

DYNAMIC CHARACTERISATION OF FRAMED STRUCTURES BASED ON THE FORMULAS FOR UNIFORM CANTILEVER

Daniel TIRELLI, Javier MOLINA
European Commission, Joint Research Centre, Applied Mechanics Unit
E.L.S.A.* , Ispra, 21020 (Va)-Italy.

ABSTRACT

This paper presents an experimental verification of the equations proposed by Shaw and Trail-Nash (Shaw et al., 1949) to find the oscillation frequencies and modes of uniform slender cantilevers where bending, shear and rotatory inertia effects are present. The solution of these equations neglecting the contribution of the rotatory inertia offers an approximate method to evaluate the main oscillation modes of structures such as multi-storey frames. Assuming that this kind of structures could be considered as a single cantilever where the displacement are only due to bending and shear, this simple method was verified for two steel structures: the first one, experimentally, a three-storey steel frame; the second, analytically, being the same structure but adding three storeys more, modelled with the finite element technique. The results were satisfactory both as regards the first oscillation mode as well as for the global flexural stiffness.

KEYWORDS

Mode; Stiffness; Framed Structure; Steel frame; Cantilever; Bending; Shear; Rotatory inertia; Dynamic Test.

1. INTRODUCTION

The calculation of vibration modes of a multi-storey steel structure, could be a tedious process using finite elements, or insufficiently accurate using the Eurocode 8** or the equivalent american code U.B.C.***. In the initial stages of design, an accurate computer model is often not justified and a more speedy, approximate, hand method is usually adequate; such a method, proposed herein consists of adimensional design curves for each mode of vibration, for example, of the steel structure composed of columns and beams welded together and supporting a given mass on each floor as shown in Fig. 4. The theoretical basis of this approximate method is that the structure, can be modelled by a sum of columns working in parallel in bending and shear like several cantilevers. The horizontal beams act increasing the shear deformation, and decreasing the flexural deformation in the columns.

The exact solution of a deformed cantilever takes into account all the effects which contribute to the curvature, hence: bending, shear, and rotatory inertia. In our case, for a simpler resolution, the rotatory inertia effect was neglected. Therefore the resolution of the equilibrium equations of a beam section submitted to bending and shear gives a transcendental equation which can be solved by iteration. The parameter used in it, is the

* European Laboratory for Structural Assessment.

** $T = \sqrt{d/5}$ where d is the top displacement in cm. of the structure loaded by its weight.

*** $T=(0.1. N)$ where N is the floor number.

ratio between the bending flexibility and the shear flexibility. The curve obtained can be fitted by a simple closed-form formula.

Initially the validity of the solutions of the general equation proposed by F. S. Shaw and R.W. Trail-Nash (Shaw et al., 1949) was verified experimentally. They give the exact frequencies of a beam with uniform section allowing for shear and rotatory inertia effects. The experiments were conducted on a steel cantilever specimen with a section type HEA (often used on multi-storey steel buildings) of different lengths. The first five modes were measured. After this experimental verification, the proposed simplified formula for several storey steel structures, was tested on a three storey steel structure built in the ELSA laboratory and on a six floors steel structure modelled with finite elements.

2. FREQUENCIES OF A CANTILEVER STEEL BEAM: ANALYTICAL EXPRESSION AND EXPERIMENTAL VERIFICATION

2.1 Exact Equation Solution

In reference to the equations given by Shaw and Trail-Nash the equilibrium of a cross section of a steel beam results in the following equation for an harmonic motion:

$$\frac{d^5 u}{dx^5} + \varphi(i^2 + \alpha) \frac{d^3 u}{dx^3} - \varphi^2 (1 - \varphi^2 i^2 \alpha) \frac{du}{dx} = 0 \quad (1)$$

where: $\varphi^2 = \sigma \frac{\omega^2}{EI}$ and $\alpha = E \frac{I}{AG}$

Having defined two kinds of stiffness terms; a flexural one equal to (EI), and a shear term equal to (G.A.l²), then applying the boundary conditions (built in-free) to an uniform cantilever beam, the exact solution is:

$$2 + (2 - \varphi^2 \alpha^2) \cos(ql) \cosh(rl) - \left(\varphi \cdot (i^2 + \alpha) / \left(\sqrt{(1 - \varphi^2 i^2 \alpha)} \right) \right) \sin(ql) \sinh(rl) = 0 \quad (2)$$

$$\text{Where: } q, r = \left(\left(\varphi \cdot \sqrt{(i^2 + \alpha) / 2} \right) \cdot \sqrt{1 + ((1 - \varphi^2 i^2 \alpha) \cdot 4) / (\varphi^2 (i^2 + \alpha)^2)} \right) \pm 1 \quad (3)$$

This transcendental equation may be solved numerically to obtain the different values of omega(j) (j modes). For a given cantilever, material, and geometry, the equations (2) and (3) are conditioned by the factor φ so a first value of omega is required to start the numeric resolution. It will converge to the nearest value of the first mode.

2.2 Experimental Verification on a Steel Cantilever

The experiments have been conducted on a steel beam with an I section of 120 (DIN normative). The beams were welded around all their perimeter to steel plates anchored onto a reaction wall; the boundary conditions were as near as possible a cantilever beam.

The acceleration measurements were performed with a piezoelectric accelerometer (model Endevco- 2213). To get different slenderness ratios five different beam lengths were used. The results are expressed in terms of the correction to apply to the observed frequency versus the pure-bending cantilever frequency value(f) which

is obtained by the common analytical expression : $f = \frac{1}{2\pi} \cdot \sqrt{E \frac{I}{\sigma l^2}}$

Fig. 1. resumes all the results of the measurements, and shows on the same graph the theoretical curves and the experimental ones.

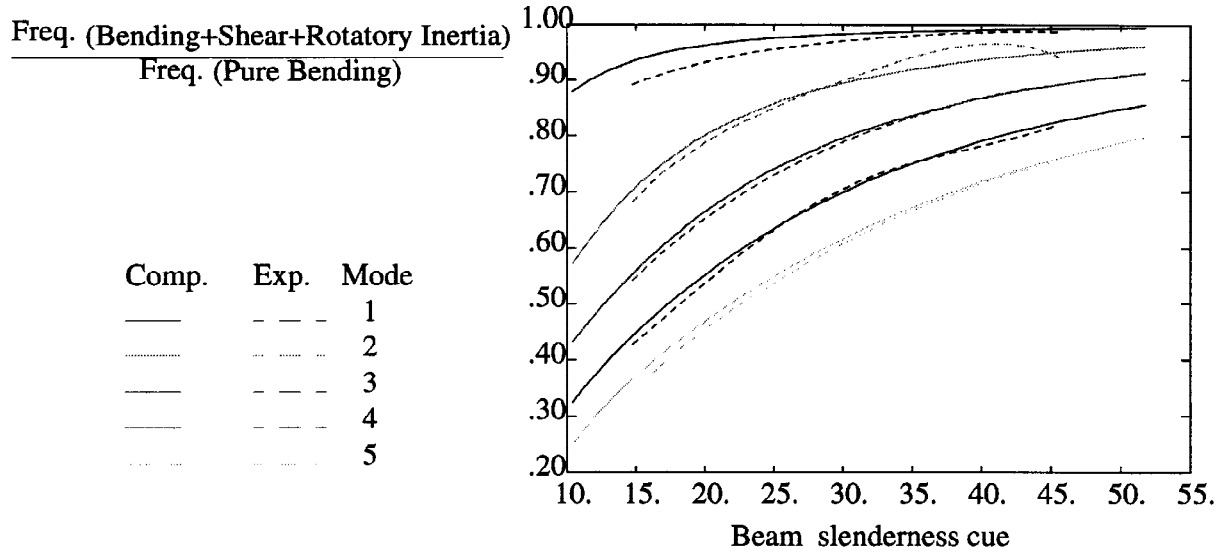


Fig. 1. Experimental and computed correction values

3. EQUATION AND APPLICATION TO THE CASE WITHOUT ROTATORY INERTIA

3.1 Equation Solution

In the previous general equations if we do not take into account of the rotatory inertia effect then equations (2) and (3) become:

$$(2 - \varphi^2 \alpha^2) \cos(ql) \cosh(rl) - \varphi \alpha \sin(ql) \sinh(rl) + 2 = 0 \quad (4)$$

$$r, q = \varphi \cdot \left(\sqrt{\frac{\alpha}{2}} \cdot \sqrt{\sqrt{1 + \frac{4}{\alpha^2 \varphi^2}} \pm 1} \right) \quad (5)$$

We are now in the case where the deflection of the cantilever is only due to bending and shear. It is convenient to introduce the expression of pure shear frequencies of a steel cantilever beam:

$$\omega_s^2 = (2n - 1)^2 \left(\frac{\pi^2}{4} \right) \left(G \frac{A}{\sigma l^2} \right) \quad (6)$$

where: n=number of the mode

If we call x the ratio $\frac{\omega}{\omega_s}$ which represents the ratio of shear in the global stiffness and $\beta = (\alpha/l^2)$

we can give another expression for φ^2 :

$$\varphi^2 = \frac{(K^2 x^2)}{\beta l^4}$$

where $K = (2n - 1)(\pi/2)$

In the equation (5) and (6) we can substitute Alfa and ϕ by Beta, x and K. We obtain:

$$2 + (2 - K^2 x^2 \beta^2) \cos(q1) \cosh(r1) - xK\sqrt{\beta} \sin(q1) \sinh(r1) = 0 \quad (7)$$

$$\text{and} \quad q1, r1 = x \frac{K}{\sqrt{2}} \left(\sqrt{\sqrt{\left(1 + \frac{4}{\beta x^2 K^2}\right)} \pm 1} \right) \quad (8)$$

The numerical solution of this equation, gives the value of the first frequency, choosing a K equal to $p/2$. It is more convenient to express these solutions with non-dimensional parameters which are x and beta; respectively the ratio of the real frequency and the pure shear frequency, and a parameter depending only on the geometry and the nature of the material used for the structure. Then, solving these equations with various values of parameter beta (geometry and nature) we get a curve of x versus Beta. Each point of the curves represents a couple (x, Beta) being a solution of equation (7). A numerical fit was found, close in value to the exact curve (see Fig.2. for $0 < \text{Beta} < 0.12$):

$$\beta = 0.148448 \left(\frac{1}{\cos \frac{\pi x}{2}} - 1 \right) \quad (9)$$

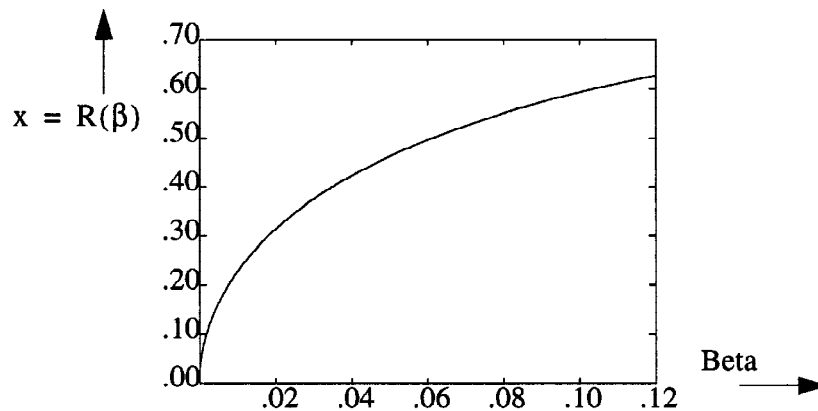


Fig. 2. Proportion of shear in the exp. frequency of a structure with its flexibility ratio.

3.2 Application To Steel Structure of "N" Storeys

3.2.1 The Model: A steel building could be considered as a number of vertical columns welded to horizontal beams at each floor, and supporting masses on each storey. For a horizontal load (wind or one direction of a earthquake, for example) we can describe the building, for reasons of symmetry, as a vertical frame where the masses and the stiffness are a ratio of the total ones depending only on the number of columns of the building in the perpendicular plane of the motion; the problem can now be solved in that direction.

The proposed model described herein, assume that the previous frame is equivalent to a cantilever beam which has a equivalent modulus of inertia, I, for which the total deflection under load is function of the ratio between the flexibility in bending and the flexibility in shear. We can then apply the solutions of equations (7) and (8) to the structure in the Fig. 3.

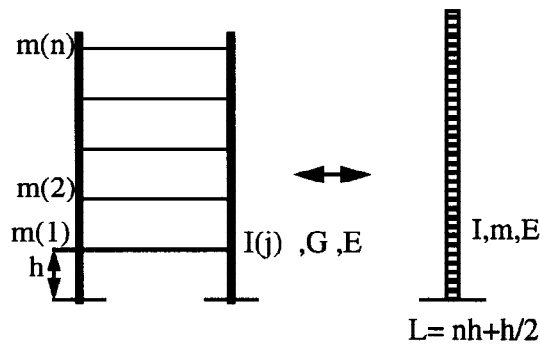


Fig. 3. Frame of n storeys modelled as an equivalent cantilever .

On the frame, the masses are concentrated on each floor and considered as discrete masses, the length of the columns attached to each floor is equal to $h/2$ above and below it, therefore the equivalent cantilever length (L) will be equal to $nh+h/2$. Two ways are possible to use the solutions found previously:

-1) The building already exists: a monitoring of the frequencies (due to wind or a snap-back test) allows to estimate, with the following formulation, the value of the global bending stiffness (EI) of the structure.

-2) The building has to be designed. The formulation below could give a good approximation of the main frequencies of the structure.

In both cases the following parameters are known: $E, m, L, h, n, I(i)$ respectively, Young's modulus, mass per unit length of the cantilever, length of the cantilever, length between two floors, number of floors and inertia modulus of the vertical columns.

3.2.2 Estimation of the Global Bending Stiffness (EI). To evaluate the value of the bending stiffness (EI) we need the experimental frequency of the first mode f_e . Monitoring an acceleration signal recorded on the structure, we can measure at least the first frequency value (f_e). In reference to paragraph 3.1 we can write the ratio $x=(f_e/f_s)$ as a function of the parameters previously defined:

$$x = \frac{(4\sqrt{mLh}f_e)}{\sqrt{12 \cdot E \cdot \sum I(j)}} \quad (10)$$

Then we obtain:
$$EI = (12 \sum I_j \cdot E \cdot L^2 \cdot R^{-1}(x))/h^2 \quad (11)$$

3.2.3 Estimation of the first frequency value of a building (f_e). In the case where the building has to be designed, the fundamental data we need is the top displacement (d) of the building under a known horizontal load (F): this is the normal outcome of the design for the static loads. To find the fundamental frequency f_e , a first stage is needed to give an expression to beta assuming that the cantilever deformation could be divided into flexural and shear deformations as follows:

$$d = d_f + d_s$$

with : $d_s = nh^3(F/(12 \sum I_j))$

and: $d_f = F \left(\frac{(nh)^3 \cdot h^2}{36 \cdot \beta \cdot L^2 \cdot E \cdot \sum I_j} \right)$

Then substituting the flexural displacement by the total displacement and the shear displacement we obtain

$$\beta = (n^3 \cdot F \cdot h^5) / ((36 \cdot L^2 \cdot d \cdot E \cdot \sum I_j) - (3 \cdot L^2 \cdot F \cdot n \cdot h^3)) \quad (12)$$

With the value of Beta we obtain the ratio f_e/f_s with the curve $x = R(\beta)$, we can write the formulation of the first frequency value:

$$f_e = \frac{R(\beta)}{4 \cdot h \cdot L} \sqrt{\frac{12 \cdot E \cdot \sum I_j}{m}} \quad (13)$$

3.3 Two Examples of this Application

3.3.1 Three Storey Steel Building : In ELSA a three storey steel building was built to be tested using the pseudo-dynamic method. It consists of four steel columns supporting three floors consisting of horizontal steel beams and concrete slabs. The characteristics of the frame (which is identical to half structure) are as follows: $I_j=57680 \text{ cm}^4$, $nh= 9.275 \text{ m.}$, $m=2159 \text{ Kg/m}$, $L=10.875 \text{ m.}$

Instead of using a routine to evaluate the displacement under an horizontal load, it is possible to push and pull it with pistons fixed to the reaction wall. With a value of the load of 137 KN., the top of the structure had a displacement of 21.7 mm. Applying these data to the equation (12) we find a $\beta=0.06533$ which give us with equation (13) a first mode at $f_e= 4.09 \text{ Hz.}$ The application of the Rayleigh method to that case conducted, with the assumption that the loads are equal to the weight of each floor, at a first frequency value of 4.02 Hz. An experimental snap-back test has shown that the distribution of the modes were: $f_1=3.96 \text{ Hz.}$, $f_2= 13.18 \text{ Hz.}$, $f_3=23.66 \text{ Hz.}$

The other possibility to use this formulation is with the value of the first mode(3.96 Hz.), estimate the global bending stiffness of the structure. The ratio x so obtained, is then equal to 0.48708, which corresponds to a Beta value of 0.060. We can deduce from equation (11) that the value of the stiffness: $EI= 2068 \text{ MNm}^2$. The measured stiffness on the real structure related on Fig.4. below, gives an experimental $EI = 2159 \text{ MNm}^2$.

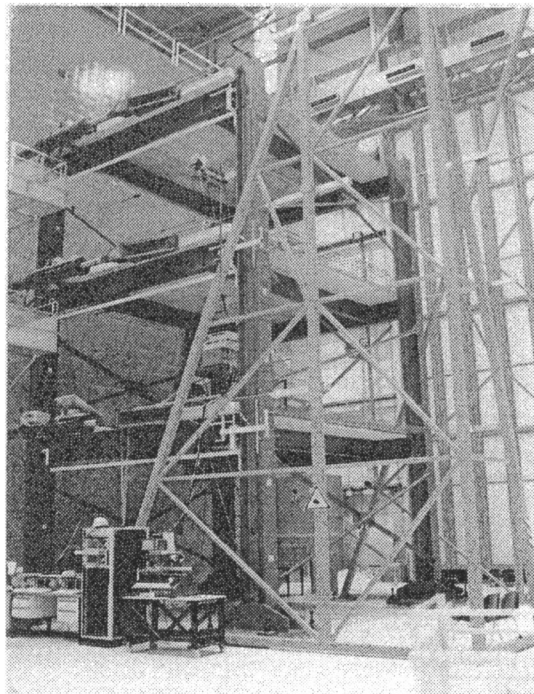
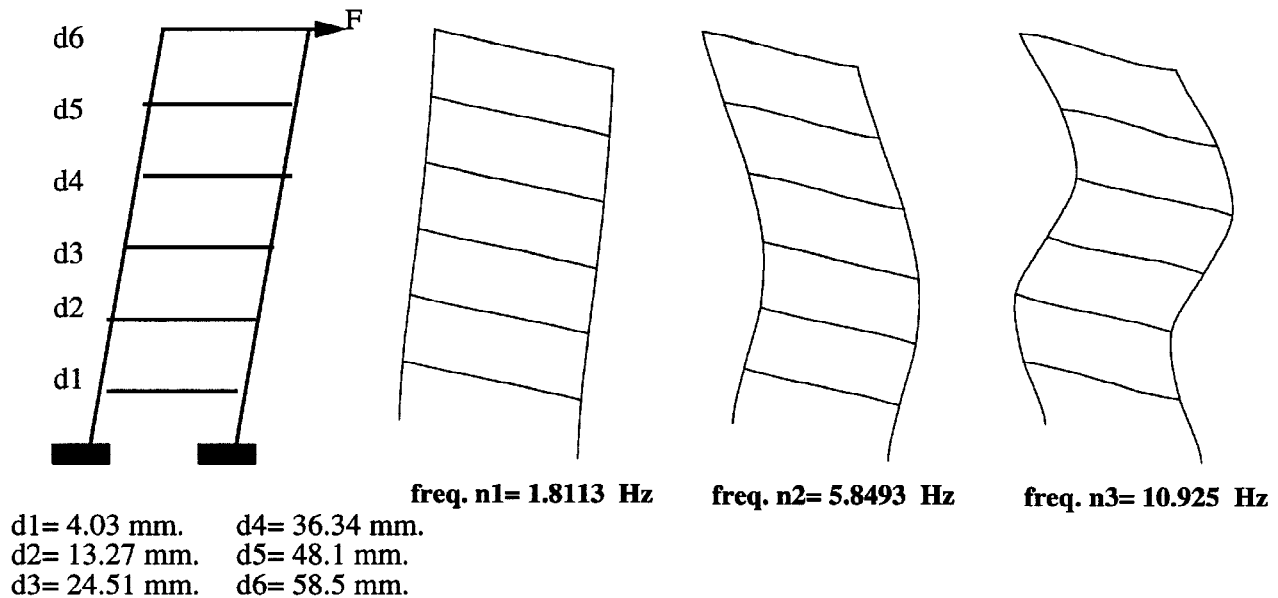


Fig. 4. Steel Frame of 3 Floors with Concrete Slabe in E.L.S.A.

3.3.2 Six Storeys Steel Building : To verify to another case the validity of the method, the same building, with three more floors was modelled by the finite element method (see Fig.5. below). Previously, the finite element model (using Timoshenko elements) was calibrated comparing it to the experimental results of the three storey building; the results were in good agreement. In that case only two parameters were changed: $nh = 18.875$ m. and $L = 20.475$ m.

With the same horizontal load (137 kN) at the top, we have a total displacement $d = 58.56$ mm. The equation (12) applied to that case give a beta equal to 0.0553. The first frequency correspondent is 1.98 Hz. The finite element give the first mode at 1.81 Hz. Rayleigh method give a first mode at 2.43 Hz.



$F = 137$ kn.

Fig.5. Results of the finite element model on a steel frame of 6 floors.

4. CONCLUSION

The experiments conducted on the “ideal” steel cantilever beam, to show the effect of the shear and rotatory inertia on short beam, are in agreement with the theoretical equation described by Shaw and Trail-Nash. Displaying these curves the engineer could have a good idea of the importance of the shear, in decreasing the frequencies expected for a structure chiefly for the higher modes, mainly in the case of small slender beams currently used in our buildings.

Satisfied by the validity of these results, the deduction of an expression of β from these equations allows the engineer to evaluate, with reasonable accuracy, the values of the modes of a building without assumptions regarding the distribution of the load on the structure. The other advantage of the method is that it doesn't need any heavy calculations like finite elements.

The experimental results in the first case and the finite element technique in the other, show that the error between the real values and those computed by the method developed here, is about 2.8% in the first case, and 8.8% in the other which is rather better than by Rayleigh method. Finally it provides also in the other sense a good approximation of the global bending stiffness expected for a building; the error is about 10.5% of the experimental one. The method must be applied to many other examples before being adopted, but to calculate dynamic parameters of a structure, the evaluation issue of static equations still leaves more often a non discarding error. This method seems to give good estimations of these parameters.

Table 1 : List of symbols

A=cross sectional area	h=floor height	ω =natural frequency
A _c = cross sectional shear area	i=Giration radius	σ =Mass per unit length
E=Young modulus	j=Number of columns	u =deflection of the beam
G= elastic shear modulus	L=Length of the equivalent beam	
H=Total height of the structure	ω_s =pur shear frequency	

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