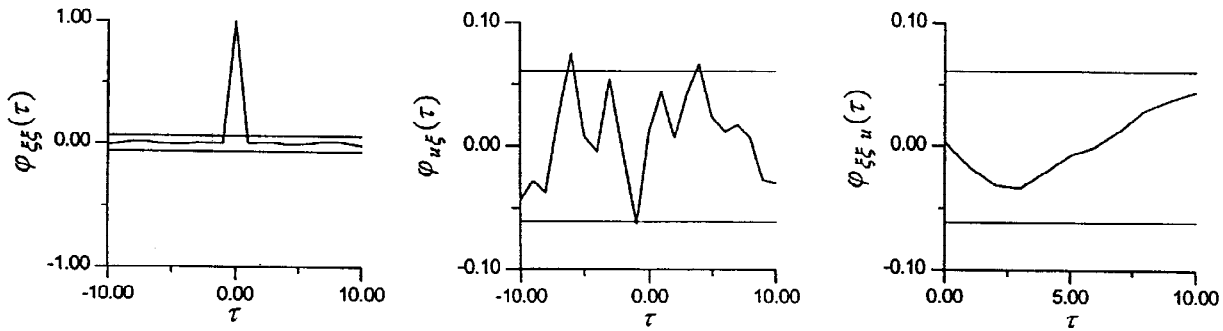


Fig. 2: Model validation test for the Mathieu equation using NARMA model.

## NONLINEAR FREQUENCY RESPONSE FUNCTIONS

The generalized frequency response functions can also be obtained from the estimated NARMA model by discarding the noise model. The probing method (Billings, 1989) can be used to obtain the analytical expressions for the frequency response function. This method can be justified by considering the steady-state output of a nonlinear system with several exponential inputs. The probing method (or harmonic input method) can be used to determine the symmetries  $n$ th order transfer function  $H_i(f_1, \dots, f_n)_{sym}$  by equating coefficients of  $n! \exp[i2\pi(f_1 + f_2 + \dots + f_n)t]$  in the system output for an input defined by

$$u(k) = \sum_{k=1}^K A_k e^{i2\pi f_k t} \quad (17)$$

Based on the probing method, for the Mathieu equation, this procedure can be applied to find at each step higher order nonlinear frequency response functions in terms of lower order functions. If a system is linear the application of a sinusoidal input will generate a sinusoidal output of the same frequency but with a different gain and phase. It will not generate new frequencies. If a system is nonlinear however new frequency components such as harmonics and intermodulation can be produced together with effects such as gain compression/expansion and desensitization. Each of the nonlinear phenomena will be briefly explained as follows:

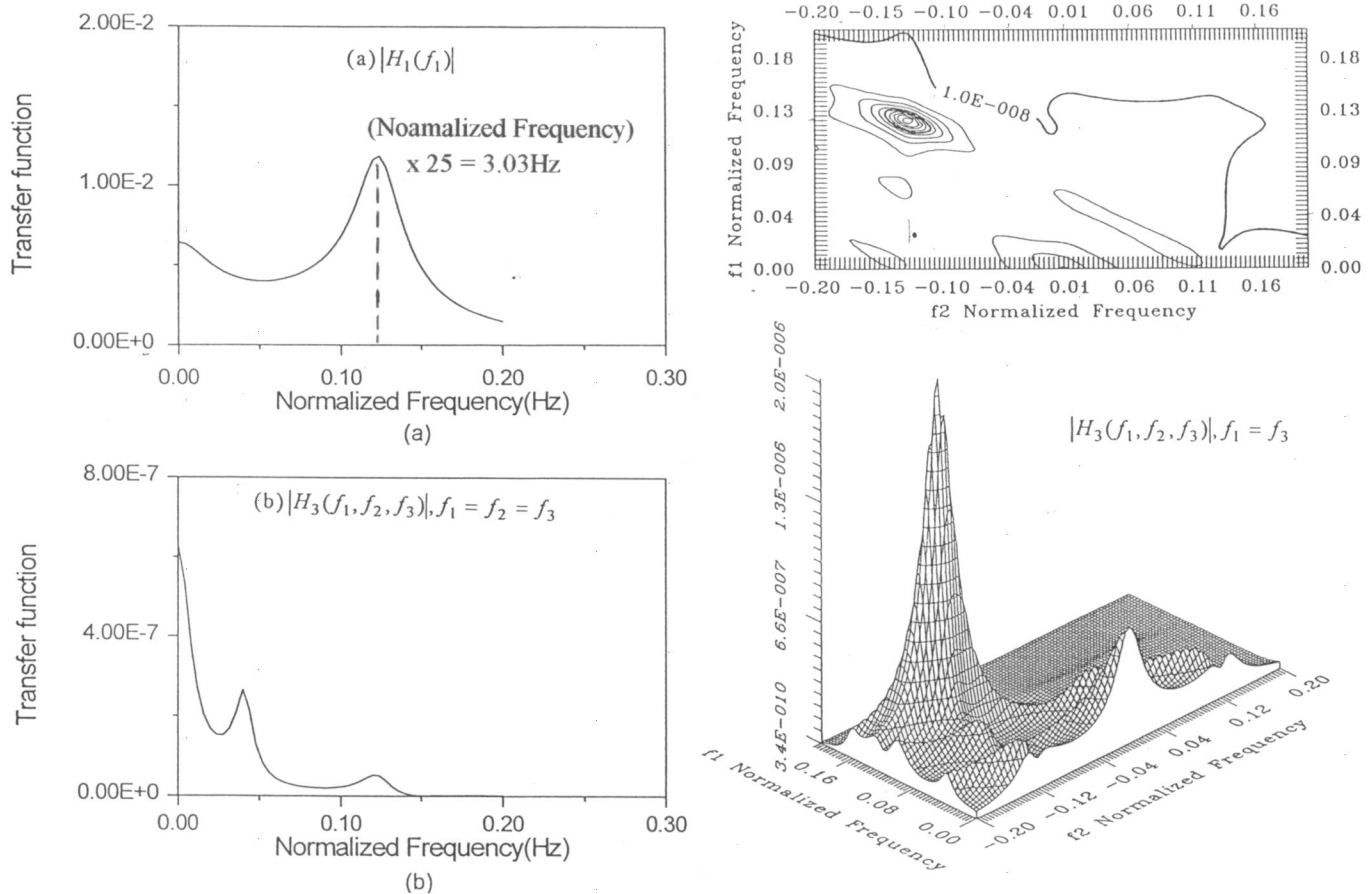


Fig. 3: Estimated Frequency Response Functions (FRF) ; (a) 1st order FRF,  $H_1$  , (2) 3rd order FRF,  $|H_3(f, f, f)|$  ; (c) Top view and 3-D plot of FRF,  $|H_3(f_1, f_2, f_3)|$   $f_1 = f_3$

1. Harmonics – Harmonics are frequency components which are equal to multiples of the fundamental input frequency.
2. Intermodulation – Intermodulation is the process by which two or more signals combine in a nonlinear manner to produce a new frequency component.
3. Gain compression or expansion – These terms are used to describe the variation in the gain of a system as the input amplitude is increased. Gain expansion occurs when the system gain is greater than the linear gain and gain compression when the system gain is less than the linear gain.
4. Desensitization – When a system is nonlinear the sinusoidal response at frequency  $f_1$  can be modified by the application of a second sinusoidal signal at another frequency  $f_2$ .

Figure 3 shows the nonlinear frequency response functions of  $|H_1(f_1)|$  and  $|H_3(f_1, f_2, f_1)|$  for the nonlinear model of Mathieu equation. This suggests that the magnitude of the 3rd harmonic is proportional to the 3rd power of the input magnitude for small input signals. It can be demonstrated by the 3rd order frequency response function  $|H_3(f_1, f_2, f_3)|$  along the line  $f_1 = f_2 = f_3$ . Also from the 3rd order frequency response function  $|H_3(f_1, f_2, f_3)|$  shows a peak at  $f_1 + f_2 + f_3 = 0.12$  for  $f_1 = f_3$  which means that gain compression or expansion is observed.

#### ANALYSIS ON SEISMIC RESPONSE DATA

The CSMIP in the Department of Conservation, Division of Mines and Geology, collected two strong-motion records from the Van Nuys building (CSMIP Sta.# 24386). The response data from Whittier earthquake (1 October, 1987) and Northridge earthquake (17 January, 1994) are used for the analysis. Damage was observed in this building from the Northridge earthquake. In this study the roof response data (relative displacement between roof and basement) is selected as output and the basement acceleration is selected as input motion for identification. Following the procedures described in the previous section, the linear model loss function from these two sets of seismic data is studied first. From the loss function the time delay is selected as  $d = 1$  and the appropriate model order would be  $n_a = n_y = 2$ . The reduction in the loss function does not converge, but for the linear model the selected order and the delay time step are acceptable. To incorporate the nonlinear damping phenomenon in the model the velocity term  $\dot{y}(t)$  was considered (i.e., energy dissipation  $F_D \approx |\dot{y}^n| \text{sgn}(\dot{y})$ ,  $n = 1$  gives viscous damping).

Table 3: Identified NARMA model of Van Nuys building (Whittier earthquake).

VAN-NUYS 7-story hotel Whittier Earthquake (10/1/87)
$y(t) = (3.6429638e-2)y(t-1) + (3.9172756e-2)\dot{y}(t-1) + (9.634620e-1)y(t-2) + (2.1649173e-4)u(t-1) - (1.9961796e-4)u(t-2)$

Table 4: Identified NARMA model of Van Nuys building (Northridge earthquake).

VAN-NUYS 7-story hotel Northridge Earthquake (1/17/94)
$y(t) = (1.53113542)y(t-1) + (4.5626041e-2)\dot{y}(t-1) - (2.7536559e-2)\dot{y}(t-2) + (5.7779633e-4)u(t-1) - (3.6078692e-4)u(t-2) - (5.3267714e-1)y(t-2) - (7.1361802e-6)u(t-2)y(t-2)$

The identified NARMA model of the system from Whittier earthquake and Northridge earthquake are shown in Tables 3 and 4, respectively. Through the probing method only the FRF of  $H_1(f_1)$  can be identified, as shown in Fig. 4a. The validation test of the model is shown in Fig. 4b showing that the results are acceptable. Different from the Whittier earthquake data the results from the analysis of Northridge earthquake is shown in Figs. 5 and 6. The validation test of the model is shown in Fig. 5a. Because the structure was damaged during the Northridge earthquake, this system can not be identified as a time-invariant system. The result of the analysis on these data does not pass the linear test, hence the 2nd order frequency response function must be identified, as shown in Fig. 6b. Although the validation tests of the model are not all within 95% confidence interval, the nonlinear phenomenon was studied.

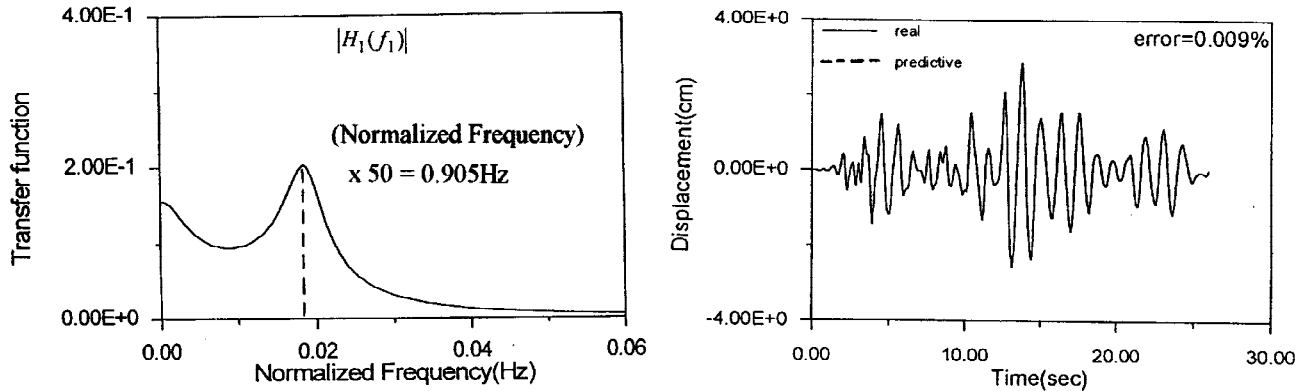


Fig. 4a: Identified 1st order Frequency Response Function of Van Nuys building from Whittier earthquake data. Comparison between recorded and predicted response is also shown.

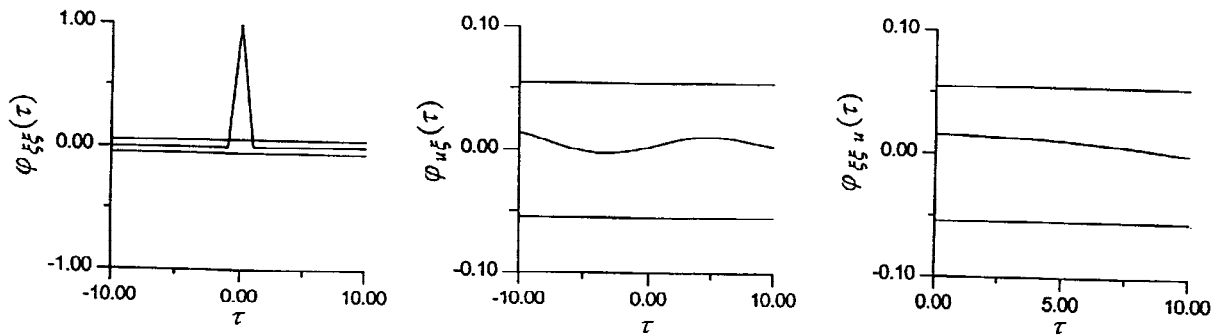


Fig. 4b: Model validation test of the predicted model (using data from Whittier earthquake).

### CONCLUSIONS

The purpose of this paper is to propose an identification scheme for the nonlinear system. The NARMA methodology was applied which includes: orthogonal estimation algorithm and forward regression algorithm was applied to identify the modal parameter of NARMA model and the probing method was also used to estimate the higher order Frequency Response Function. Application of the method to a nonlinear Mathieu system was investigated. Finally, based on the seismic response data of the building,

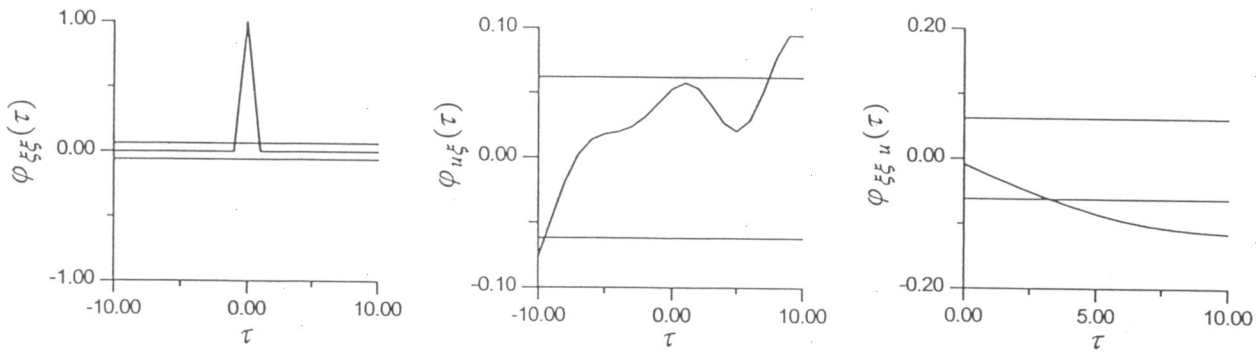


Fig. 5: Model validation test of the predicted model (using data from Northridge earthquake).

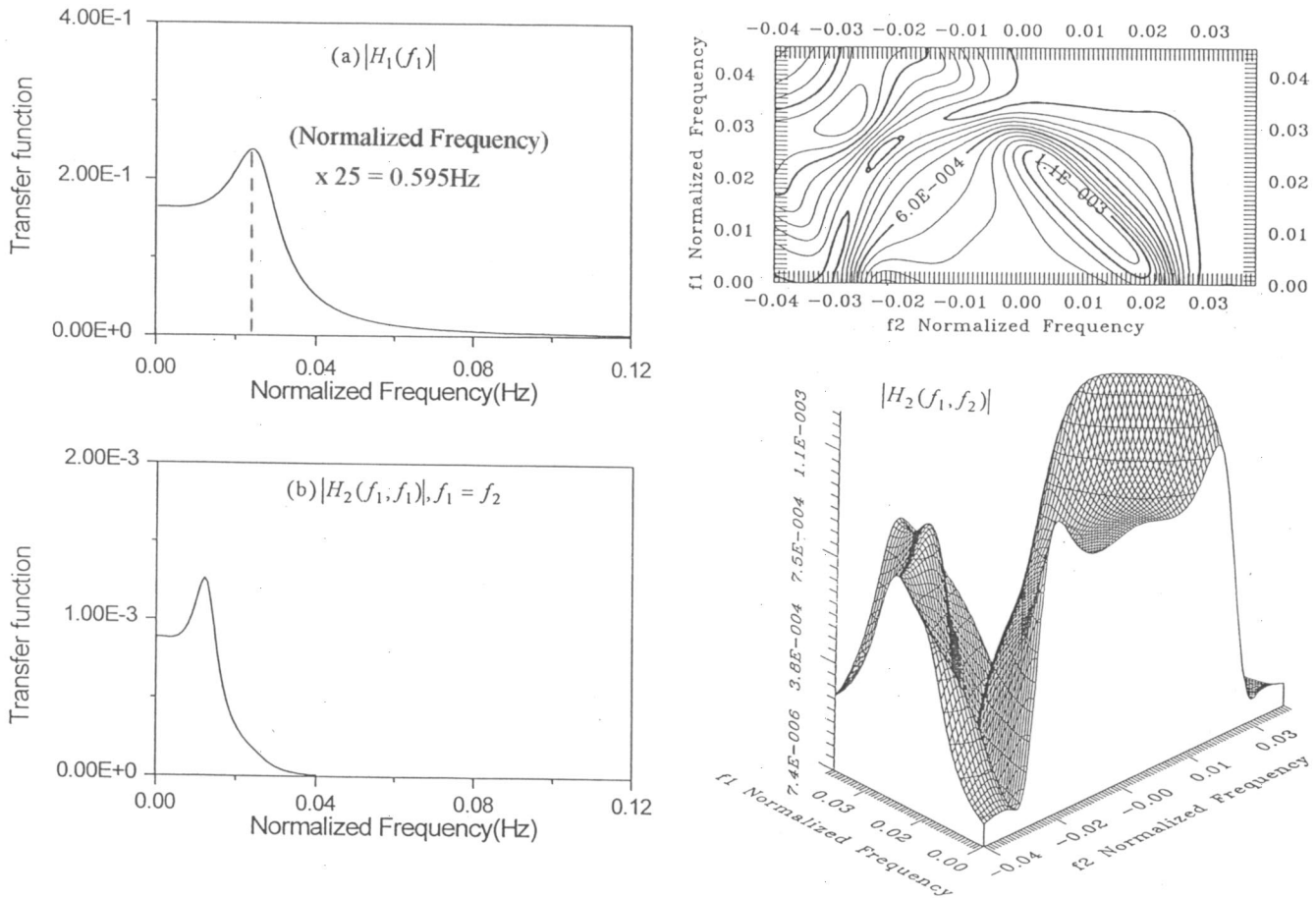


Fig. 6: Estimated FRF from Northridge earthquake data of Van Nuys building; (a) 1st order FRF, (b) 2nd order FRF, (c) 3-D plot of 2nd FRF. Frequency contents below 4Hz are considered.

identification was performed with NARMA model. The following conclusions are made: (1) Application of Hilbert transformation can provide a measure of system nonlinearity, from which the NARMA model can be used to present the nonlinear system; (2) The NARMA methodology provides a systematic method to identify the nonlinear system. If the system can pass the validation test (auto-correlation function of residuals and cross-correlation between residuals and inputs), then the proposed model is a good estimate; (3) Incorporate the velocity term,  $\dot{y}(t)$ , in NARMA model provides a much better estimate of the nonlinear system, because the velocity damping of the system can be considered; (4) From the analysis of Whittier and Northridge earthquake data of the Van Nuys Building, significant nonlinearity was observed from the Northridge earthquake data.

#### ACKNOWLEDGEMENTS

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