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OPTIMAL INSTRUMENTATION OF STRUCTURAL SYSTEMS

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ABSTRACT

A criterion is presented to make decisions concerning the optimal instrumentation of structural systems subjected to earthquake ground motion. It is considered that a set of response components, to be recorded for the identification of uncertain structural parameters, needs to be chosen given some constraints on the number of recording instruments. A measure of goodness of an instrumentation scheme is defined in terms of a Bayesian loss function which is related to the so-called Fisher information matrix. The criterion to determine the optimum instrumentation alternative is then to select the one which minimizes the expected value of the loss function. The criterion is applied to the case of structural systems subjected to seismic ground motions modeled as stochastic processes.

KEYWORDS

Instrumentation; Bayesian loss function; Fisher information; stochastic ground motion; system identification

INTRODUCTION

Although several methods have been developed for identification of structural properties based on available response records, only a few researchers have studied the problem of optimizing the location of the recordings instruments. The first studies on this topic are probably those of Shah and Udwadia (1978). More recently, Udwadia (1984) has proposed a methodology for the optimum sensor location problem which is based on the use of information matrices and efficient estimators.

In this paper, a criterion for determining the optimal location of a number of recording instruments is proposed based on a Bayesian approach. The structural properties to be identified with an instrumentation scheme are supposed to be uncertain with known a-priori distributions. The criterion to choose an optimal instrument location is based on minimizing a Bayesian loss function. First, the Bayesian approach is presented and the use of efficient estimators and information matrices is discussed. The approach is then applied to the case of MDOF systems with uncertain parameters subject to random ground motions. Expressions are derived for the computation of the Bayesian loss function and two examples of MDOF structural systems are then given to illustrate the use of the criterion. Results from the examples are discussed and some concluding remarks are given at the end.

A BAYESIAN APPROACH FOR OPTIMAL INSTRUMENTATION

Consider the case where a vector of uncertain parameters Θ is to be estimated based on the observation of a set of random vectors $Y_o = \{Y_1, Y_2, ..., Y_N\}$. Suppose that given certain constraints on the number of vectors that can be observed a subset Y of M vectors out of the N vectors in Y_o has to be chosen for the estimation of Θ. We would like to establish a criteria for the selection of the M vectors that should be observed so that the "best" estimate of Θ is obtained.

Let $f(Y/\Theta)$ denote the joint probability density function of the M random vectors in the subset Y given Θ. An unbiased estimator $\theta(Y)$ of Θ is said to be efficient if its covariance matrix is given by,

$$Cov[\theta(Y)] = M_\theta^{-1} \tag{1}$$

where M_Θ is the Fisher information matrix defined as

$$M_\theta = E_{Y/\theta}\left[\frac{\partial}{\partial\theta}\ln f(Y/\theta)\right]\left[\frac{\partial}{\partial\theta}\ln f(Y/\theta)\right]^T \tag{2}$$

The inverse of the Fisher information matrix (FIM) is the so called Cramer-Rao lower bound, and represents the minimum covariance that an unbiased estimator can achieved (Goodwin and Payne, 1977). Equation (1) suggests that the greater the "size" of the FIM, the smaller the "size" of the estimator covariance matrix. The size of the FIM becomes larger as the random vector Y becomes more sensitive to changes in the values of the vector parameter Θ.

Suppose now that we can assign to Θ a prior distribution which represents the relative likelihood of the possible values of Θ. The prior distribution summarizes the information and knowledge that one has on the likelihood of the possible values of Θ before Y is observed. Each of the possible subsets Y will be considered here as a possible "instrumentation" for the estimation of Θ. In order to compare between the different instrumentation alternatives, a measure of the goodness of each alternative -related to the expected accuracy of the parameter estimates to be obtained from the observed Y- is required. A suitable measure of goodness can be defined in terms of a Bayesian loss function $L(\Theta, \theta(Y))$ for which the expectation $E[L(\Theta, \theta(Y))]$ is to be minimized. A commonly used loss function is the so called squared error loss function,

$$L(\theta, \theta(Y)) = (\theta(Y) - \theta)^T(\theta(Y) - \theta) \tag{3}$$

By definition of conditional expectation, we have that

$$E[L(\theta, \theta(Y))] = E_\theta E_{Y/\theta}[L(\theta, \theta(Y))] \tag{4}$$

Substituting (3) into (4),

$$\begin{aligned} E[L(\theta, \theta(Y))] &= E_\theta E_{Y/\theta}[(\theta(Y) - \theta)^T(\theta(Y) - \theta)] \\ &= E_\theta tr E_{Y/\theta}[(\theta(Y) - \theta)(\theta(Y) - \theta)^T] \end{aligned} \tag{5}$$

where *tr* denotes the trace of a matrix. Since $E_{Y/\Theta}[\theta(Y)] = \Theta$ and assuming an efficient estimator is used, it follows from the definition in (1) that,

$$E_{Y/\theta}[(\theta(Y) - \theta)(\theta(Y) - \theta)^T] = M_\theta^{-1} \tag{6}$$

Substituting (6) into (5) we obtain

$$E[L(\theta, \theta(Y))] = E_\theta[tr M_\theta^{-1}] \tag{7}$$

The criterion to select the optimal instrumentation alternative is then to choose for observation the subset Y for which the expected loss in (7) is a minimum. Notice that the expected value in the right hand side of (7) is taken with respect to Θ, *i.e.* with respect to the prior distribution of Θ. Therefore, the ability to select the optimal alternative depends on the knowledge that one has a-priori about the properties of the system.

It may certainly be of interest not only to select the optimal subset Y but also to assess the goodness of the selected Y for the estimation of Θ. A measure of such goodness is related to the amount of uncertainty in the prior knowledge of Θ that can be reduced by observing Y. Let Cov [Θ/Y=y] denote the posterior covariance of Θ when the values Y=y are observed. Prior to the observation of Y, Cov[Θ/Y] becomes a random matrix. Then the overall posterior covariance matrix, averaged over all possible values of Y is E_Y [Cov [Θ/Y]]. It is important to point out that E_Y [Cov [Θ/Y]] is the posterior covariance matrix associated with the complete process of observing Y and then estimating Θ, before any particular value Y=y has been observed. On the other hand, if the vector of parameters Θ was to be predicted without observing Y, the best estimator would be the prior mean, E[Θ], with prior covariance matrix Cov[Θ]. However, if Y can be observed for estimating Θ, the reduction of uncertainty in Θ, as measured by its covariance matrix, is then,

$$Cov[\theta]-E_Y[Cov[\theta/Y]] \tag{8}$$

Such reduction provides a measure of the usefulness of Y for estimating Θ. Suppose we choose the trace of a covariance matrix as a norm of its size; then, the reduction of uncertainty in Θ, say ε(Θ), could be measured by,

$$\epsilon(\theta)=trCov[\theta]-trE_Y[Cov[\theta/Y]] \tag{9}$$

By definition of conditional expectation,

$$\begin{aligned} trE_Y[Cov[\theta/Y]] &= trE_Y E_{\theta/Y}[(\theta-\boldsymbol{\theta}(Y))(\theta-\boldsymbol{\theta}(Y))^T] \\ &= trE[(\theta-\boldsymbol{\theta}(Y))(\theta-\boldsymbol{\theta}(Y))^T] \\ &= E[(\theta-\boldsymbol{\theta}(Y))^T(\theta-\boldsymbol{\theta}(Y))] \end{aligned} \tag{10}$$

Then from (7) and (10), it follows that,

$$trE_Y[Cov[\theta/Y]]=E_\theta[trM_\theta^{-1}] \tag{11}$$

Substituting (11) into (9), ε(Θ) can be written as,

$$\epsilon(\theta)=trCov[\theta]-E_\theta[trM_\theta^{-1}] \tag{12}$$

The expression in (12) allows the FIM to be used for evaluating the usefulness of Y in estimating Θ. Thus, once the optimal M vectors have been selected from the available N in Y_o, one can assess how much uncertainty in Θ is reduced from the observation of Y. If prior to the instrumentation a criteria is set on the amount of uncertainty one expects to reduce from the observation of Y, it is possible then to evaluate the appropriateness of observing just the M vectors in Y for the estimation of Θ. If the number M of vectors was not appropriate, other optimal instrumentations can be chosen for different number of vectors in Y. From these, the minimum number of vectors in the subset Y to achieve the expected amount of uncertainty reduction can be determined.

APPLICATIONS TO MDOF STRUCTURAL SYSTEMS

Let $Y_1, Y_2, \ldots, Y_K$ be M dimensional, independent, Gaussian, zero mean, random vectors. Let $f(Y_1, Y_2, \ldots, Y_K / \Theta)$ denote the conditional joint probability density function given the set of parameters Θ and let Θ_k be an uncertain parameter in Θ; then

$$\frac{\partial \ln f(Y/\theta)}{\partial\theta_k} = -\frac{1}{2}\sum_{i=1}^{K} \frac{\partial \ln\Delta_i}{\partial\theta_k} + Y_i^T \frac{\partial C_i^{-1}}{\partial\theta_k} Y_i \tag{13}$$

where $C_i=E[Y_i\, Y_i^T]$ is a (MxM) covariance matrix and Δ_i denotes its determinant. If M_{kl} denotes an element in the *k-th* row and *l-th* column of the Fisher information matrix, then by definition

$$\begin{aligned} M_{kl} &= E_{Y/\theta}\left[\frac{\partial \ln f}{\partial\theta_k}\frac{\partial \ln f}{\partial\theta_l}\right] \\ &= E_{Y/\theta}\frac{1}{4}\sum_{i=1}^{K}\sum_{j=1}^{K}\left[\frac{\partial \ln\Delta_i}{\partial\theta_k} + Y_i^T\frac{\partial C_i^{-1}}{\partial\theta_k}Y_i\right]\left[\frac{\partial \ln\Delta_j}{\partial\theta_l} + Y_j^T\frac{\partial C_j^{-1}}{\partial\theta_l}Y_j\right] \end{aligned} \tag{14}$$

It can be shown that,

$$E_{Y/\theta}\left[Y_i\frac{\partial C_i^{-1}}{\partial\theta_k}Y_i\right] = -\frac{\partial \ln\Delta_i}{\partial\theta_k} \tag{15}$$

Thus, given that for i not equal to j, Y_i and Y_j are independent, M_{kl} can be written as follows,

$$M_{kl} = E_{Y/\theta}\frac{1}{4}\sum_{i=1}^{K}\left[Y_i^T\frac{\partial C_i^{-1}}{\partial\theta_k}Y_i + \frac{\partial \ln\Delta_i}{\partial\theta_k}\right]\left[Y_i^T\frac{\partial C_i^{-1}}{\partial\theta_l}Y_i + \frac{\partial \ln\Delta_i}{\partial\theta_l}\right] \tag{16}$$

Provided that for Gaussian random variables fourth order moments can be obtained from the second order ones, it can be shown that

$$E_{Y/\theta}\left[Y_i^T\frac{\partial C_i^{-1}}{\partial\theta_k}Y_iY_i^T\frac{\partial C_i^{-1}}{\partial\theta_l}Y_i\right] = 2C_i^{-1}\frac{\partial C_i}{\partial\theta_k}\bullet\frac{\partial C_i}{\partial\theta_l}C_i^{-1} + \frac{\partial \ln\Delta_i}{\partial\theta_k}\frac{\partial \ln\Delta_i}{\partial\theta_l} \tag{17}$$

where the dot denotes an scalar matrix product. Substituting (17) into (16), one obtains for M_{kl}

$$M_{kl} = \frac{1}{2}\sum_{i=1}^{K} C_i^{-1}\frac{\partial C_i}{\partial\theta_k}\bullet\frac{\partial C_i}{\partial\theta_l}C_i^{-1} \tag{18}$$

Consider now a linear structural system with Q degrees of freedom subjected to an earthquake ground motion modeled as a zero-mean stationary Gaussian stochastic process. Suppose there are $M < Q$ recording instruments available to record the response of the system. Let $X_i(t_n)$, i=1,2,...,M, and n=1,2,...,K, be the random lateral displacement of the system at a recording point, where $t_n=(n-1)\,\Delta t$, $1/\Delta t$ is the sampling frequency of the record and $(K-1)\Delta t$ is the duration of the record. The response displacement $X_i(t_n)$ can be expressed as follows,

$$X_i(t_n) = \sum_{k=1}^{K} A_{ik}\cos(\omega_k t_n) + B_{ik}\,sen(\omega_k t_n) \tag{19}$$

where the Fourier coefficients A_{ik}, B_{ik} are independent Gaussian variables corresponding to frequency $\omega_k = 2\pi\,(k-1)/K\,\Delta t$. As has been shown elsewhere (Vanmarcke et al., 1993),

$$E[A_{ik}A_{jk}] = \begin{cases} \frac{\Delta\omega}{2}[S_{ij}(\omega_k) + S_{ij}(\omega_{K-k+2})], & k=2,3,\ldots,K/2 \\ 2\Delta\omega S(\omega_k), & k=1+K/2 \\ \Delta\omega S(\omega_k), & k=1 \end{cases} \tag{20}$$

and $E[B_{ik}\,B_{jk}]=E[A_{ik}A_{jk}]$ for k=2,3,...,K/2, $E[B_{ik}\,B_{jk}]=0$ for k=1 and k=1+N/2. The cross spectral density functions between displacement responses X_i and X_j, $S_{ij}(\omega)$, in (20) are given by,

$$S_{ij}(\omega_k) = \sum_{q=1}^{Q} \alpha_{ij}^{q} |H_q(\omega_k)|^2 S(\omega_k) \tag{21}$$

where the parameters α_{ij}^{q} are effective modal participation factors related to the responses X_i and X_j, $H_q(\omega)$ is the *q-th* modal transfer function, and $S(\omega)$ is the ground acceleration spectral density function (Heredia-Zavoni, 1993).

Let F_k denote the vector of Fourier coefficients for frequency ω_k at the recording points, $F_k=\{A_{ik},B_{ik};i=1,2,3,...M\}$. The set of Fourier coefficients $F_1,F_2, ...F_K$ are independent zero-mean Gaussian vectors. Thus, equation (18) provides an expression for computing the elements of the corresponding Fisher information matrix, where the covariance matrices for each of the vectors F_n, n=1,2,...,K, can be assembled using the expressions in (20) for the covariances between Fourier coefficients A_{in},B_{jn}. The procedure then consists of evaluating for different combinations of response components to be recorded, say X_i, i=1,2,...,M, -i.e. for different instrumentation schemes-, the expected trace of the Fisher information matrix inverse as given in (7). The expected value is taken with respect to the uncertain structural parameters to be identified. One then chooses that set of response components which minimizes the expected loss.

EXAMPLES

Consider a shear building with three degrees of freedom having masses concentrated at the floor levels of m_1=2 k s²/in, m_2=1.5 k s²/in and m_3=1 k s²/in, where the subscripts indicate the corresponding floor level. For the purpose of illustration consider here the case where the floor lateral stiffnesses are uncertain and perfectly correlated with each other. It should be noticed though, that the methodology presented above is not restricted to the case of perfectly correlated uncertain structural parameters. The first floor lateral stiffness, K_1, will be taken as the uncertain structural parameter to be identified with an instrumentation of the building. The lateral stiffnesses of the other two floors can be expressed as $K_2=c_2\,K_1$, $K_3 = c_3\,K_1$, where the constants c_2 and c_3 are given values of 2/3 and 1/3, respectively. The a-priori probability density function of the first floor stiffness K_1 will be taken as Gaussian with a mean value μ of 900 k/in. The system is subjected to a ground acceleration at the base modeled as a Gaussian White Noise process.

Table 1. Expected Loss Function for various Coefficients of Variation

C.V.	1st. Floor	2nd. Floor	3rd. Floor
0.05	65.39	76.96	117.46
0.10	65.52	77.10	117.66
0.15	65.80	77.44	118.19

First, consider the case where only one recording instrument is available and one wants to choose its optimum location. Table 1 lists values of the expected loss function for coefficients of variation in the a-priori density function of K_1 equal to 0.05, 0.10 and 0.15. Each column shows results obtained when the recording instrument is located at the first, second or third floor levels. Regardless of the a-priori uncertainty in K_1, as measured by its coefficient of variation, it is found that the best location for the instrument is always at the first floor level. Table 2 lists results for the case where two instruments are available. In such case, it is always better to place the instruments at the first and second floor levels. The effect of the recordings duration is shown in Table 3. Durations of approximately 5 and 10 secs are considered. In both cases, as obtained before, the best location for a single recording instrument is the first floor level. Notice that the longer the duration of the recording, the smaller the expected value of the loss function. Thus, a greater uncertainty reduction is achieved with longer records. The results presented so far do not account for the effect of noise in the records.

Table 2. Expected Loss Function given Two Recording Instruments

C.V.	1st-2nd Floors	1st-3rd Floors	2nd-3rd Floors
0.05	133.58	160.54	171.33
0.10	133.97	161.01	171.81
0.15	134.52	161.67	172.54

Table 3. Expected Loss Function for different Record Durations

Duration (sec)	1st. Floor	2nd. Floor	3rd Floor
10.24	185.06	217.69	331.37
5.12	371.94	438.16	670.02

Consider now a second example of a three degree of freedom shear building with floor masses m1=m2=m3=1 k s^2/in. The lateral stiffnesses are taken to be uncertain and perfectly correlated. Let these stiffnesses be expressed as $K_1=c_1K$, $K_2=c_2K$, $K_3=c_3K$, where c_1, c_2, c_3 are constants and K is a Gaussian variable, $K \sim N(\mu_K=900$ k/in, $\sigma_K=90$ k/in). The system is subjected to a Gaussian ground acceleration with the Kanai-Tajimi spectral density function shown in Fig. 1. The characteristic frequency, damping and White Noise intensity in the Kanai-Tajimi model have been taken from an estimation based on records from the Smart 1 Array (Hao, 1989).

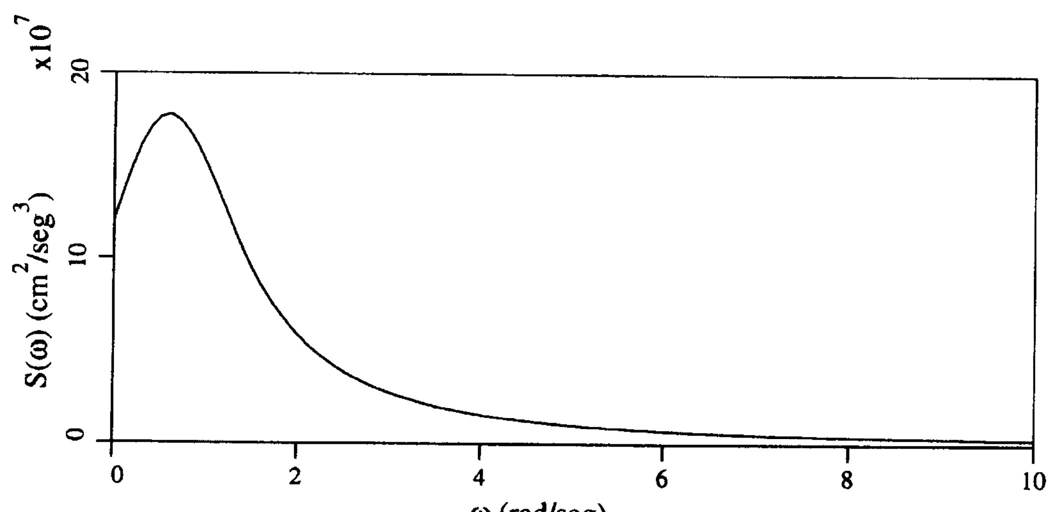

Fig. 1. Kanai-Tajimi Spectral Density Function

Suppose one recording instrument is available and one is interested in determining its best location for the identification of the uncertain stiffness K. As a test of the methodology, Table 4 lists the expected loss function for three combinations of values for the constants c_1, c_2, and c_3, corresponding to different distributions of stiffness along the height of the building. When $c_1=1, c_2=c_3=100$, the responses of all three floors tend to be highly correlated and the results show that it is optimal to place the recording instrument at any of the three floor levels. When the responses of the first and second floors are highly correlated ($c_1=c_2=100, c_3=1$) it is approximately the same to place the recording instrument at the first or second floor levels. In the third case, when the correlation between the second and third floor responses have a high correlation, the expected values of the loss function indicate that the recording instrument should be located at any of these two floor levels.

Table 4. Expected Loss Function for different Stiffness Distributions

c_1	c_2	c_3	1st. Floor	2nd. Floor	3rd. Floor
1	100	100	183.86	183.89	183.86
100	100	1	14.95	15.13	94.82
100	1	100	28.96	149.62	149.32

To analyze the effect of noise in the recordings, this will be modeled as a band limited White Noise process. It will be assumed that noises at different level recordings are statistically independent from each other. Figure 2 shows the variation of the expected loss function versus the amplitude of the noise spectral density function. Each curve in Fig. 2 corresponds to the expected value of the loss function for the recording instrument placed at a given floor level. For low noise intensities, it is best to place the recording instrument at the first floor level. However, as the noise intensity increases, the best location for the recording instrument becomes the third floor level. Notice that it is never the best option to instrument the second floor level. Thus, one should take into account that the results previously discussed above for the purpose of illustration, may not always apply in the presence of noise.

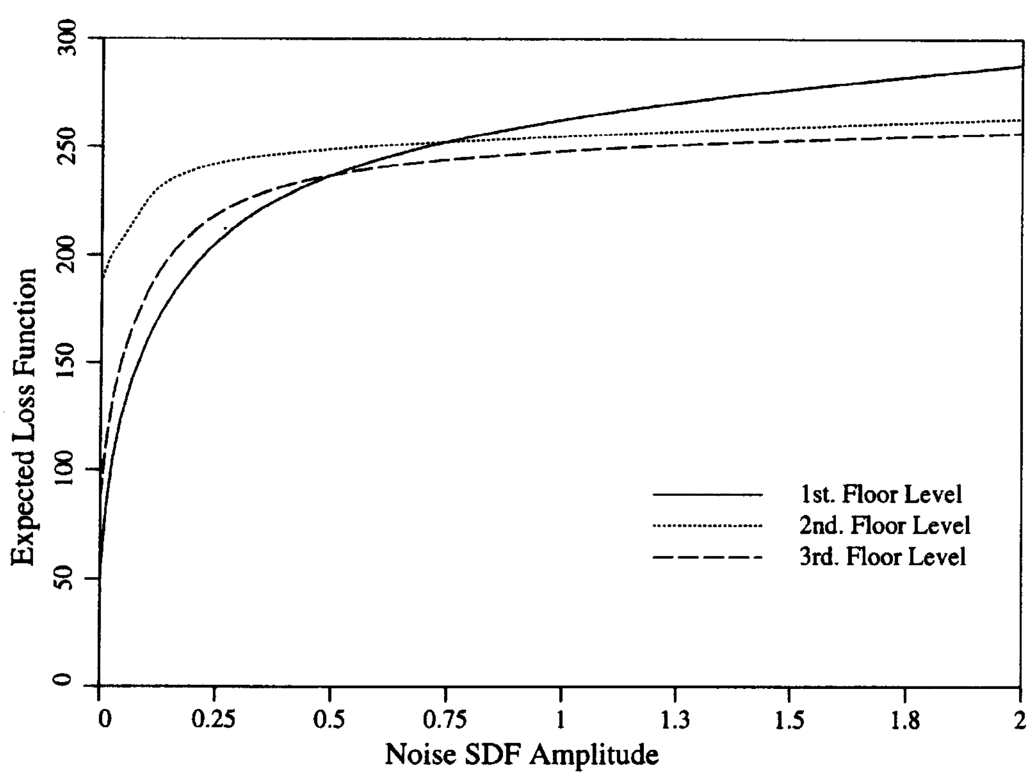

Fig. 2. Expected Loss Function versus Noise Spectral Density Amplitude

CONCLUSIONS

A criterion has been presented to make decisions concerning the optimal location of recording instruments for the identification of uncertain structural parameters given some constraints on the number of available instruments. The approach is based on the use of a Bayesian loss function whose expected value is expressed in terms of the expected trace of the Fisher information matrix inverse. The criterion consists of selecting for recording those response components which minimize the expected loss. The expected loss is evaluated with respect to the prior probability distributions of the uncertain structural parameters to be identified. Thus, the ability to select the optimal solution depends on the knowledge that one has a-priori about the properties of the structural system. It is shown that the loss function can also be used to assess the reduction of uncertainty in the structural parameters to be identified by recording some response components. An expression has been derived to evaluate the Fisher information matrix for a set of independent Gaussian random vectors in terms of their covariance matrices.

The criterion is applied to the case of MDOF structural systems subjected to stochastic earthquake ground motions by expressing the response components to be recorded in terms of their Fourier coefficients. Two examples of MDOF shear buildings with uncertain lateral stiffnesses have been given. The application of the criterion has been illustrated through the analysis of the effect of such factors as the uncertainty in the a-priori knowledge of structural parameters, the duration of the recordings and the distribution of stiffness with heigth, on the optimum location of recording instruments. It has been shown that in the presence of noise in the records, the optimum instrument locations depend on the noise level. Results have showed the usefulness of the criterion to make decisions regarding the optimal instrumentation of structural systems.

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