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CONSTITUTIVE LAWS OF REINFORCED CONCRETE MEMBRANE ELEMENTS

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ABSTRACT

This paper summarizes a set of constitutive laws for concrete and steel bars, established in conjunction with the development of the softened truss model. It consists of three average stress-strain relationships: concrete in tension, reinforcing bars stiffened by concrete, and softened concrete in compression. Observations regarding these three relationships are: 1) concrete in tension develops substantial tensile stresses even after extensive cracking; 2) stress-strain curve of mild steel embedded in concrete does not exhibit a yield plateau after yielding, as in the case of bare bars, but has an "apparent yield stress" lower than the yield stress of a bare bar; and 3) the softening of concrete in compression is expressed by a softening coefficient which is a function of the tensile strains (smeared cracking) in the perpendicular direction. This softening coefficient is also found to be inversely proportional to $\sqrt{f_c'}$, for f_c' up to 100 MPa. The proposed set of "accurate constitutive laws" is compared to two sets of constitutive laws currently in use: (1) the conventional set of "simplified constitutive laws" in which the tensile stress of concrete is neglected and the stress-strain curve of steel bars is assumed to exhibit the elastic-perfectly-plastic behavior; and (2) the "modified constitutive laws" in which the tensile stress of concrete is taken into account but the stress-strain curve of steel bars remains elastic-perfectly-plastic.

KEYWORDS

Biaxial loads; cracking (fracturing); strain compatibility; equilibrium; membrane elements; reinforced concrete; reinforcing steel; stiffening, strains; stresses; stress-strain relationships.

INTRODUCTION

Wall-type and shell-type reinforced concrete structures have received considerable attention in recent years. An element isolated from such a structure is subjected to membrane stresses. Since the understanding of the behavior of a reinforced concrete element is the key to the analysis of the whole structure, a softened truss model theory has been developed for the nonlinear analysis of such membrane elements (Hsu, 1993). The softened truss model incorporates the three fundamental principles of the mechanics of materials: stress equilibrium, strain compatibility and constitutive laws of materials. An accurate prediction by the softened truss model depends strongly on the constitutive laws of the concrete and steel in the elements.

Using the universal panel tester constructed at the University of Houston (Hsu *et al.*, 1995), fifty-five full-size reinforced concrete panels, 1.4 m square and 17.8 cm thick, have been tested to determine the stress-strain relationships of concrete and steel in the membrane elements. Thirty panels were subjected to biaxial tension-compression in which the reinforcement is oriented in the same direction as the principal stresses, and the loads were applied sequentially (with tension first) or proportionally. The remaining twenty-five panels were subjected to in-plane shear in which the reinforcement is oriented at a 45° angle with respect to the principal stress directions. Of the fifty-five panels, thirty-five were made with normal-strength concrete, $f'_c = 42$ MPa, six were made with medium-strength concrete, $f'_c = 65$ MPa, and fourteen were made with high-strength concrete, $f'_c = 100$ MPa.

SOFTENED TRUSS MODEL FOR MEMBRANE ELEMENTS

A reinforced concrete membrane element is subjected to in-plane shear stresses and normal stresses as shown in Fig. 1(a). The stresses σ_ℓ, σ_t and $\tau_{\ell t}$ are defined in the ℓ-t coordinate of the longitudinal and transverse steel bars. This set of in-plane stresses σ_ℓ, σ_t and $\tau_{\ell t}$ is equivalent to a set of principal stresses, σ_2 and σ_1, acting along the principal 2-1 coordinate system, Fig. 1(d). The angle between the 2-1 coordinate and the ℓ-t coordinate is called the fixed-angle α_2 because this angle remains unchanged when the applied stresses σ_ℓ, σ_t and $\tau_{\ell t}$ increase proportionally.

The first set of cracks occurs when the principal tensile stress σ_1 reaches the tensile strength of concrete. These diagonal cracks will separate the concrete into a series of concrete struts. In general, when an element is reinforced with different amounts of steel in the ℓ- and t-directions, the direction of the principal stresses in concrete after cracking will continuously deviates from the direction of the applied principal stresses, as the applied load increases proportionally. The post-cracking principal stresses in the concrete are defined by the d-r coordinate in Fig. 1(e). The angle α between the d-axis and the ℓ-axis continues to rotate away from the initial angle α_2 throughout the loading history. As such, the angle α is called the rotating-angle. The average principal compressive stress and the average principal tensile stress in the concrete are designated σ_d and σ_r, respectively.

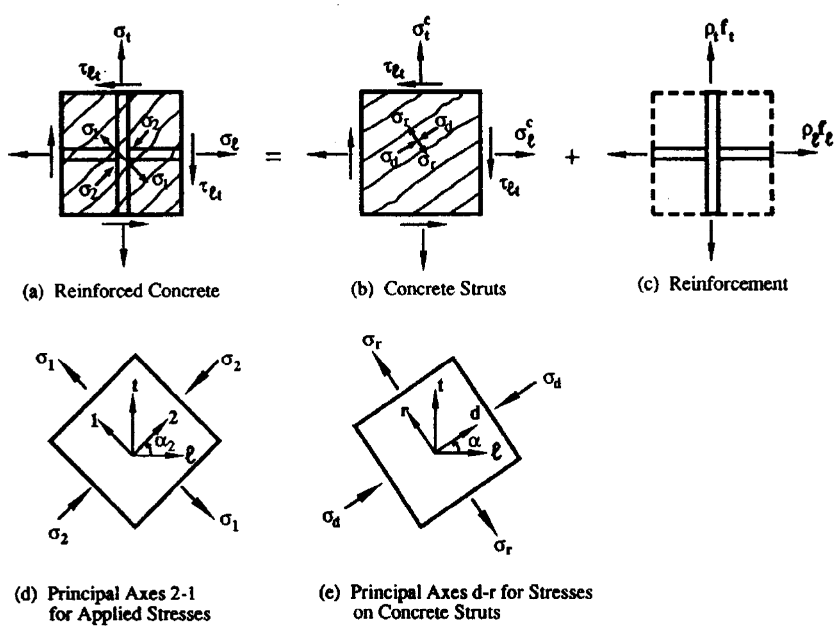


Fig. 1. Stress conditions in a reinforced concrete membrane element

Equilibrium Equations

The two-dimensional equilibrium condition relates the average internal stresses in the concrete (σ_d and σ_r) and in the reinforcement (f_ℓ and f_t) to the applied stresses (σ_ℓ, σ_t and $\tau_{\ell t}$). Utilizing the transformation of concrete stresses and assuming that the steel bars can resist only axial stresses, the superposition of concrete stresses and steel stresses results in (Hsu, 1993):

$$\sigma_\ell = \sigma_d \cos^2\alpha + \sigma_r \sin^2\alpha + \rho_\ell f_\ell \quad (1)$$

$$\sigma_t = \sigma_d \sin^2\alpha + \sigma_r \cos^2\alpha + \rho_t f_t \quad (2)$$

$$\tau_{\ell t} = (-\sigma_d + \sigma_r) \sin\alpha \cos\alpha \quad (3)$$

in which ρ_ℓ and ρ_t are the percentages of reinforcement in the ℓ-and t-directions, respectively.

Compatibility Equations

The two-dimensional compatibility condition expresses the state of strains within the element. Assuming that the d- and r- are also principal axes for strains, then the transformation of average strains between the ℓ-t coordinate system (ϵ_ℓ, ϵ_t, $\gamma_{\ell t}$) and the d-r principal axes (ϵ_d, ϵ_r) gives:

$$\epsilon_\ell = \epsilon_d \cos^2\alpha + \epsilon_r \sin^2\alpha \quad (4)$$

$$\epsilon_t = \epsilon_d \sin^2\alpha + \epsilon_r \cos^2\alpha \quad (5)$$

$$\gamma_{\ell t} = 2\ (-\epsilon_d + \epsilon_r) \sin\alpha \cos\alpha \quad (6)$$

Constitutive Laws

Based on the principles of mechanics, the stresses in the equilibrium equations need to be related to the strains in the compatibility equations through the constitutive laws of materials. The analysis/design of the membrane element of Fig. 1(a) requires four stress-strain relationships, i.e., concrete in compression, concrete in tension, and mild steel in longitudinal and transverse directions as follows:

$$\sigma_d = f_1(\epsilon_d, \epsilon_r) \quad (7)$$

$$\sigma_r = f_2(\epsilon_r) \quad (8)$$

$$f_\ell = f_3(\epsilon_\ell) \quad (9)$$

$$f_t = f_4(\epsilon_t) \quad (10)$$

Because the equilibrium and compatibility equations are derived for a continuous material, the stress-strain relationships of concrete and reinforcement must relate average stresses to average strains. The derivation of functions f_1 through f_4 in Eqs. (7) through (10) was the focus of an extensive study carried out at the University of Houston (Belarbi and Hsu, 1994, 1995; Pang and Hsu, 1995; Zhang, 1992, 1995). Fifty-five full-size reinforced concrete panels were tested in a universal panel tester and subjected to various types of membrane stresses, including biaxial tension-compression and pure shear loadings. The overall objectives of the research were threefold: 1) to experimentally study the variables that may affect the constitutive laws of cracked concrete; 2) to develop a physical understanding of the observed phenomenon so that the influence of each variable can be quantified; and 3) to improve the mathematical expression for the constitutive laws required in the softened truss model. Based on this research, the following constitutive laws were derived.

Concrete in Compression

The primary characteristic of the constitutive law of concrete in compression is the softening of peak stress in comparison to the companion cylinder. The variables that may affect the softening phenomenon were studied in a systematic manner. These variables include the tensile strain, the tensile stress, the load path and the nature of applied loads (biaxial tension-compression vs. pure shear), the percentage of steel, the spacing of steel bars, the ratio of longitudinal to transverse reinforcements, and the concrete strength. Among the variables investigated, the severity of cracking expressed in terms of ϵ_r, the concrete strength and to a certain extent the load path were found to be the main variable. The softening coefficient was also found to be inversely proportional to $\sqrt{f_c'}$. The graphic representation of the stress-strain relationship of the softened concrete struts is shown in Fig. 2. The function f_1 of Eq. (7) is mathematically expressed as follows:

$$\sigma_d = \zeta f_c' \left[2\left(\frac{\epsilon_d}{\zeta\epsilon_o}\right) - \left(\frac{\epsilon_d}{\zeta\epsilon_o}\right)^2 \right] \qquad \epsilon_d/\zeta\epsilon_o \le 1 \tag{11a}$$

$$\sigma_d = \zeta f_c' \left[1 - \left(\frac{\epsilon_d/\zeta\epsilon_o - 1}{2/\zeta - 1}\right)^2 \right] \qquad 1 < \epsilon_d/\zeta\epsilon_o \le 1.5/\zeta \tag{11b}$$

where ϵ_o is the strain at the peak stress of standard concrete cylinder taken usually as 0.002 for normal strength concrete (42 MPa) and 0.0024 for high-strength concrete (100 MPa), and ζ is the softened coefficient taken as follows (Zhang, 1995):

$$\zeta = \frac{5.8}{\sqrt{f_c'(\mathrm{MPa})}} \frac{1}{\sqrt{1 + 400\epsilon_r}} \tag{12}$$

Concrete in Tension

Based on the tests of seventeen full-size reinforced concrete panels subjected to uniaxial tension (with longitudinal reinforcement placed along the applied stresses), average stress-strain relationship for concrete in tension was derived. Prior to the yielding of reinforcement, the average stress in steel can be estimated by multiplying the measured average strain by the elastic modulus of steel. The difference between the average steel stress and the applied stress is attributed to concrete and is considered as the average stress in concrete. Based on this average stress approach, concrete was found to develop substantial tensile stresses even after extensive cracking, and the function f_2 of Eq. (8) is expressed mathematically as follows (Belarbi and Hsu, 1994):

$$\sigma_r = E_c \epsilon_r \qquad \epsilon_r \le \epsilon_{cr} \tag{13a}$$

$$\sigma_r = f_{cr} \left(\frac{\epsilon_{cr}}{\epsilon_r}\right)^{0.4} \qquad \epsilon_r > \epsilon_{cr} \tag{13b}$$

where E_c is the elastic modulus of concrete given as $E_c = 3900\sqrt{f_c'(\mathrm{MPa})}$; f_{cr} is the tensile strength of the concrete given as $f_{cr} = 0.31\sqrt{f_c'(\mathrm{MPa})}$; and ϵ_{cr} is the average tensile strain at which the concrete begins cracking, given as 0.00008. Equation (13) is expressed graphically in Fig 3.

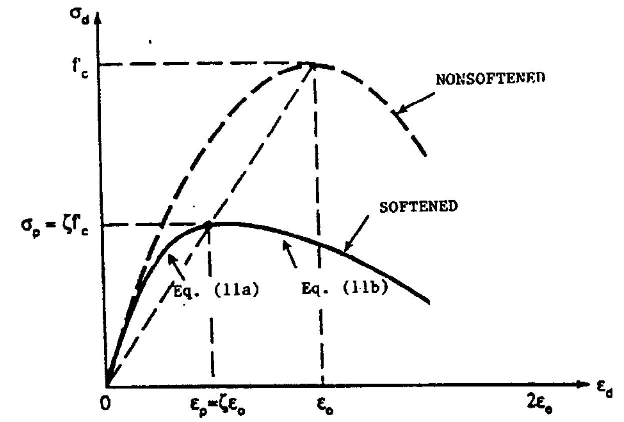


Fig. 2. Proposed softened stress-strain curve of concrete

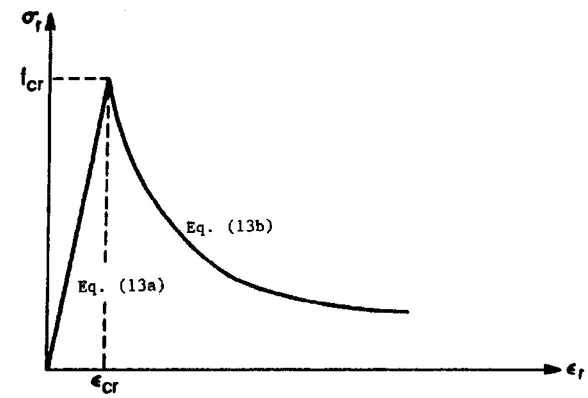


Fig. 3. Proposed tensile stress-strain curve of concrete

Reinforcing Bars in Tension

The stress-strain relationship of mild steel bar tested in a bare condition exhibits a long plateau after yielding. However, the average stress-strain curve of steel bars embedded in concrete does not show such a yield plateau. The average stress at first yield called "apparent yield stress f_y^*" and the average stresses in the post-yield range were found to be lower than those of a bare bar. This reduction of steel stress was found (Belarbi and Hsu, 1994) to be directly related to a parameter $B = (1/\rho)(f_{cr}/f_y)^{1.5}$ expressed in terms of steel and concrete tensile strengths (f_y and f_{cr}) as well as the reinforcement ratio (ρ).

Furthermore, in the pure shear case where the reinforcing bars are laid out at a 45° angle with respect to the principal applied stresses, there will be kinking of steel bars at the cracks. This kinking phenomenon was found to lower even further the average stress of reinforcement (Pang and Hsu, 1995). The kinking factor is related to the angle α_2, Fig. 1(d). In the case of biaxial tension-compression tests ($\alpha_2 = 90°$), the reinforcing bars do not experience any kinking, whereas in the case of pure shear tests ($\alpha_2 = 45°$, the reinforcing bars undergo significant kinking.

The average stress-strain relationship of reinforcing bars embedded in concrete is shown graphically in Fig. 4 as a bilinear model. This relationship is valid for both the longitudinal and transverse reinforcements indicated by functions f_3 and f_4 in Eqs. (9) and (10) and is expressed mathematically as follows:

$$f_s = E_s \epsilon_s \qquad \epsilon_s \le \epsilon_n \tag{14a}$$

$$f_s = f_y \left[(0.91 - 2B) + (0.02 + 0.25B)\frac{\epsilon_s}{\epsilon_y} \right] \left(1 - \frac{2 - \alpha_2/(45°)}{1000\rho} \right) \qquad \epsilon_s > \epsilon_n \tag{14b}$$

where

$$\epsilon_n = \epsilon_y \, (0.93 - 2B) \left(1 - \frac{2 - \alpha_2/(45°)}{1000\rho} \right) \tag{15}$$

In Eq. (14b), the factor $[(0.91 - 2B) + (0.02 + 0.25B)(\epsilon_s/\epsilon_y)]$ takes care of the averaging of steel stresses in the post-yield branch. The factor $[1 - (2 - \alpha_2/45°)/1000\rho]$ takes into account the kinking of reinforcing bars at the cracks. When $\alpha_2 = 45°$, this "kinking factor" is equal to $[1 - 1/1000\rho]$. When $\alpha_2 = 90°$, it becomes unity.

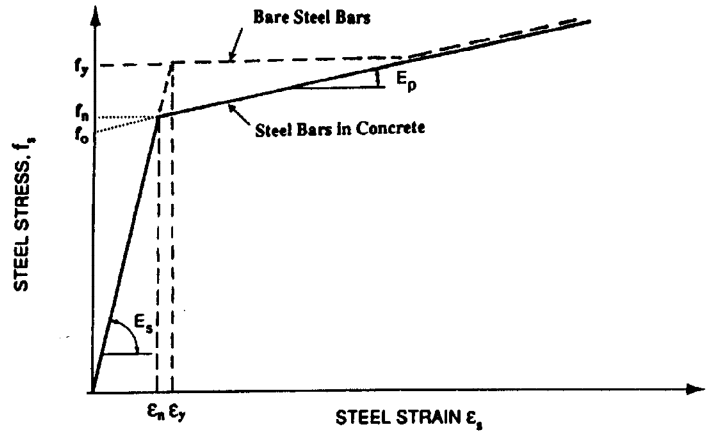


Fig. 4. Average stress-strain curve of mild steel bars using bilinear model

Simplified and Modified Versions of the Constitutive Laws

The set of constitutive laws relating average stresses to average strains, Eqs. (11) through (15), are referred to as the accurate constitutive laws. They can be used in the analysis when the deformations and the yield strength of the structure are both important.

If a structure is subjected to static loads and the deformation of the structure is not important, then the constitutive laws can be simplified by (1) neglecting the tensile stress of concrete, giving

$$\sigma_r = 0 \tag{16}$$

and (2) assuming the elastic-perfectly-plastic characteristic of bare mild steel bars, giving

$$f_s = E_s\epsilon_s \qquad \epsilon_s \leq \epsilon_y \tag{17a}$$

$$f_s = f_y \qquad \epsilon_s > \epsilon_y \tag{17b}$$

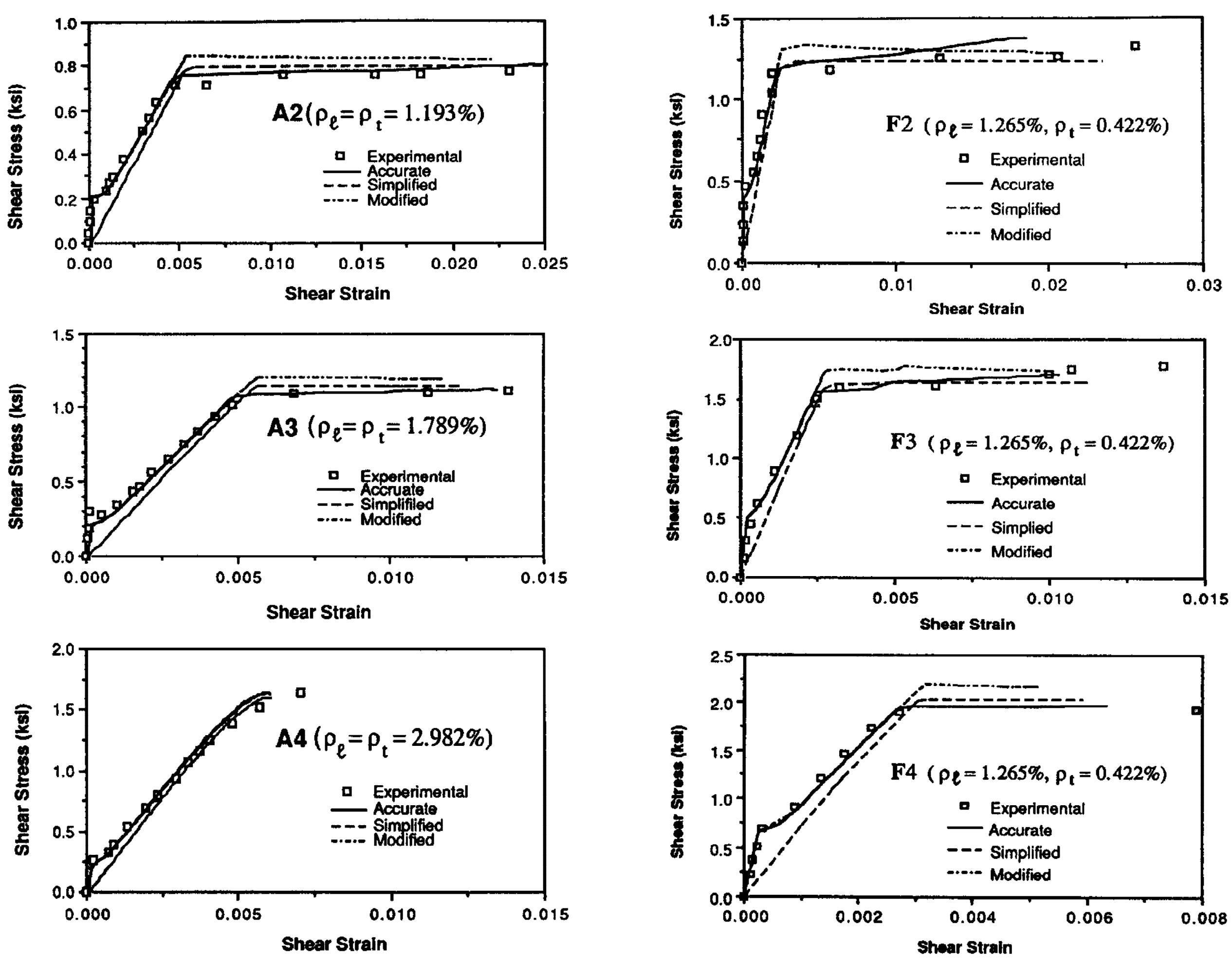
Equations (11), (12), (16) and (17) are referred to as "the simplified constitutive laws". Since the use of Eq. (16) is conservative in terms of the yield strength of an element and Eq. (17) is unconservative, the errors induced by these two relationships cancel each other so that the yield strength is correctly predicted. However, the deformations will be overestimated because the tension stiffening effect is neglected. Physically, this simplification implies that the average tensile stress-strain relationships of concrete and steel are replaced by the local stress-strain relationships at the cracks. Indeed, the tensile strength of concrete is zero at the cracks, and the stress-strain relationship of mild steel bars at the cracks does exhibit the elastic-perfectly-plastic characteristic of the bare bars.

A modified version of the constitutive laws is a simultaneous employment of the average tensile stress-strain curve of concrete, Eq. (13), and the stress-strain curve of bare steel bars, Eq. (17). The modified constitutive laws will take into account the tension stiffening effect so that the deformations are correctly evaluated but it will overestimate the strength at the first yielding of steel. Physically, this overestimation of yield strength is caused by the incorrect matching of the "local" stress-strain relationship of steel at the cracks and the "average" tensile stress-strain relationship of concrete over a length that traverses several cracks. As a result, this combination of two constitutive laws will produce an unwarranted "concrete strengthening", in addition to a correct reduction in deformations due to tension stiffening effect.

COMPARISON WITH EXPERIMENTS

The three sets of constitutive equations given above are used in the softened truss model, and the theoretical predictions are compared to the experimental results of six full-size reinforced concrete panels tested at the University of Houston. The six panels include three panels in F-series (Belarbi and Hsu, 1995) and three panels in A-series (Pang and Hsu, 1995). In the panels of F-series, the longitudinal reinforcement was placed perpendicular to the direction of the compressive loads ($\alpha_2 = 90°$). The number following the letter "F" gives the ratio of the compressive stress to the tensile stress. In the panels of A-series, the longitudinal reinforcement was oriented at an angle of 45° to the principal compressive loads ($\alpha_2 = 45°$). Compressive and tensile loadings were applied proportionally with equal magnitude until failure. The predicted responses of the softened truss model using the three sets of constitutive laws (simplified, modified and accurate) are compared with the experimental responses of the six panels in Fig. 5 (Hsu and Zhang, 1996).

When the set of simplified constitutive equations is employed, Fig. 5 shows that good agreement is obtained in terms of yield strength. However, the post-cracking deformations are overestimated due to the neglect of the tension stiffening effect. This overestimation of deformations is especially severe before the cracking of concrete, and is about 20% in the post-cracking service load stage. When the set of modified constitutive laws is utilized, Fig. 5 shows that correct predictions are obtained for load-deformation curves up to yielding. The predictions, however, significantly overestimate the shear stresses at the first yielding of steel by an average of 13%. Figure 5 also shows that excellent agreement is obtained throughout the loading history using the set of accurate constitutive equations. The agreement occurs not only in terms of yield strengths but also in terms of deformations.



(a) A-series (b) F-series

Fig. 5. Shear stress vs. shear strain relationships for panels in A- and F-series (1 ksi = 6.895 MPa)

CONCLUSIONS

(1) Softened truss model was developed for the nonlinear analysis of concrete membrane elements. This model involves three equilibrium equations, three compatibility equations and four equations for constitutive laws of concrete and steel. Based on the test results of fifty-five full-size reinforced concrete panels, average stress-strain relationships have been determined for concrete in compression, concrete in tension, and steel reinforcement embedded in concrete.

(2) The stress-strain relationship of concrete in compression can be expressed by Eqs. (11) and (12). The peak-softening coefficient is primarily a function of the principal tensile strain, ϵ_r and the concrete strength f'_c. In tension, the concrete carries tensile stresses even after yielding of steel. The stress-strain curve for concrete in tension is expressed by Eq. (13). For reinforcing steel, the first yield stress of the average stress-strain curve is lower than that of a bare bar. Kinking of inclined bars at crack further reduces the average stress. The average stress-strain relationship of steel bars embedded in concrete is given by a bilinear model expressed by Eqs. (14) and (15). Incorporating this set of accurate constitutive laws into the softened truss model will produce an accurate prediction of load-deformation response throughout the loading history.

(3) In a modified version of the constitutive laws, the average tensile stress-strain curve of concrete is used in conjunction with the elastic-perfectly-plastic stress-strain relationship of bare mild steel bars. This combination of constitutive laws will take care of the tension stiffening effect on deformations, but will also result in a significant overestimation of the yield strength. If the concrete in tension is neglected, however, as in the simplified version of constitutive laws, the deformation at service load will be significantly overestimated.

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