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# DISPERSION OF TWO-DIMENSIONAL WAVES THROUGH CONTINUUM WITH RANDOMLY VARIABLE PARAMETERS

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## ABSTRACT

The paper is concerned with the properties of a longitudinal (compression) harmonic wave propagating in a 3D continuum the properties of which are burdened by random deviations from normal values, considered constant. Thanks to random perturbations the initially isotropic environment acquires a slightly anisotropic character. Thanks to the hypothesis of high correlation of random parts of the individual components of elasticity tensor on local scale and the low correlation in space, the continuum may be considered as locally isotropic with random-variable Lamé elasticity parameters. Further solution, therefore, concentrates on the Helmholtz equation of amplitude of the potential of the longitudinal wave propagating in the environment of random, but locally isotropic, properties. The solution itself is oriented to the spatial amplitude in longitudinal direction. The initial Laplace operator in space thus degenerates to Laplace operator in transverse direction of wave propagation. In the longitudinal direction the equation becomes parabolic. These formulations provide the possibility of qualitative analyses by means of variation differentiation and the application of the Furutsu formula. This gives rise to the partial differential equation of the mathematical mean value of response. Analogous steps lead to the development of equations for all other stochastic moments of response.

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## KEYWORDS

Stochastic Mechanics, Waves Propagation, Stochastic Continuum, Longitudinal Waves, Gaussian Response, Lateral Diffusion of Energy, Upper Limit Frequency, Degradation of Compress waves.

## INTRODUCTION

The parameters of the environment in which waves of seismic origin propagate are burdened by a strong fluctuation component and can be considered as random functions of spatial coordinates. In space the random component of response is of the character of a heavily non-homogeneous process dependent on the processes of perturbations of environmental parameters. The problem of wave propagation in stochastic environment has been afforded considerable attention. The approach to its solution is based usually on the experience from the deterministic region and the methods are selected accordingly. An extensive survey of these activities with extensive references to further papers has been done by Mester and Benaroya (1993) and Manolis (1993). From the number of further works directly connected with seismicity, mention should be made of e.g. Chu *et al.* (1981), Askar and Cakmak (1988), Harada (1991) etc.; special problems of non-linear character are dealt with e.g. Ostoja-Starzewski (1991), Belyaev and Irschik (1995) etc. Beginning with a certain distance from the excitation point the solution cannot be based on perturbation theory (e.g. Manolis, 1993), as it clashes with the energy equilibrium law (Náprstek, 1994). This is testified to also by the analyses of the records of seismic events at various distances from their epicentres.

Although the study of the one-dimensional problem has yielded numerous quantitative achievements differentiating stochastic environment principally from deterministic environment even in case of small parameter fluctuations, it cannot be used for the investigation of such a serious phenomenon as transverse wave diffusion in a two- and three-dimensional environment, where it is necessary to take into account also the factor of wave front shape, the scope of excitation coherence, etc.

## MATHEMATICAL MODEL

The fluctuation of parameters turns the continuum into a random anisotropic one. The perturbation processes are modelled usually as random functions of the vector of position. In nominal state the continuum is considered as isotropic with constant parameters. In such a case the equations of motion have the form of:

$$(\rho^0 + \rho^\varepsilon(\mathbf{x}))\ddot{u}_i = (\lambda^0\delta_{ij}\delta_{kl} + \mu^0(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}))\frac{\partial^2 u_k}{\partial x_j \partial x_l} + \frac{\partial}{\partial x_j} c^\varepsilon_{ijkl}(\mathbf{x})\frac{\partial u_k}{\partial x_l} \tag{1}$$

$c^\varepsilon_{ijkl}(\mathbf{x})$ - continuous random homogeneous centered Gaussian processes of perturbations of parameters of elasticity or density, as the case may be,

$\lambda^0, \mu^0, \rho^0$ - nominal values of Lamé constants of elasticity or of the density of the material,

$$u_i = u_i^0 + u_i^\varepsilon \tag{2}$$

$u_i^0, u_i^\varepsilon$ -mathematical mean value or the random component of response, as the case may be.

As this study is concerned with the qualitative and not quantitative analysis, following currently used simplifying assumptions concerning the character of perturbations $c^\varepsilon_{ijkl}(\mathbf{x})$, $\rho^\varepsilon(\mathbf{x})$ can be adopted:

a) Mutual correlation of perturbations of individual components $c^\varepsilon_{ijkl}(\mathbf{x})$ in point $\mathbf{x}$ is high and, consequently, the environment can be considered as locally isotropic in every point $\mathbf{x}$; its perturbations can be described by the continuous centered Gaussian processes $\lambda^\varepsilon(\mathbf{x})$, $\mu^\varepsilon(\mathbf{x})$, i. e.

$$\lambda(\mathbf{x}) = \lambda^0 + \lambda^\varepsilon(\mathbf{x}) \; ; \; \mu(\mathbf{x}) = \mu^0 + \mu^\varepsilon(\mathbf{x}) \tag{3}$$

b) The perturbations are small in comparison with nominal parameter values; the derivatives $c^\varepsilon_{ijkl}(\mathbf{x})$ according to spatial coordinates may be neglected; with sufficient accuracy it holds that

$$(\lambda(\mathbf{x}) + 2\mu(\mathbf{x}))^{-1} \approx (\lambda^0 + 2\mu)^{-1}\left(1 - \frac{\lambda^\varepsilon(\mathbf{x}) + 2\mu^\varepsilon(\mathbf{x})}{\lambda^0 + 2\mu}\right) \tag{4}$$

c) Mutual correlation of $\rho^\varepsilon(\mathbf{x})$ and $\lambda^\varepsilon(\mathbf{x})$, $\mu^\varepsilon(\mathbf{x})$ in point $\mathbf{x}$ is high.

The presence of variable coefficients of above mentioned structure produces in (1) the interaction of longitudinal and shear wave motions on the level of the random component of response. The assumptions a) and b), however, make it possible to divide the stress field into independent longitudinal and shear parts, each of which has its nominal and random components.

If the assumption c) has been complied with, it is possible to write the Helmholtz equation for the complex amplitude of the longitudinal monochromatic wave in the form of:

$$\Delta\varphi(\mathbf{x}) + \omega^2 c^{-2}(\mathbf{x})\varphi(\mathbf{x}) = 0 \; ; \; \mathbf{u}^{long}(\mathbf{x}, t) = e^{i\omega t}\mathrm{grad}\varphi(\mathbf{x}) \tag{5}$$

where: $c$ - propagation velocity in point $\mathbf{x}$

With reference to adopted assumptions it is possible to write approximately:

$$c^{-2}(\mathbf{x}) \approx c_0^{-2}(1 + \varepsilon(\mathbf{x}))$$

$$c_0^{-2} = \frac{\rho^0}{\lambda^0 + 2\mu^0} \; ; \; \varepsilon(\mathbf{x}) = \frac{\rho^\varepsilon(\mathbf{x})}{\rho^0} - \frac{\lambda^\varepsilon(\mathbf{x}) + 2\mu^\varepsilon(\mathbf{x})}{\lambda^0 + 2\mu^0} \tag{6}$$

which means that

$$\Delta\varphi(\mathbf{x}) + q^2(1 + \varepsilon(\mathbf{x}))\varphi(\mathbf{x}) = 0 \; ; \; q^2 = \omega^2 c_0^{-2} \tag{7}$$

Similar equation as Eq. (7) holds also for the shear part of the wave motion.

It can be assumed that in the direction of its propagation a wave can be described in the form of a product of its amplitude and the harmonic function:

$$\varphi(\mathbf{x}) = \Phi(\mathbf{x}) \cdot e^{i\alpha x} \; ; \; \mathbf{x} = (x, \mathbf{r}) \tag{8}$$

$x$ - direction of wave propagation

$\mathbf{r}$ - coordinates in the wave front propagation, describing the position of the investigated point with reference to the axis $x$.

The function $\Phi(\mathbf{x})$ changes in the direction $x$ far more slowly than $\exp(i\alpha x)$. If the mean length of correlation of imperfections is smaller than the wave length (super critical length, see e.g. Náprstek, 1994), it can be assumed that

$$\left|\frac{\partial^2\Phi(\mathbf{x})}{\partial x^2}\right| \ll q \left|\frac{\partial\Phi(\mathbf{x})}{\partial x}\right| \tag{9}$$

The substitution of (8) into (7) with the use of (9) gives the equation for $\Phi(\mathbf{x})$:

$$2iq\frac{\partial\Phi(\mathbf{x})}{\partial x} + \Delta_r\Phi(\mathbf{x}) + q^2\varepsilon(\mathbf{x})\Phi(\mathbf{x}) = 0 \; ; \; \Delta_r \equiv \partial^2/\partial y^2 + \partial^2/\partial z^2 \tag{10}$$

Eq. (10) is in the variable $x$ of the first order. It follows from this structure that $\Phi(\mathbf{x})$ in the direction $x$ may be only of diffusion character and does not respect certain local phenomena observed in the one-dimensional continuum (Náprstek, 1995; Belyaev and Irschik, 1995).

## MATHEMATICAL MEAN VALUE OF LONGITUDINAL WAVE POTENTIAL

The operator of mathematical mean value in (10) can be differentiated according to realizations:

$$2iq\frac{\partial\Phi_0(\mathbf{x})}{\partial x} + \Delta_r\Phi_0(\mathbf{x}) + q^2\mathbf{E}\left\{\varepsilon(\mathbf{x})\Phi(\mathbf{x})\right\} = 0 \tag{11}$$

$\Phi_0(\mathbf{x}) = \mathbf{E}\left\{\Phi(\mathbf{x})\right\}$ ; $\mathbf{E}\{\cdot\}$ - operator of mathematial mean value,

and the last addend of Eq. (11) can be developed by means of the Furutsu formula:

$$\mathbf{E}\left\{F(\alpha(t))G(\alpha(t) + g(t)\right\} = \mathbf{E}\left\{F\left(\alpha(t) + \int d\tau \cdot K(t,\tau)\frac{\delta}{\delta g(t)}\right)\right\}\mathbf{E}\left\{G(\alpha(t) + g(t))\right\} \tag{12}$$

$\alpha(t)$ - Gaussian centered process
$g(t)$ - deterministic variation
$K(t,\tau)$ - correlation functions of the process $\alpha(t)$
$\delta$ - symbol of variation differentiating operation
$F, G$ - functionals of the same process $\alpha(t)$

The function $\Phi(\mathbf{x})$ may be understood as a functional in the meaning of the composite function of the process $\varepsilon(x)$, i. e. $\Phi(\mathbf{x}) = \Phi(\varepsilon(\mathbf{x}))$. Consequently, the substitution "$F(\alpha(t)) = \varepsilon(\mathbf{x})$" in Eq. (12), thus replacing the differentiation according to $g(t)$ with the differentiation according to $\alpha(t)$, and the substitution $g(t) = 0$ gives, after several rearrangements:

$$\mathbf{E}\left\{\varepsilon(\mathbf{x})\Phi(\varepsilon(\mathbf{x}))\right\} = \int dV \; K(\mathbf{x},\mathbf{x}')\mathbf{E}\left\{\frac{\delta\Phi(\varepsilon(\mathbf{x}))}{\delta\varepsilon(\mathbf{x}')}\right\} \; ; \; dV = dx \cdot dy \cdot dz \tag{13}$$

Adopting the hypothesis of delta correlation of imperfections in longitudinal direction and homogeneity in transverse direction, it follows from Eq. (13) that

$$\mathbf{E}\left\{\varepsilon(\mathbf{x})\Phi(\varepsilon(\mathbf{x}))\right\} = \frac{1}{2}\int d\mathbf{r}K_r(\mathbf{r}'-\mathbf{r})\mathbf{E}\left\{\frac{\delta\Phi(\varepsilon(\mathbf{x}))}{\delta\varepsilon(\mathbf{x})'}\right\} = i\frac{q}{4}K_r(0)\Phi_0(\mathbf{x}) \tag{14}$$

With reference to Eq. (14), the Eq. (11) can be written in the form of

$$2iq\frac{\partial\Phi_0(\mathbf{x})}{\partial x} + \Delta_r\Phi_0(\mathbf{x}) + i\frac{q^3}{4}K_r(0)\cdot\Phi_0(\mathbf{x}) = 0 \tag{15}$$

with the initial condition:

$$\Phi_0(0,\mathbf{r}) = \Phi_{0S}(\mathbf{r}) \tag{16}$$

Now, the Eq. (10) can be re-written in the form of integro-differential equation:

$$\Phi(\mathbf{x}) = \Phi_0(0,\mathbf{r})\cdot\exp(i\frac{q}{2}\int_0^x d\xi\cdot\varepsilon(\xi,\mathbf{r})) + \frac{i}{2q}\int_0^x d\xi\exp(i\frac{q}{2}\int_\xi^x d\theta\varepsilon(\theta,\mathbf{r}))\Delta_r\Phi(\xi,\mathbf{r}) \tag{17}$$

Now the mathematical mean value operator and the hypothesis of delta-correlated imperfections in the direction $x$ can be applied to this equation. After a number of re-arrangements it results that

$$\Phi_0(x,\mathbf{r}) = \Phi_0(0,\mathbf{r})\cdot\Gamma(x,\mathbf{r}) + \frac{i}{2k}\Gamma(x,\mathbf{r})\int_0^x\frac{d\xi}{\Gamma(\xi,\mathbf{r})}\Delta_r\Phi(\xi,\mathbf{r}) \tag{18}$$

$$\Gamma(x,\mathbf{r}) = \mathbf{E}\left\{\exp(i\frac{q}{2}\int_0^x d\theta\cdot\varepsilon(\theta,\mathbf{r}))\right\}$$

which gives the following differential equation for $\Phi_0(x,\mathbf{r})$:

$$\frac{\partial}{\partial x}\Phi_0(x,\mathbf{r}) = \frac{i}{2q}\Delta_r\Phi_0(x,\mathbf{r}) + \Phi_0(x,\mathbf{r})\frac{\partial}{\partial x}\lg\left(\Gamma(x,\mathbf{r})\right) \tag{19}$$

Introducing the characteristic functional of the field $\varepsilon(x,\mathbf{r})$:

$$\Psi(\eta(\xi,\mathbf{r}')) = \mathbf{E}\left\{\exp(i\int_0^x d\xi\int d\mathbf{r}'\varepsilon(\xi,\mathbf{r}')\eta(\xi,\mathbf{r}'))\right\} = \exp(\Theta(\eta(x',\mathbf{r}'))) \tag{20}$$

the Eq. (13) can be re-written in the form of:

$$\frac{\partial}{\partial x}\Phi_0(x,\mathbf{r}) = \frac{i}{2q}\Delta_r\Phi_0(x,\mathbf{r}) + \frac{\partial}{\partial x}\Theta(\frac{q}{2}\delta(\mathbf{r}-\mathbf{r}'))\cdot\Phi_0(x,\mathbf{r}) \tag{21}$$

which fully conforms with Eq. (15).

## FLUCTUATING PART OF RESPONSE

Taking again the Eq. (10) or the integro-differential Eq. (17) as the basis, and using analogous procedure, the equations for the stochastic moments of response can be written as:

$$M_{n,m}(x,\{\mathbf{r}_i\},\{\mathbf{r}_j\}) = \mathbf{E}\left\{\Phi(x,\mathbf{r}_1)\cdots\Phi(x,\mathbf{r}_n)\cdot\overline{\Phi(x,\mathbf{r}_1)}\cdots\overline{\Phi(x,\mathbf{r}_m)}\right\}\ ;\ \{\mathbf{r}_i\}\equiv\mathbf{r}_1,\cdots,\mathbf{r}_n \tag{22}$$

Writting firstly the differential equation for the quantity $\Phi(x,\mathbf{r_1})\cdots\overline{\Phi(x,\mathbf{r_m})}$, converting it into an integral equation of the type of Eq. (17) and modifying the operator of mathematical mean value, an integro-differential equation of the type of Eq. (18) will be obtained, after a number of modifications using the hypothesis of delta correlation of $\varepsilon(x, \mathbf{r})$ in the direction $x$, viz:

$$\left.\begin{aligned} & M_{n,m}(x, \{\mathbf{r}_i\}, \{\mathbf{r}'_j\}) = \\ & = M^0_{n,m}(\{\mathbf{r}_i\}, \{\mathbf{r}_j\}) \cdot \mathbf{E}\left\{\exp\left(i\tfrac{q}{2}\int\limits_0^x d\xi(\textstyle\sum_{i=1}^n \varepsilon(\xi, \mathbf{r}_i) - \sum_{j=1}^m \varepsilon(\xi, \mathbf{r}'_j))\right)\right\} + \\ & + \tfrac{i}{2q}\int\limits_0^x d\xi \cdot \mathbf{E}\left\{\exp(i\tfrac{q}{2}\int\limits_\xi^x d\theta\left(\textstyle\sum_{i=1}^n \varepsilon(\theta, \mathbf{r}_i) - \sum_{j=1}^m \varepsilon(\theta, \mathbf{r}'_j))\right)\right\} \cdot \\ & \cdot \left(\textstyle\sum_{i=1}^n \Delta_{ri} - \sum_{j=1}^n \Delta_{r'j}\right) M_{n,m}(x, \{\mathbf{r}_i\}, \{\mathbf{r}_j\}) \end{aligned}\right\} \tag{23}$$

$$M^0_{n,m} = \mathbf{E}\{\Phi(0, \mathbf{r}_1)\cdots\Phi(0, \mathbf{r}_n)\overline{\Phi(0, \mathbf{r}'_1}\cdots\overline{\Phi(0, \mathbf{r}'_m}\}$$

Hence it is possible to derive the appropriate differential equation:

$$\frac{\partial}{\partial x}M_{n,m} = \frac{i}{2q}\left(\sum_{i=1}^n \Delta_{ri} - \sum_{j=1}^n \Delta_{r'j}\right) M_{n,m} + \frac{\partial}{\partial x}\Theta\left(\frac{q}{2}\sum_{i=1}^n \delta(\mathbf{r}' - \mathbf{r}_i) - \frac{q}{2}\sum_{j=1}^m \delta(\mathbf{r}' - \mathbf{r}'_j)\right) M_{n,m} \tag{24}$$

The perturbations $\varepsilon(x, \mathbf{r})$ form part of the argument of the functional $\Theta$ (compare with Eq. (20)) and so far have not been specified in detail. If they are of Gaussian character and if they are delta-correlated in the direction $x$, then

$$\mathbf{E}\{\varepsilon(x, \mathbf{r}_1)\varepsilon(x', \mathbf{r}'_1)\} = \delta(x_1 - x'_1)K(\mathbf{x}_1, \mathbf{x}'_1) \tag{25}$$

As in a statically homogeneous field $\varepsilon(x, \mathbf{r})$ the correlation will not depend on $x$, the functional $\Theta$ will be:

$$\Theta(\eta(\xi, \mathbf{r})) = -\frac{1}{2}\int\limits_0^x d\xi \int d\mathbf{r}_1 d\mathbf{r}'_1 K(\mathbf{r}_1 - \mathbf{r}'_1)\eta(\xi, \mathbf{r}_1)\eta(\xi, \mathbf{r}_2)$$

so that Eq. (24) will obtain the form of:

$$\left.\begin{aligned} & \tfrac{\partial}{\partial x}M_{n,m} = \tfrac{i}{2q}\left(\textstyle\sum_{i=1}^n \Delta_{ri} - \sum_{j=1}^m \Delta_{r'j}\right) M_{n,m} - \\ & \tfrac{k^2}{8}\left(\textstyle\sum_{i,j=1}^{n,m} K(\mathbf{r}_i - \mathbf{r}_j) - 2\sum_{i,j=1}^{n,m} K_r(\mathbf{r}_i - \mathbf{r}'_j) + \sum_{i,j=1}^{n,m} K_r(\mathbf{r}'_i - \mathbf{r}'_j)\right) M_{n,m} \end{aligned}\right\} \tag{26}$$

Developing Eq. (26) for $n = 1$, $m = 0$, the Eq. (15) or (21) for the mathematical mean value of the response potential $\Phi_0(x, \mathbf{r})$ will be obtained. For $n = m = 1$ and $n = m = 2$ it will be:

a) $n = m = 1$ ; $M_{1,1} = \Phi_2(x, \mathbf{r}_1, \mathbf{r}'_1) = \Phi_2$

$$\frac{\partial \Phi_2}{\partial x} = \frac{i}{2q}(\Delta_{r1} - \Delta_{r'1})\Phi_2 - \frac{q^2}{4}(K_r(0) - K_r(\mathbf{r}))\Phi_2 \tag{27}$$

The Eq. (27) can be derived also directly from Eq. (10), multiplying the Eq. (10) by $\overline{\Phi_2(\mathbf{x}'_1)}$:

$$2iq\frac{\partial \Phi_2(\mathbf{x}_1)}{\partial x} \cdot \overline{\Phi_2(\mathbf{x}'_1)} + \Delta_{r1}\Phi_2(\mathbf{x}_1)\overline{\Phi(_2\mathbf{x}'_1)} + q^2\varepsilon(\mathbf{x}_1)\Phi_2(\mathbf{x}_1)\overline{\Phi_2(\mathbf{x}'_1)} = 0 \tag{28}$$

and the equation conjugate with Eq. (27) by $\Phi_2(\mathbf{x}_1)$:

$$-2iq\frac{\partial \overline{\Phi_2(\mathbf{x}'_1)}}{\partial x} \cdot \Phi_2(\mathbf{x}'_1) + \Delta_{r'1}\overline{\Phi_2(\mathbf{x}'_1)}\Phi_2(\mathbf{x}_1) + q^2\varepsilon(\mathbf{x}'_1)\overline{\Phi_2(\mathbf{x}'_1)}\Phi_2(\mathbf{x}_1) = 0 \tag{29}$$

Subtracting Eq. (29) and Eq. (28) gives, after the application of the mathematical mean value operator:

$$2iq\frac{\partial \Phi_2}{\partial x} + (\Delta_{r1} - \Delta_{r'1})\Phi_2 + q^2\mathbf{E}\left\{(\varepsilon(\mathbf{x}_1) - \varepsilon(\mathbf{x}'_1))\Phi_2(\mathbf{x}_1)\overline{\Phi_2(\mathbf{x}'_1)}\right\} = 0 \tag{30}$$

The last term in Eq. (30) can be developed once again using the Furutsu formula (12), into

$$\mathbf{E}\left\{(\varepsilon(\mathbf{x}_1)-\varepsilon(\mathbf{x'}_1))\Phi_2(\mathbf{x}_1)\overline{\Phi_2(\mathbf{x'}_1)}\right\} = -\frac{iq}{4}(K_r(0)-K_r(\mathbf{r}))\Phi_2 \tag{31}$$

After substitution of (31) in (30) the Eq. (27) will be once again obtained..

At the same time Eq. (27) describes the mean value of wave intensity and can be solved for any continuous function $K_r(\mathbf{r})$, if it complies with the conditions imposed on correlation function.

b) $n=m=2$ ; $M_{2,2}=\Phi_4(x,\mathbf{r}_1,\mathbf{r}_2,\mathbf{r'}_1,\mathbf{r'}_2)=\Phi_4$

$$\left.\begin{array}{r}\frac{\partial\Phi_4}{\partial x}=\frac{i}{2q}(\Delta_{r1}+\Delta_{r2}-\Delta_{r'1}-\Delta_{r'2})\Phi_4-\frac{q^2}{4}(2K(0)+\\ +K(\mathbf{r}_1-\mathbf{r'}_1)+K(\mathbf{r}_2-\mathbf{r'}_2)+K(\mathbf{r}_1-\mathbf{r'}_2)+K(\mathbf{r}_2-\mathbf{r'}_1)-K(\mathbf{r}_2-\mathbf{r}_1)-K(\mathbf{r'}_2-\mathbf{r'}_1))\Phi_4\end{array}\right\} \tag{32}$$

## SPECIAL CASES

Eq. (15) with the initial condition (16) can be solved without any specific problems. Its solution has the form of:

$$\Phi_0(\mathbf{x})=H(x,\mathbf{r})\exp(-\gamma x)\ ;\ \gamma=\frac{1}{8}q^2K_r(0)=2\pi^2q^2\cdot\int_0^\infty S(\kappa)d\kappa \tag{33}$$

$S(\kappa)$ - spectral density of imperfections ascertained experimentally; it is possible to consider e. g. the diffusion process, see e. g. Nigam and Narayanan (1994), Lin and Cai (1995) and many others:

$$S(\kappa)=\frac{a}{\kappa^2+a^2}\cdot\frac{\sigma_0^2}{\pi};$$

$\sigma_0^2$ - dispersal of the process $\varepsilon(x,\mathbf{r})$ in the direction $\mathbf{r}$.

Putting (33) in (15), the equation for $H(x,\mathbf{r})$ will be:

$$2iq\frac{\partial H(x,\mathbf{r})}{\partial x}+\Delta_r H(x,\mathbf{r})=0 \tag{34}$$

for the initial condition (16).

Eq. (34) is parabolic and describes wave propagation in perfect environment ($\varepsilon(x,\mathbf{r})\equiv 0$). Such problem may be solved by a number of analytical or numerical methods used in deterministic problems. Consequently, Eq. (33) can be considered as the solution of Eq. (15). It shows the drop of the mathematical mean value of response with the distance from the surface $x=0$ in which this mathematical mean value has been specified by the initial conditions (16). By way of example the solution of the mathematical mean value of waves of two types can be shown:

a) Plane coherent wave

$$\Phi_{0s}(\mathbf{r})=C\Rightarrow H(x,\mathbf{r})=C\ ;\ \Phi_0(\mathbf{x})=C\cdot e^{-\gamma x} \tag{35}$$

b) Cylindrical wave

$$\Phi_{0s}(\mathbf{r})=C\Rightarrow H(x,\mathbf{r})=\frac{C}{x}e^{i\frac{q|\mathbf{r}|^2}{2x}}\ ;\ \Delta_r\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\ ;\ x>0\ ;\ \Phi_0(\mathbf{x})=\frac{C}{x}\exp(i\frac{qr^2}{2x}-\gamma x) \tag{36}$$

A comparison of (35) and (36) reveals that the mathematical mean value of the cylindrical wave disappears faster than that of the plane wave. However, this is not due to the stochastic character of the environment, but to usual reasons known from the deterministic region.

The simple exponential drop of the mathematical mean value according to (35) is determined by the delta-correlated imperfections in the direction $x$; if the process is e. g. of diffusion character, this quantity starts dropping exponentially only at a certain distance in the direction $x$ and has zero derivative in point $x = 0$, as it has been successfully proved for a semi-infinite bar (Náprstek, 1994; 1995) which represents a case approaching most Eq. (35).

The Eq. (27) can be solved for the initial condition:

$$\Phi_2(0, \mathbf{r}_1, \mathbf{r}'_1) = \Phi_{2s}(\mathbf{r}_1, \mathbf{r}'_1) \tag{37}$$

Introducing the transformation $\mathbf{r}_s = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}'_1)$ ; $\mathbf{r}_d = \mathbf{r}_1 - \mathbf{r}'_1$ in the Eq. (27), it will be changed into:

$$\left(2iq\frac{\partial}{\partial x} + 2\left(\frac{\partial}{\partial \mathbf{r}_s} \cdot \frac{\partial}{\partial \mathbf{r}_d}\right) + \frac{iq^3}{2}(K(0) - K(\mathbf{r}_d))\right)\Phi_2(x, \mathbf{r}_s, \mathbf{r}_d) = 0 \tag{38}$$

$$\mathbf{r}_s = y_s \cdot \mathbf{e}_y + z_s \mathbf{e}_z \; ; \; \mathbf{r}_d = y_d \cdot \mathbf{e}_y + z_d \mathbf{e}_z$$

Applying now the Fourier transformation according to $\mathbf{r}_s$ to the Eq. (38), which will not interfere with the coefficient of the third term:

$$\left(2iq\frac{\partial}{\partial x} + 2\rho_s \cdot \frac{\partial}{\partial \mathbf{r}_d} + \frac{iq^2}{2}(K(0) - K(\mathbf{r}_d))\right) F_2(x, \rho, \mathbf{r}_d) = 0 \tag{39}$$

$$\Phi_2(x, \mathbf{r}_s, \mathbf{r}_d) = \int F_2(x, \mathbf{p}_s, \mathbf{r}_d)\exp(i\mathbf{p}_s \cdot \mathbf{r}_s)d\mathbf{p}_s \tag{40}$$

it can be seen that the Eq. (39) is a linear equation of the first order and can be solved by the method of characteristics. Taking into account simultaneously Eq. (40), the solution can be written in the form of

$$\Phi_2(x, \mathbf{r}_s, \mathbf{r}_d) = \int d\mathbf{p}_s \cdot F_2(0, \mathbf{p}_s, \mathbf{r}_d - \mathbf{p}_s\frac{x}{q})\exp\left(i\mathbf{p}_s\mathbf{r}_s - \frac{q^2}{4}\int_0^x d\xi\left(K(0) - K(\mathbf{r}_d - \mathbf{p}_d - \mathbf{p}_s\frac{\xi}{q})\right)\right) \tag{41}$$

The Eq. (41) can be illustrated on similar examples as previously, i. e. for a plane wave and a cylindrical wave:

a) Plane coherent wave
$\Phi_2(0, \mathbf{r}_s, \mathbf{r}_d) = K_0$ ; $K_0$ - real positive value

$$F_2(0, \mathbf{p}_s, \mathbf{r}_d) = K_0\delta(\mathbf{p}_s)$$

Eq. (41) obtains the simple form of

$$\Phi_2(x, \mathbf{r}_s, \mathbf{r}_d) = K_0 \cdot \exp\left(-\frac{q^2 x}{4}(K(0) - K(\mathbf{r}_d))\right) \tag{42}$$

b) Cylindrical wave

$$\Phi_2(0, \mathbf{r}_s, \mathbf{r}_d) = K_0\left(\frac{2\pi}{q}\right)^2\delta(\mathbf{r}_s)\delta(\mathbf{r}_d) \; ; \; \Phi_2(x, \mathbf{r}_s, \mathbf{r}_d) = \frac{K_0}{x^2}\exp\left(i\frac{q}{x}\mathbf{r}_s \cdot \mathbf{r}_d - D\right) \tag{43}$$

$$D(x) = \frac{q^2}{4}\int_0^x d\xi\left(K(0) - K(\mathbf{r}_d - \mathbf{p}_s\frac{\xi}{q})\right) \tag{44}$$

Eqs. (42), (43) show the decrement of correlation with the distance from the observer's point. In special cases this decrement is described by simple relations which determine the character of fluctuations of the parameters of the environment itself.

Eq. (32) or Eq. (26) for $n > 1$, $m > 1$ describe higher statistical moments of response. The general solution of Eq. (32) obviously cannot be obtained by analytical methods. However, the equation can be solved approximately for some special cases.

## CONCLUSION

Qualitative analyses of compression wave propagation in the environment with random perturbations shows a strong tendency to Gaussian response, as long as the perturbations of environmental parameters are also Gaussian. In the proximity of the point of excitation, however, the response is visibly of non-Gaussian character, especially if it concerns a more complex wave than a coherent plane wave. This characteristic, however, disappears relatively fast with distance and the response acquires an approximately Gaussian character.

There emerge practical boundaries of applicability of the purely analytical solution to stochastic moments of the first and second orders. Higher moments, decisive for the character of response, can be assessed only with reference to basic asymptotic characteristics. If real solution is required, the equations for higher order moments must be integrated numerically. It is possible to start with the moment equations described in this paper. The adopted hypotheses mean nothing but an idealization by the application of Markov processes, especially in longitudinal direction. The potential amplitude, which has been selected as the principal descriptive quantity, can be certainly considered as a Markov process.

The study of the first moment (mathematical mean value or the effective part of the wave) and the second moment (dispersal of wave potential or mathematical mean value of energy level as a special case) has revealed that the form of the wave front does not exercise primary influence on the stochastic response component. The disappearance of the mathematical mean value is determined primarily by the local level of energy in local volume.

## REFERENCES


Askar A. and Cakmak A.S. (1988). Seismic waves in random media. *Probabilistic Engineering Mechanics*, **3**, 124-129.

Belyaev A.K. and Irschik H. (1995). Non linear waves in complex structures modelled by elastic-viscoplastic stochastic media. In: *Proc. EUROMECH 329 Methods for Nonlinear Stochastic Structural Dynamics* (G.I. Schuëller edt), Univ. of Innsbruck, Innsbruck, Austria.

Chu L., Askar A. and Cakmak A.S. (1981). Earthquake waves in a random medium. *Int.Jour.Num. Anal.Meth.Geomech.*, **5**, 79-96.

Harada T. (1991). Seismic response of stochastic ground. In: *Proc. W.I.T. Conf. Computational Stochastic Mechanics*(P.D.Spanos and C.A.Brebbia eds), pp.649-660. Comp. Mech.Publ. and Elsevier, Southampton-London.

Lin Y.K. and Cai G.Q. (1995). *Probabilistic Structural Dynamics - Advanced Theory and Applications.* Mc Graw - Hill, New York.

Manolis G.D. (1993). Acoustic and seismic wave propagation in random media. In: *Computational Stochastic Mechanics*(A.H-D.Cheng and C.Y.Yang eds), Chap. 26, pp.597-622. Comp.Mech.Publ. and Elsevier, Southampton-London.

Mester S.S. and Benaroya H. (1993). Wave propagation in,periodic structures with imperfections. In: *Computational Stochastic Mechanics*(A.H-D.Cheng and C.Y.Yang eds), Chap. 19, pp.425-449. Comp. Mech.Publ. and Elsevier, Southampton-London.

Náprstek J. (1994). Influence of randomly variable parameters on the stochastic character of longitudinal wave propagation (in Czech). *Eng. Mechanics (Inž. mechanika)*, **2**, pp. 71-88.

Náprstek J. (1995). Dispersion of longitudinal waves propagating in a continuum with randomly perturbated parameters. In: *Proc. 3rd Int. Conf. on Recent Advances in Geotechnical Earthquake Engineering and Soil Dynamics* (S. Prakash edt.). pp. 705-708, Univ. of Missouri Rolla, St. Louis.

Nigam N.C. and Narayanan S. (1994). *Application of Random Vibrations.* Addison-Wesley, Singapore.

Ostoja-Starzewski M. (1991). On wavefront propagation in random nonlinear media. In: *Proc. W.I.T. Conf. Computational Stochastic Mechanics*(P.D.Spanos and C.A.Brebbia eds), pp.687-698. Comp. Mech.Publ. and Elsevier, Southampton-London.