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Paper No. 1413. (quote when citing this article)
Eleventh World Conference on Earthquake Engineering
ISBN: 0 08 042822 3

PROPERTY MATRICES IDENTIFICATION OF UNBOUNDED MEDIUM FROM UNIT-IMPULSE RESPONSE FUNCTIONS USING LEGENDRE POLYNOMIALS

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ABSTRACT

A systematic procedure to construct the (symmetric) static-stiffness, damping and mass matrices representing the unbounded medium is presented addressing the unit-impulse response matrix corresponding to the degrees of freedom on the structure-medium interface. The unit-impulse response matrix is first diagonalized which then permits each term to be modelled independently from the others using expansions in a series of Legendre polynomials in the time domain. This leads to a rational approximation in the frequency domain of the dynamic-stiffness coefficient. Using a lumped-parameter model which provides physical insight the property matrices are constructed.

KEYWORDS

Dynamic stiffness; Legendre polynomials; property matrices identification; rational approximation; soil-structure-interaction; system theory; unbounded medium-structure-interaction; unit-impulse response.

INTRODUCTION

To analyse the dynamic interaction of a structure with the adjacent unbounded (semi-infinite) medium, the two substructures are coupled on the structure-medium interface.

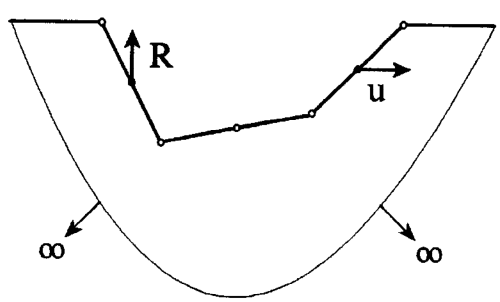

Fig. 1. Interaction force-displacement relationship on structure-medium interface of unbounded medium

The modelling of the bounded non-linear structure with finite elements is well understood resulting in the banded static-stiffness, damping and mass matrices called the property matrices, corresponding to a finite number of degrees of freedom. The representation of the unbounded linear medium is also possible, introducing the *unit-impulse response matrix* $[S(t)]$. The interaction force $\{R(t)\}$-displacement $\{u(t)\}$

relationship with respect to the degrees of freedom of the nodes on the structure-medium interface of the unbounded medium is global in space and time (Fig. 1) (Wolf 1988)

$$\{R(t)\} = [K_\infty]\{u(t)\} + [C_\infty]\{\dot{u}(t)\} + \{R_r(t)\} \tag{1}$$

The first two terms on the right-hand side representing the instantaneous response define the singular part with $[K_\infty]$ and $[C_\infty]$ denoting the high-frequency limit $(\omega \to \infty)$ of the dynamic-stiffness matrix $[S(\omega)]$ The third term describing the lingering response is equal to the regular part (subscript r) consisting of the convolution integral of the corresponding unit-impulse response matrix $[S_r(t)]$ and the displacement vector

$$\{R_r(t)\} = \int_0^t [S_r(t-\tau)]\{u(\tau)\}d\tau \tag{2}$$

The interaction forces at a specific time depend on the time histories of the displacements in all nodes from the start of the excitation onwards. In this rigorous formulation a large computational effort (proportional to the square of the number of time stations) and storage requirement result.

To reduce the computational effort, concepts of *linear system theory* can be applied. These consist of introducing a *rational approximation of the dynamic-stiffness matrix* $[S(\omega)]$, i.e. each coefficient is a ratio of two polynomials in $i\omega$. In Paronesso and Wolf (1995) a procedure is described to construct the property matrices of the unbounded medium starting from $[S(\omega)]$. A diagonalization is first performed which permits scalars to be addressed without any approximation. After the rational approximation the property matrices are constructed using lumped-parameter models without introducing any additional approximation.

It is the goal of the present paper to summarise an analogous formulation using as starting point the regular part of the unit-impulse response matrix $[S_r(t_j)]$, which for computational efficiency is only available for $t_j \le t_{\max}$.

The diagonalization transforms $[S_r(t)]$ of order NxN to the diagonal matrix $[S_r^m(t)]$ of order Nx(N+1)/2 rigorously with the matrix $[T]$ which is time independent

$$[S_r(t)] = [T]\,[S_r^m(t)]\,[T]^T \tag{3}$$

$[T]^T$ plays the role of a kinematic matrix. This corresponds to the following transformation

$$\{u^m(t)\} = [T]^T\{u(t)\} \quad (4a) \qquad\qquad \{R_r(t)\} = [T]\{R_r^m(t)\} \quad (4b)$$

where $\{u^m(t)\}$ and $\{R_r^m(t)\}$ are the input and the output vectors of the diagonal form. After the rational approximation to be discussed in the next section the property matrices $[M]$, $[C]$ and $[K]$ are determined yielding the symmetric second-order differential equation

$$[M]\begin{Bmatrix}\{\ddot{w}(t)\}\\ \{\ddot{u}(t)\}\end{Bmatrix} + [C]\begin{Bmatrix}\{\dot{w}(t)\}\\ \{\dot{u}(t)\}\end{Bmatrix} + [K]\begin{Bmatrix}\{w(t)\}\\ \{u(t)\}\end{Bmatrix} = \begin{Bmatrix}\{\ 0\ \}\\ \{R(t)\}\end{Bmatrix} \tag{5}$$

with the internal variables $\{w(t)\}$. For details of the diagonalization and the construction of the property matrices Paronesso and Wolf (1995) should be consulted.

Rational Approximation

The input-output relationship is formulated for each term of $[S_r^m(t)]$ in (3) as

$$R_r^m(t) = \int_0^t S_r^m(t-\tau)\, u^m(\tau)\; d\tau \tag{6}$$

To construct a rational function in the frequency domain representing an approximation for the dynamic-stiffness coefficient, elements of *linear system identification* are applied. For a chosen input $u^m(t)$ $(t_{j\leq} t_{\max})$, the output $R_r^m(t)$ $(t_{j\leq} t_{\max})$ can be calculated evaluating the convolution integral of (6). An input-output pair is thus available for $t_{j\leq} t_{\max}$, which is the starting point in system identification where it is customary to measure the output for a given input. This theory leads to a linear dynamic system described by a finite number of parameters. The corresponding dynamic-stiffness coefficient will be a rational function. Both $u^m(t)$ and $R_r^m(t)$ are expanded in a series of Legendre polynomials (Chang and Wang, 1982). The coefficients of this series permit the unknown coefficients of the rational function to be determined.

Starting from the basic polynomials 1, t, t^2,..., and applying orthogonalization for $t_{j\leq} t_{\max}$ yield the set of (shifted) Legendre polynomials $\varphi_i(t)$ $(i = 0,1,2,....)$. They can be constructed for $t_{j\leq} t_{\max}$ recursively using

$$\varphi_0(t) = 1 \quad \text{(7a)} \qquad\qquad \varphi_1(t) = 2\frac{t}{t_{\max}} - 1 \quad \text{(7b)}$$

$$\varphi_{i+1}(t) = \frac{2i+1}{i+1}\left(2\frac{t}{t_{\max}} - 1\right)\varphi_i(t) - \frac{i}{i+1}\varphi_{i-1}(t) \qquad i \geq 1 \tag{7c}$$

The Legendre polynomials form a complete set. They are orthogonal

$$\int_0^{t_{\max}} \varphi_i(t)\, \varphi_j(t) dt = \frac{t_{\max}}{2i+1}\delta_{ij} \tag{8}$$

where δ_{ij} is the Kroneker delta (δ_{ij}=1 for i=j and =0 for $i \neq j$). Any function $f(t)$ which is square integrable over the interval $0 \leq t \leq t_{\max}$ can be expanded in a Legendre series with ℓ terms

$$f(t) \cong \sum_{i=0}^{\ell-1} c_i\, \varphi_i(t) \tag{9}$$

where based on the orthogonality property

$$c_i = \frac{2i+1}{t_{\max}} \int_0^{t_{\max}} f(t)\, \varphi_i(t) dt \tag{10}$$

In vector form, (9) is formulated as

$$f(t) \cong \{c\}^T \{\varphi(t)\} \tag{11}$$

The integral of $\{\varphi(t)\}$ can be written as

$$\int_0^t \{\varphi(t)\}\, dt = [\, [L]\{L_\ell\}\,] \begin{Bmatrix} \{\varphi(t)\} \\ \varphi_\ell(t) \end{Bmatrix} \tag{12}$$

with the so-called operational matrix of integration for the Legendre polynomials $[L]$ of order ℓxℓ

$$[L]=\begin{bmatrix} 1/2 & 1/2 & & & & \cdots & \\ -1/6 & & 1/6 & & & \cdots & \\ & -1/10 & & 1/10 & & \cdots & \\ \vdots & \vdots & \vdots & \ddots & & \cdots & \\ & & & \dfrac{-1}{2(2\ell-3)} & & & \dfrac{1}{2(2\ell-3)} \\ & & & & \dfrac{-1}{2(2\ell-1)} & & \end{bmatrix} \qquad (13) \qquad \{L_\ell\}=\begin{Bmatrix} \\ \\ \\ \vdots \\ \\ \dfrac{1}{2(2\ell-1)} \end{Bmatrix} \qquad (14)$$

(12) is derived based on properties of the Legendre polynomials. It can be verified straightforwardly through integration. From the last row of (12) it follows that the integration of a Legendre polynomial of degree $\ell-1$ results in a linear combination of Legendre polynomials of degrees $\ell-2$ and ℓ with the coefficients $-1/(2(2\ell-1))$ and $1/(2(2\ell-1))$ respectively (last lines of equations (13) and (14)).In the derivation $\{L_\ell\}$ is suppressed yielding from (12)

$$\int_0^t \{\varphi(t)\}\, dt \cong [L]\,\{\varphi(t)\} \qquad (15)$$

For a large ℓ the neglected term $1/(2(2\ell-1))$ in (14) tends to zero. Integrating (15) k times results in

$$\underbrace{\int_0^t}_{k\ times} \{\varphi(t)\}\, dt \cong [L]^k\,\{\varphi(t)\} \qquad (16)$$

The regular part of the unit-impulse response coefficient $S_r^m(t)$ in (6) is approximated as that corresponding to an ordinary differential equation of order M for the output $R_r^m(t)$, whereby derivatives up to M-1 for the input $u^m(t)$ are present

$$\begin{aligned} &q_0\, R_r^m(t) + q_1\frac{dR_r^m(t)}{dt} + q_2\frac{d^2R_r^m(t)}{dt^2} + \ldots + \frac{d^M R_r^m(t)}{dt^M} = \\ &p_0\, u^m(t) + p_1\frac{du^m(t)}{dt} + p_2\frac{d^2u^m(t)}{dt^2} + \ldots + p_{M-1}\frac{d^{M-1}u^m(t)}{dt^{M-1}} \end{aligned} \qquad (17)$$

All $2M$ unknown coefficients q_0, ... p_{M-1} are constant and real. Note that the coefficient of $d^M R_r^m(t)/dt^M$ is selected as one. In the algorithm the order M must be chosen.

The Fourier transformation of (17) leads to the input-output relationship in the frequency domain

$$R_r^m(\omega) = S_r^m(\mathrm{i}\omega)u^m(\omega) \qquad (18)$$

where the approximated regular part of the dynamic-stiffness coefficient equals

$$S_r^m(\mathrm{i}\omega) = \frac{p_0 + p_1(\mathrm{i}\omega) + p_2(\mathrm{i}\omega)^2 + p_{M-1}(\mathrm{i}\omega)^{M-1}}{q_0 + q_1(\mathrm{i}\omega) + q_2(\mathrm{i}\omega)^2 + \ldots + (\mathrm{i}\omega)^M} \qquad (19)$$

$S_r^m(\mathrm{i}\omega)$ is a rational function in $\mathrm{i}\omega$ with the coefficients q_0, ... p_{M-1} where the degrees of the polynomials in the denominator and the numerator are equal to M and M-1, respectively. For the limit of $\mathrm{i}\omega \to \infty$ the approximation of the regular part tends to zero. The approximate dynamic-stiffness coefficient is thus exact in the high-frequency limit (asymptotic behaviour).

For a specified $u^m(t)$, the output $R_r^m(t)$ is calculated by evaluating the convolution integral in (6). Both $u^m(t)$ and $R_r^m(t)$ are then expanded in a Legendre series with ℓ terms (11)

$$u^m(t) \cong \{c_u\}^T \{\varphi(t)\} \quad (20) \qquad R^m(t) \cong \{c_R\}^T \{\varphi(t)\} \quad (21)$$

with the coefficients c_u and c_R determined from (10).

To determine the coefficients $q_0, \ldots p_{M-1}$, (17) is integrated M times, which transforms the differential equation of M-th order to an integral equation. For vanishing initial conditions (17) is transformed to

$$\begin{aligned} & q_0 \underbrace{\int_0^t R_r^m(t)dt}_{M\ times} + q_1 \underbrace{\int_0^t R_r^m(t)dt}_{M-1\ times} + q_2 \underbrace{\int_0^t R_r^m(t)dt}_{M-2\ times} + \cdots + R_r^m(t) = \\ & p_0 \underbrace{\int_0^t u^m(t)dt}_{M\ times} + p_1 \underbrace{\int_0^t u^m(t)dt}_{M-1\ times} + p_2 \underbrace{\int_0^t u^m(t)dt}_{M-2\ times} + \ldots + p_{M-1}\int_0^t u^m(t)dt \end{aligned} \quad (22)$$

Substituting the Legendre series expansions of $u^m(t)$ (20) and $R_r^m(t)$ (21) in (22) and using (16) lead to

$$\left[-([L]^T)^M\{c_R\} \vdots -([L]^T)^{M-1}\{c_R\} \vdots \cdots \vdots -[L]^T\{c_R\} \vdots ([L]^T)^M\{c_u\} \vdots ([L]^T)^{M-1}\{c_u\} \vdots \cdots \vdots [L]^T\{c_u\}\right] \begin{Bmatrix} q_0 \\ q_1 \\ \vdots \\ q_{M-1} \\ p_0 \\ p_1 \\ \vdots \\ p_{M-1} \end{Bmatrix} = \{c_R\} \quad (23)$$

The coefficient matrix of (23) is of order ℓ x$2M$. For a solution $\ell \geq 2M$ must be selected. For $\ell > 2M$ the overdeterminated equation is solved using the least-squares procedure yielding the coefficients of the rational approximation $q_0, \ldots p_{M-1}$ in (19). For a well conditioned system $M \leq 12$ must be chosen, as the numerical rank of the eigenvector matrix of $[L]^T$ (which can be diagonalized) does not exceed 24.

Optimum Implementation

The selected input is formulated as

$$u^m(t) = a\frac{\alpha}{t_{\max}} e^{\frac{-\alpha}{t_{\max}}t} H(t) \quad (24)$$

with the Heaviside step function $H(t)$. α is dimensionless. The constant a, with the dimension length times time, represents the integral $\int_0^\infty u^m(t)\,dt$ which is selected as one and is thus independent from α. For the limit $\alpha \to \infty$, $u^m(t)$ tends to the Dirac delta function $\delta(t)$. The more α diminishes, the more emphasis is placed on $u^m(\omega)$ at small ω at the cost of that at large ω.

An input defined as a Dirac-delta function $\delta(t)$ can be selected as an alternative. For an expansion of $\delta(t)$ in a Legendre series, the coefficients equal (10)

$$c_{ui} = \frac{2i+1}{t_{\max}} \varphi_i(0) \qquad i{=}0,..\ell-1 \tag{25}$$

with the Legendre polynomials at t=0 (7)

$$\varphi_i(0) = (-1)^i \tag{26}$$

The convolution integral to determine the output (6) is avoided, as

$$R_r^m(t) = S_r^m(t) \tag{27}$$

applies, i.e. an expansion in a Legendre series is calculated directly for $S_r^m(t)$.

Numerical experience indicates that the i-th row of the overdeterminated system (23) in the case of the input being equal to $\delta(t)$ must be multiplied by $1/|c_{ui}| = t_{\max}/(2i+1)$ .Thus, a weighted least-squares approximation is performed with a diagonal weighting matrix.

The static-stiffness coefficient K^m can also be enforced, making the rational approximation doubly asymptotic. For the implementation, enforcing K^m corresponds to equating $S_r^m(\mathrm{i}\omega = 0)$ in (19) to $K^m - K_\infty^m$. This yields

$$\frac{p_0}{q_0} = K^m - K_\infty^m \tag{28}$$

The number of unknowns is thus reduced by one. This condition can be directly introduced in (23) by e.g. eliminating p_0.

As a stringent test of a dispersive system with a cutoff frequency, the one-dimensional semi-infinite rod with area A, modulus of elasticity E, mass density ρ resting on an elastic foundation with the spring stiffness k_g (Fig. 2) is analysed. The analytical solutions for $S(a_0)$ and $S_r(\bar{t})$ at the beginning of the rod in point 0 are derived in Wolf (1988) with the dimensionless frequency $a_0 = \omega\sqrt{A\rho/k_g}$ and time $\bar{t} = t\sqrt{k_g/(A\rho)}$. The cutoff frequency equals a_0=1.

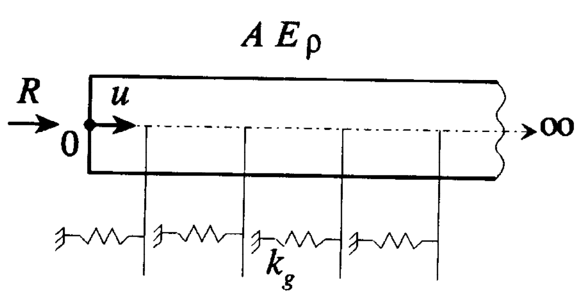

Fig. 2. Semi-infinite rod on elastic foundation

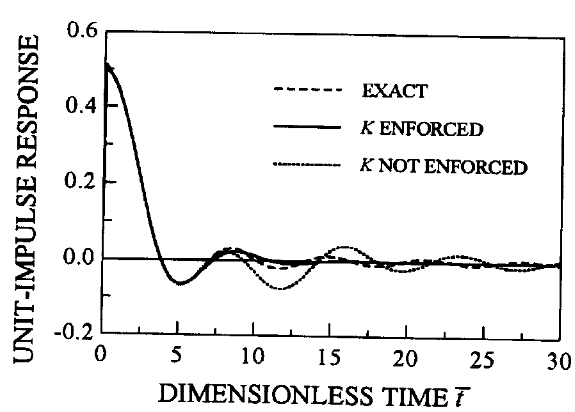

Fig. 3. Regular part of unit-impulse response coefficient

The analysis is performed for M=4 and ℓ=30 with $\bar{t}_{\max}$=5, which is a small value. As input the Dirac delta function $\delta(t)$ is used. The influence of enforcing the static-stiffness coefficient K is examined. The regular part of the unit-impulse response coefficient $S_r(\bar{t})$ is compared in Fig. 3 and the total dynamic-stiffness coefficient $S(a_0)$ normalized by K and decomposed in the spring coefficient $k(a_0)$ and the damping coefficient $c(a_0)$ in Fig. 4. When K is not enforced in the rational approximation, large deviations exist in

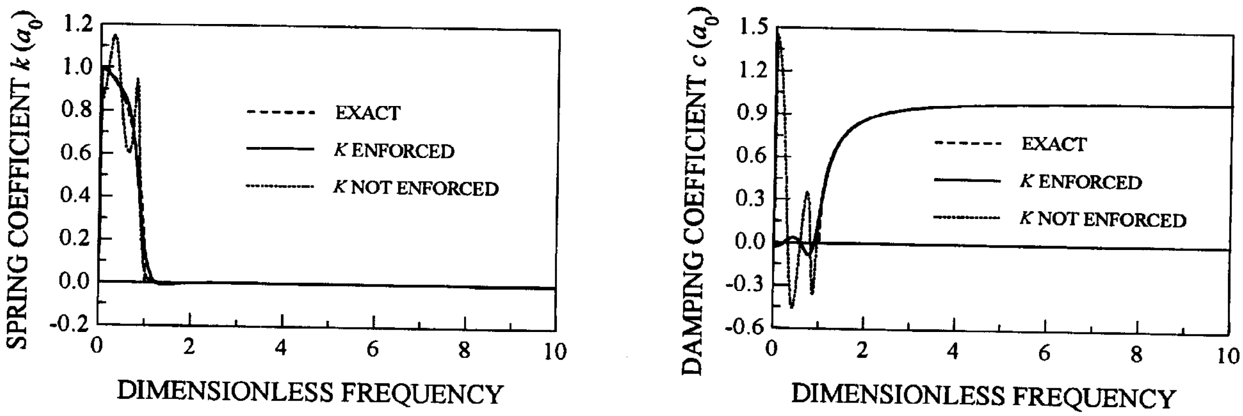

Fig 4. Total dynamic-stiffness coefficient

$S_r(\bar{t})$ for $\bar{t}>5$ which results in inaccurate $S(a_0)$ for $a_0<1$. A drastic improvement results when the static-stiffness coefficient K is enforced.

IN-PLANE MOTION OF LAYER FIXED AT ITS BASE

The in-plane motion of a semi-infinite layer with a free and a fixed boundary extending to infinity of constant depth d, shear modulus G, Poisson's ratio =1/3 and mass density ρ is examined (Fig. 5). On the vertical structure-medium interface 8 line finite elements, each with 3 nodes, are introduced (not shown) in the consistent infinitesimal finite-element cell method (Song and Wolf, 1996). This discretization permits an adequate modelling up to the dimensionless frequency $a_0 = \omega\, d / c_s = 2.5\pi$ ($c_s = \sqrt{G/\rho}$). To reduce the data for the examination, 4 nodes with the numbers shown in Fig. 5 with piecewise linear displacements are introduced, and the corresponding reduction is performed based on virtual-work considerations. This leads to the corresponding matrices $[S_r(\bar{t})]$ and $[S(a_0)]$ of order 8 x 8 with the dimensionless time $\bar{t} = t\, c_s / d$. These results are denoted as rigorous. The cutoff frequency of the layer corresponds to $a_0 = \pi / 2$.

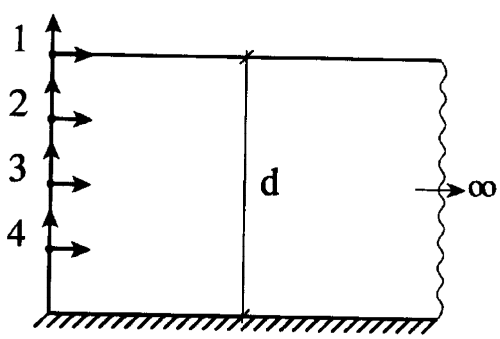

Fig. 5. Semi-infinite layer fixed at its base

The analysis is performed for the degree of the rational approximation M=12, and the number of terms in the Legendre expansion is selected as ℓ=40. The maximum dimensionless time is equal to $\bar{t}_{\max}$=10. As input, the exponential function specified in (24) with α=10 is chosen.

To check the accuracy in the frequency domain, the total dynamic-stiffness coefficient $S_{11}(a_0)$, relating the horizontal displacement in node 1 to the horizontal interaction force in the same node, and $S_{18}(a_0)$, relating the vertical displacement in node 4 to the horizontal interaction force in node 1, are examined. Both dynamic-stiffness coefficients are non-dimensionalized by the static-stiffness coefficient K_{11} and then decomposed into a spring coefficient $k(a_0)$ and a damping coefficient $c(a_0)$. From the comparison shown in Figs. 6 and 7, it follows that although the rigorous results vary significantly the rational approximation up to a_0=10 is good. The range a_0>10 should hardly affect the seismic response.

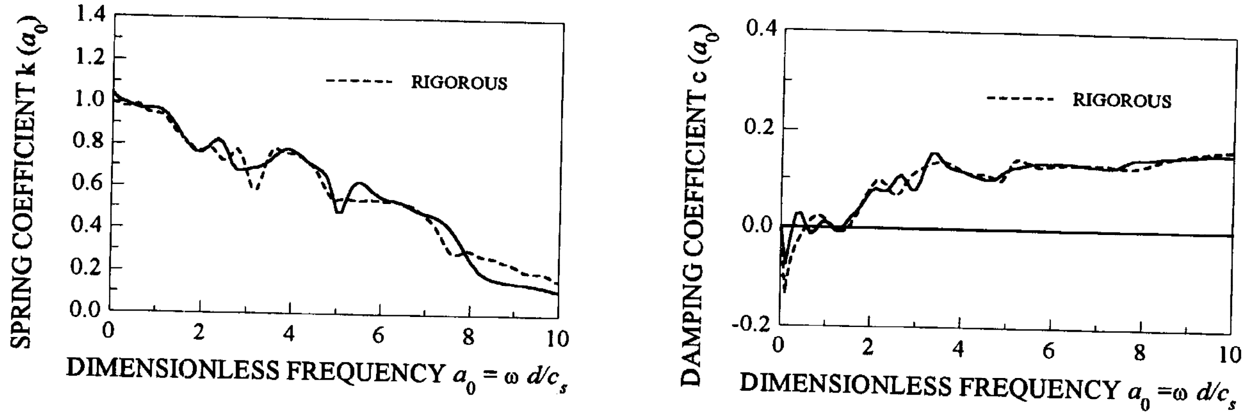

Fig. 6. Total dynamic-stiffness coefficient $S_{11}(a_0)$

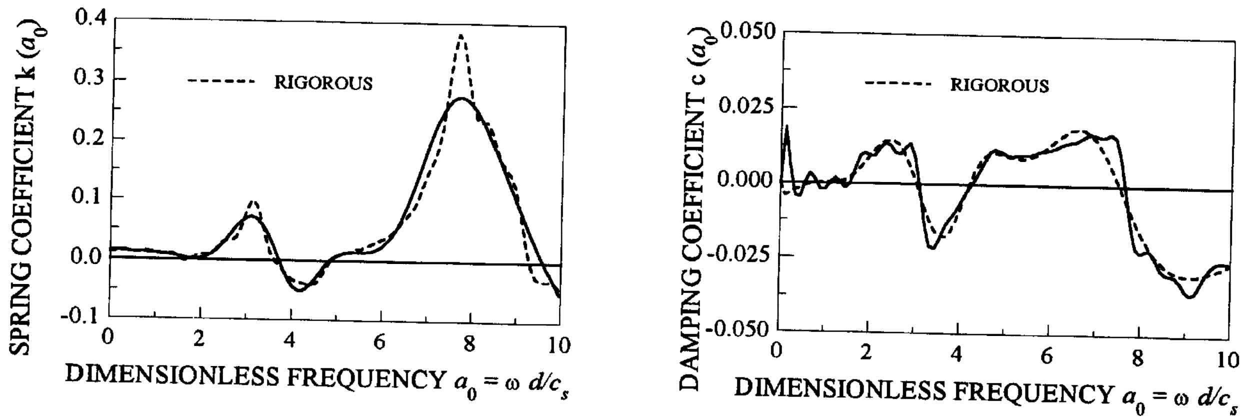

Fig. 7. Total dynamic-stiffness coefficient $S_{18}(a_0)$

CONCLUSIONS

The presented procedure using the unit-impulse response matrix in the time domain with Legendre polynomials is analogous to the least-squares method addressing the dynamic-stiffness matrix in the frequency domain, both yielding a rational approximation in the frequency domain.

The unbounded medium is modelled in the same manner as the structure consisting of (symmetric) static-stiffness, damping and mass matrices. The same computer program can be used for dynamic unbounded medium-structure-interaction analysis as for structural dynamics.

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