Paper No. 1010. (quote when citing this article)
Eleventh World Conference on Earthquake Engineering
ISBN: 0 08 042822 3
11 WCEE

A GLOBAL DESIGN SENSITIVITY ANALYSIS IN STRUCTURAL DYNAMICS

H.A. JENSEN
Department of Civil Engineering, Federico Santa Maria University
Casilla 110 V - Valparaiso - Chile

ABSTRACT

An approach to study the global sensitivity of the dynamic response of structural systems as a function of a set of design parameters is presented. The sensitivity of the system is evaluated by considering the global behavior of the system response when the design parameters vary within a given design space. The sensitivity is computed globally by means of response surfaces, which are evaluated by using approximation concepts. The approximation is based on modal analysis and it is valid for general linear damped systems. Intermediate design variable and intermediate response quantity concepts are used to enhance the accuracy of the approximation. Numerical results that illustrate the usefulness and effectiveness of the method are presented. Great insight into the behavior of the system can be gained using this methodology.

KEYWORDS

Sensitivity; structural dynamics; parameter uncertainties; response surfaces; approximation concepts.

INTRODUCTION

Sensitivity analysis of structural systems to variations in their parameters plays a critical role in the design and analysis of structural systems. The basic concepts of sensitivity analysis of structural response are well documented in a number of publications (Frank (1978), Arora and Huag (1979), Adelman and Haftka (1986)). In general, the sensitivity of the system response is evaluated by partial derivatives of some response functions with respect to the system parameters. Methods for computing partial derivatives of structural response include finite difference methods, direct differentiation methods and adjoint methods. All these methods consider the variability in the system response to local variation of the design parameters, that is, they establish a measure of the way in which the response varies with changes in the parameters in the neighborhood of their nominal values.

Sensitivities have been derived for a number of structural systems with respect to a wide range of design parameters. For example, sensitivity with respect to material properties (Dems and Morz (1984), Pedersen (1987)), sectional parameters which describe beams, plates and shells (Cheng and Olhoff (1982), Brockman and Lung (1988)), and shape parameters which describe the body's geometry (Mota *et al* (1984), Dopker *et al* (1988), Chieu (1989)). Sensitivities have also been derived for nonlinear structural systems (Mroz Z. *et al* (1985), Haftka R.T and Morz Z. (1986), Cardoso, J.B. and Arora J.S (1988)). Eigenvalue and frequency response sensitivity has also been investigated (Nelson R.B. (1976), Ojalvo I.U. (1986)) as well as sensitivities for elastodynamic systems (Meric R.A (1988), Tortorelli *et al* (1990)). This information can be used to predict how response function value varies for small perturbations in the model parameters without performing a reanalysis. Thus, the sensitivities offer an efficient means of predicting the local performance of modified systems. Finally, sensitivity analyses also appear for other classes of problems such as thermal systems, fluid dynamic systems, rigid body mechanics, structural optimization, identification studies, reliability analyses, and general field problems.

The objective of this paper is to generalize the standard sensitivity analysis to a global sensitivity approach in the context of structural dynamics. The method is based on modal analysis and it is valid for general damped structures. The sensitivity of the system is evaluated by considering the global behavior of the system response when the design parameters vary within a given design space. The sensitivity is computed globally by means of response surfaces, which are evaluated by using local approximation concepts. This type of approach takes into account the fact that some system parameters are difficult to determine and are usually estimated with some margin of error. If the sensitivity is obtained without quantifying the effects of these errors in the modeling of the system parameters, then the results can lead to misleading conclusions. The proposed method can also identify the more influential design variables on the global behavior of the system and the less influential variables from a global point of view.

DYNAMIC RESPONSE

The general matrix equation of motion for an n-degree-of-freedom linear structure is given by

$$M\ddot{u} + C\dot{u} + Ku = p \,, \tag{1}$$

where M, C, and K are the mass, damping and stiffness matrices, respectively, u is the vector of dynamic displacements and p is the excitation vector. Defining the state space variables as

$$q = \begin{Bmatrix} \dot{u} \\ u \end{Bmatrix} \,, \tag{2}$$

Eq.(1) leads to the equations of motion in first order form, namely

$$M^*\dot{q} + K^*q = p^* \,, \tag{3}$$

where M^*, K^* and p^* can be defined directly from Eqs.(1) and (2).

In this study, the modal solution of the dynamic response problem will be used. In the modal approach, it is assumed that the dynamic state space response can be represented as a linear combination of complex mode shapes of the form

$$q = \sum_{i=1}^{2n} \phi_i \, \eta_i(t) \,, \tag{4}$$

where $\phi_i, i = 1, ...2n$ are the complex right eigenvectors corresponding to Eq.(3). Substituting Eq.(4) in Eq.(3), pre-multiplying by the complex left eigenvector χ_r^t , and using the orthogonality of the left and right eigenvector, leads to

$$T_r^* \dot{\eta}_r(t) + U_r^* \eta_r(t) = \chi_r^t p^* \,, \tag{5}$$

where $T_r^* = \chi_r^t M^* \phi_r$ and $U_r^* = \chi_r^t K^* \phi_r$. From the definition of the right and left eigenvectors, it is easily shown that the corresponding eigenvalue λ_r satisfies $\lambda_r = -U_r^*/T_r^*$, and therefore Eq.(5) can be written as

$$\dot{\eta}_r(t) - \lambda_r \eta_r(t) = \frac{\chi_r^t p^*}{T_r^*} \,, \quad r = 1, ...2n \,. \tag{6}$$

If the complex eigenvectors are partitioned in velocity and position parts, then the following identities are obtained: $T_r^* = 2\lambda_r T_r + S_r$ and $U^* = -\lambda_r^2 T_r + U_r$, where T_r, U_r and S_r are the modal energies given by

$$T_r = \chi_{pr}^t M \phi_{pr} \;, \; U_r = \chi_{pr}^t K \phi_{pr} \;, \; S_r = \chi_{pr}^t C \phi_{pr} \,, \tag{7}$$

where ϕ_{pr} and χ_{pr} are the position parts of the right and left eigenvector, respectively. Using these definitions, the complex eigenvalues can be written as

$$\lambda_r = \frac{-S_r \pm \sqrt{S_r^2 - 4T_rU_r}}{2T_r} \ , \quad r = 1, ...2n \ . \tag{8}$$

Introducing the definition of T_r^* and p^* in Eq.(6), the differential equation for the modal participation coefficients can be written as

$$\dot{\eta}_r(t) - \lambda_r \eta_r(t) = \frac{\chi_{pr}^t p}{(2\lambda_r T_r + S_r)} \ , \quad r = 1, ...2n \ , \tag{9}$$

with initial conditions that can be obtained directly in terms of the initial displacement u_0 and velocity $\dot{u}_0$. Finally, it is noted that the coefficients of Eq.(9) appear in complex conjugate pairs, that is, $\eta_{n+r}(t) = \bar{\eta}_r(t), r = 1, ...n$.

The general solution of Eq.(9), with initial condition at $t = t_0$, is given by

$$\eta_r(t) = e^{\lambda_r(t-t_0)}\eta_r(t_0) + \int_{t_0}^{t} \frac{e^{\lambda_r(t-\xi)}\, \chi_{pr}^t\, p(\xi)}{(2\lambda_r T_r + S_r)} d\xi \ . \tag{10}$$

If general load vectors are considered, numerical integration of Eq.(10) is unavoidable to obtain $\eta_r(t)$. However, in the context of earthquake engineering, the load vector can be assumed to be a piecewise linear function in time, and therefore closed form solutions can be obtained. For the interval $[t_i, t_{i+1}]$ the forcing vector is given by $p(t) = a_i + b_i t$, where

$$a_i = \frac{p(t_i)t_{i+1} - p(t_{i+1})t_i}{t_{i+1} - t_i} \ , \ b_i = \frac{p(t_{i+1}) - p(t_i)}{t_{i+1} - t_i} \ . \tag{11}$$

The solution for $\eta_r(t)$ in the interval $[t_i, t_{i+1}]$ is denoted by $\eta_r^i(t)$, and it is given by

$$\eta_r^i(t) = e^{\lambda_r(t-t_i)}(\eta_r^{i-1}(t_i) - \alpha_r^i - \beta_r^i\, t_i) + \alpha_r^i + \beta_r^i\, t \ , \tag{12}$$

where

$$\alpha_r^i = -\frac{\chi_{pr}^T(b_i - \lambda_r a_i)}{\lambda_r^2(2\lambda_r T_r + S_r)} \ , \ \beta_r^i = -\frac{\chi_{pr}^T b_i}{\lambda_r(2\lambda_r T_r + S_r)} \ . \tag{13}$$

Equation (12) gives a recursive formula to evaluate $\eta_r^i(t)$ for each interval $[t_i, t_{i+1}]$. Finally, from Eq.(4) and using the fact that the complex eigenvectors appear in complex conjugate pairs, the vector of dynamic displacements is given by

$$u(t) = 2\, Re\, \Big(\sum_{r=1}^{n} \phi_{pr}\, \eta_r(t) \Big) \ , \tag{14}$$

where $\eta_r(t)$ is given by Eq.(12).

APPROXIMATION CONCEPTS

From Eq.(14) it is clear that the response of the system u depends on its spectral properties, that is, ϕ_r, χ_r and $\lambda_r, r = 1, ...2n$. At the same time, these properties depend on the vector of design variables y $(y_j, j = 1, ...m)$. A general system response $R(t, y)$ can be written as

$$R(t, y) = H(f, x, y) \ , \tag{15}$$

where f $(f_i, i \in I)$ denote intermediate response quantities, and x $(x_j, j \in J)$ denote intermediate system parameters. In Eq.(15) it is assumed that H is explicit in f, x, and y, $f_i, i \in I$ are implicit functions of x, and $x_j, j \in J$ are explicit functions of y.

Approximations are constructed by approximating the intermediate response quantities f_i explicitly in terms of the intermediate design variables x. Once these approximations have been obtained, the system response $R(t, y)$ can be written explicitly in terms of the set of original design variables y. In this approach, the modal energies T_r, S_r, U_r, and the position parts of the right and left eigenvectors, ϕ_{pr} and χ_{pr}, are chosen as intermediate response quantities and they are approximated locally in Taylor series with respect to selected intermediate system parameters as

$$\tilde{T}_r = T_{r0} + \sum_j \frac{\partial T_r(x_0)}{\partial x_j}(x_j - x_{j0}) + \ldots , \quad \tilde{S}_r = S_{r0} + \sum_j \frac{\partial S_r(x_0)}{\partial x_j}(x_j - x_{j0}) + \ldots ,$$

$$\tilde{U}_r = U_{r0} + \sum_j \frac{\partial U_r(x_0)}{\partial x_j}(x_j - x_{j0}) + \ldots , \quad \tilde{\phi}_{pr} = \phi_{pr0} + \sum_j \frac{\partial \phi_{pr}(x_0)}{\partial x_j}(x_j - x_{j0}) + \ldots ,$$

$$\tilde{\chi}_{pr} = \chi_{pr0} + \sum_j \frac{\partial \chi_{pr}(x_0)}{\partial x_j}(x_j - x_{j0}) + \ldots , \tag{16}$$

where $x_0 = x(y_0)$, and y_0 is the vector of base line design variables.

Introducing these approximations in Eq.(10) gives

$$\tilde{\eta}_r(t) = e^{\tilde{\lambda}_r(t-t_0)}\tilde{\eta}_r(t_0) + \int_{t_0}^{t} \frac{e^{\tilde{\lambda}_r(t-\xi)}\, \tilde{\chi}^t_{pr}\, p(\xi)}{(2\tilde{\lambda}_r\tilde{T}_r + \tilde{S}_r)} d\xi , \tag{17}$$

with initial conditions approximated in a similar manner as T_r, U_r, S_r , ϕ_{pr} , and χ_{pr}. In this way, the approximation for the transient dynamic displacements is constructed using Eq.(14) with a truncated set of modes and the approximated modal participation coefficients given by Eq.(17). Then

$$\tilde{u}(t) = 2\, Re\, \Big(\sum_{r=1}^{N} \tilde{\phi}_{pr}\, \tilde{\eta}_r(t) \Big) , \tag{18}$$

where N denotes the retained number of modes. It is noted that for this response function, the H function in Eq.(15) corresponds to the linear combination defined in Eq.(18). For response functions other than displacements, appropriate H functions can be defined.

In summary, the quantities T_r, S_r, U_r, ϕ_{pr} and χ_{pr} for all the retained modes are chosen as intermediate response quantities and they are approximated in terms of appropriate intermediate design variables. The approximation of these quantities requires a standard eigenvalue and eigenvector sensitivity analysis. Finally, it is noted that the approximations for the modal energies given in Eq.(16) have been used extensively in the area of structural optimization. It was found that high quality approximations can be generated by using this type of approximations (Schmit and Farshi (1974), Sepulveda and Schmit (1993)).

GLOBAL SENSITIVITY

Once the system response has been obtained in terms of the design parameters, global sensitivity estimates can be defined directly from the representation of $R(t, y)$. For example, the global sensitivity can be measured by the dispersion of the response about the base or nominal response through the coefficient of sensitivity

$$\Upsilon_{R,y_j}(t) = \frac{\sqrt{\frac{1}{\mu(y_j)} \int_{y_j} \left(R(t,y) - \bar{R}(t)\right)^2 dy_j}}{max_t[\bar{R}(t)]} , \tag{19}$$

where $\mu(y_j)$ is the measure of the range of variation of the design variable y_j, $\bar{R}(t)$ is the base line response, and $max_t[\bar{R}(t)]$ is the maximum base line response in time. This coefficient is evaluated numerically by using the characterization of the approximated response $R(t, y)$. The coefficient of Eq.(19) can also be used to define a global sensitivity or coupling matrix as

$$S(t) = \left(S_{ij}(t) = \Upsilon_{R_i,y_j}(t)\right) , \tag{20}$$

where R_i denotes the system response R_i. This matrix determines the degree of functional coupling in the set of design variables with respect to different response functions.

Finally, global coefficients of sensitivity and coupling matrices can also be defined for peak responses. In this case, the coefficient of sensitivity is defined as

$$\Upsilon_{R,y_j}^{max} = \frac{\sqrt{\frac{1}{\mu(y_j)}\int_{y_j}\left(max_t[R(t,y)-\bar{R}(t)]\right)^2 dy_j}}{max_t[\bar{R}(t)]}\,, \tag{21}$$

where the peak response $max_t[R(t,y)]$ can be identified directly by evaluating in time the approximated response and then choosing its maximum value.

NUMERICAL EXAMPLE

In order to illustrate the applicability of the method, a three-story two-bay shear building, shown in Fig. 1, is considered. The values of the various parameters describing the structure and loading are set as follows: elastic modulus for all columns = $2 \times 10^9\ kg/m^2$; rectangular cross-section for all columns with nominal height = $0.6m$ and nominal width = $0.3m$; total weight on the first and second floor = $3.5 \times 10^5 kg$ and on the third floor = $2.8 \times 10^5 kg$; and a base acceleration given by the 1971 San Fernando earthquake S82E record (IIG110 - Caltech Catalog).

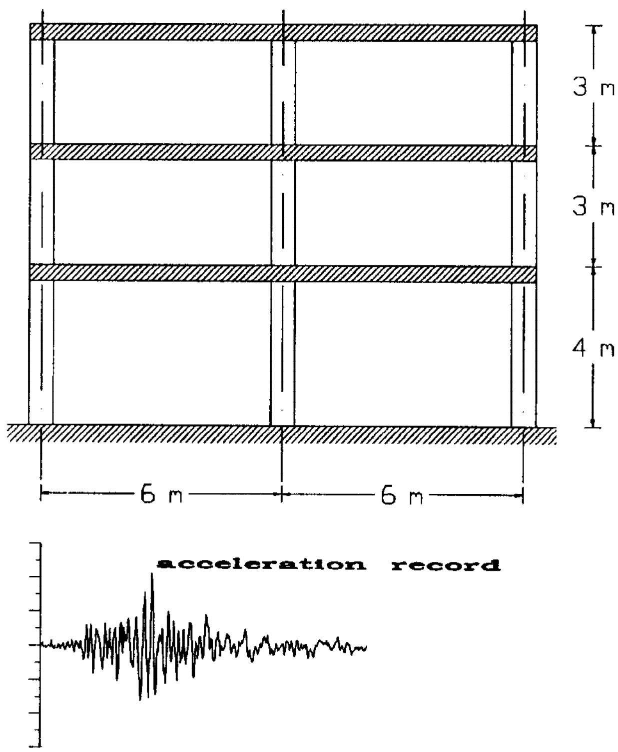

Fig. 1. Three-story two-bay shear building.

The design variables are the dimensions of the cross-section of the column elements. Three set of elements are considered: columns corresponding to the first level with design variables b_1 and h_1; columns corresponding to the second level with design variables b_2 and h_2; and columns corresponding to the third level with design variables b_3 and h_3. Therefore, the vector of design variables is given by $y^T = (b_1, h_1, b_2, h_2, b_3, h_3)^T$. The response functions to be considered for the global sensitivity analysis are the maximum displacement at the top of the building (R_1), the maximum base shear (R_2), the maximum story drift at the third floor (R_3), and the maximum acceleration at the top of the building (R_4). Expansions of first- and second-order in terms of intermediate design variables are used for the approximation of the intermediate response functions. In this example problem, the intermediate design variables are the moments of inertia of the column elements.

As previously mentioned, one way to illustrate the global sensitivity of the system response is through a global sensitivity matrix. In this case, the following matrix is defined

$$S = \begin{pmatrix} \Upsilon^{max}_{R_1,b_1} & \Upsilon^{max}_{R_2,b_1} & \Upsilon^{max}_{R_3,b_1} & \Upsilon^{max}_{R_4,b_1} \\ \Upsilon^{max}_{R_1,h_1} & \Upsilon^{max}_{R_2,h_1} & \Upsilon^{max}_{R_3,h_1} & \Upsilon^{max}_{R_4,h_1} \\ \Upsilon^{max}_{R_1,b_2} & \Upsilon^{max}_{R_2,b_2} & \Upsilon^{max}_{R_3,b_2} & \Upsilon^{max}_{R_4,b_2} \\ \Upsilon^{max}_{R_1,h_2} & \Upsilon^{max}_{R_2,h_2} & \Upsilon^{max}_{R_3,h_2} & \Upsilon^{max}_{R_4,h_2} \\ \Upsilon^{max}_{R_1,b_3} & \Upsilon^{max}_{R_2,b_3} & \Upsilon^{max}_{R_3,b_3} & \Upsilon^{max}_{R_4,b_3} \\ \Upsilon^{max}_{R_1,h_3} & \Upsilon^{max}_{R_2,h_3} & \Upsilon^{max}_{R_3,h_3} & \Upsilon^{max}_{R_4,h_3} \end{pmatrix}, \tag{22}$$

where the coefficients of the matrix are defined as before.

Equation (23) shows the corresponding sensitivity matrix for a range of variation of 20% of the design variables with respect to their nominal values. A second-order approximation is considered in this case and some amount of damping is added to the system.

$$S = \begin{pmatrix} 0.1046 & 0.1314 & 0.1383 & 0.0151 \\ 0.2643 & 0.2775 & 0.2798 & 0.0566 \\ 0.0105 & 0.0199 & 0.0138 & 0.0083 \\ 0.0648 & 0.1124 & 0.0985 & 0.0276 \\ 0.0086 & 0.0043 & 0.1240 & 0.0083 \\ 0.0261 & 0.0176 & 0.4064 & 0.0265 \end{pmatrix} \tag{23}$$

This matrix shows the more influential and the less influential design variables from a global point of view with respect to different response functions. For example, the design variable h_1 is the most significant with respect to the maximum displacement at the top of the building (R_1). In this case a coefficient of sensitivity of 26% is obtained. In the same manner, the design variable h_3 is the most significant with respect to story drift at the third floor (R_3). Note that a coefficient of sensitivity of 40% is obtained in this case. Therefore, the height of the cross-section of the columns of the third floor shows an important influence on this response. It should be noted that the maximum coefficient for the response functions R_1, R_2 and R_3 is greater than the corresponding parameter variability. The global sensitivity matrix can also be used to evaluate the degree of functional coupling in the set of design variables. For example, the design variable b_1 shows a strong coupling with respect to the response functions R_1, R_2 and R_3 and a weak coupling with respect to the response R_4. Similar analyses can be performed with other response functions and design variables.

An alternative way to present and evaluate the global sensitivity of the system response is through a graphical representation of response surfaces, that is, the response as a function of the design parameters. Figs. 2 and 3 show the maximum displacement at the top of the building and the maximum base shear as a function of the design variable h_1, respectively. A parameter variability of 30% is consider in these figures, and the value of the design variable is normalized by its nominal value. In order to validate the approximations used in the formulation, the exact response is compared with the response obtained by using first- and second-order expansions in the approximation of the intermediate response functions. The results show that these approximations give excellent results. The solution for the case of second-order expansions is almost coincident with the exact response (direct simulation).

CONCLUSIONS

A global design sensitivity analysis, in which the sensitivity of the system is evaluated by considering the global behavior of the system response when the design parameters vary within a given design space has been described. The method is based on the approximation of response surfaces and it permits the analyst to quantify the response sensitivity from a global point of view. The analysis can identify the more influential system parameters on the global behavior of the system and the less influential variables. In addition, the proposed methodology can be very useful in optimum redesign analysis. For example, if a given system does not meet the performance requirements, then the global sensitivity analysis can help to identify the critical design variables. Great insight into the global behavior of the system response can be obtained using the proposed approach. At the same time, it provides a valuable information for rational decision making in the design and analysis of structural systems. It can be used in combination with reliability analysis, structural optimization and fuzzy analysis. Finally, validation calculations show that the results from the method agree very well with those obtained by direct evaluation of the response surfaces.

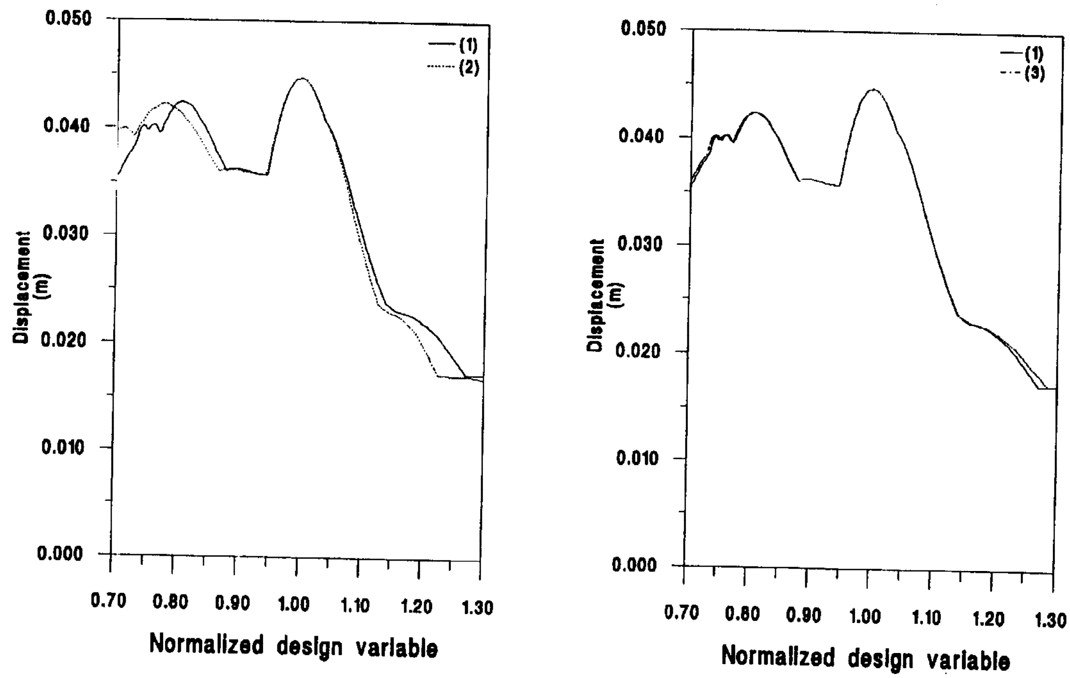

Fig. 2. Maximum displacement at the top of the building as a function of the design variable h_1. (1) Exact response; (2) First-order expansion response; (3) Second-order expansion response.

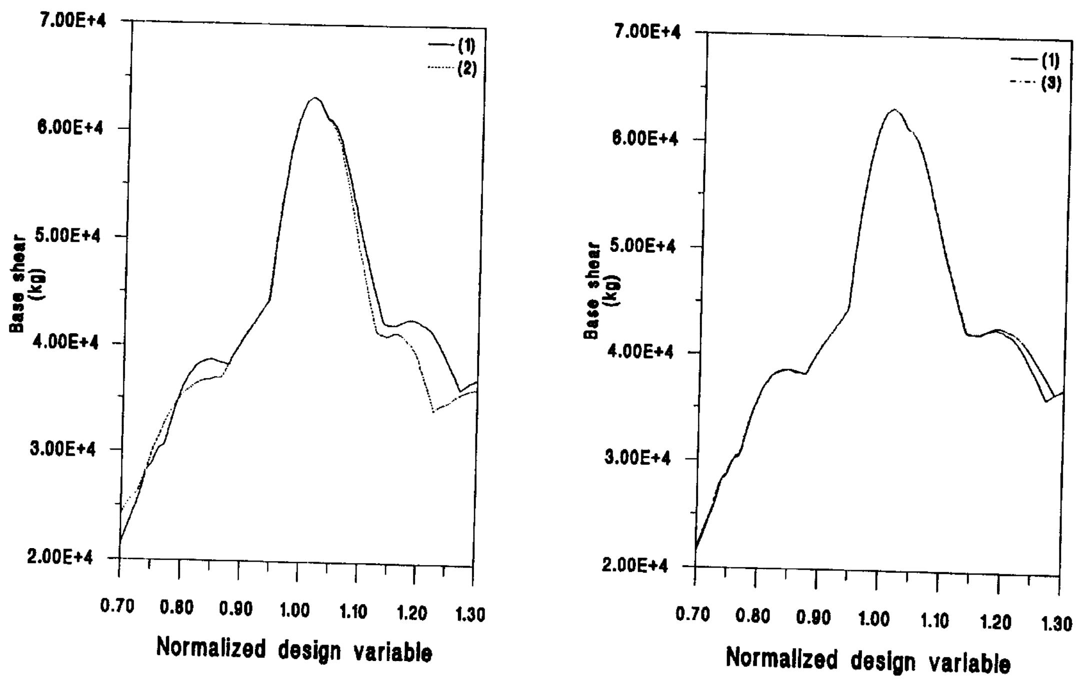

Fig. 3. Maximum shear base as a function of the design variable h_1. (1) Exact response; (2) First-order expansion response; (3) Second-order expansion response.

ACKNOWLEDGMENT

The research reported here was supported in part by CONICYT under Grant No. 1950710. The conclusions presented are those of the writer and do not necessarily represent the views of the sponsor.

REFERENCES

Adelman, H.M. and Haftka, R.T. (1986), Sensitivity analysis for discrete structural systems, *AIAA Journal*, **24** (5), 823-831.

Arora, S.J. and Huag, E.J. (1979), Methods of design sensitivity analysis in structural optimization, *AIAA Journal* **17** (9), 970-974.

Brockman, R.A. and Lung, F.Y. (1988), Sensitivity analysis with plate and shell finite elements, *Int J. for Numerical Methods in Engineering*, **26**, 1129-1143.

Cardoso, J.B. and Arora, J.S. (1988), Variational Method for Design Sensitivity Analysis in Nonlinear Structural Mechanics, *AIAA Journal* **26**, 595-603.

Cheng, K.T. and Olhoff, N. (1982), Regularized formulation for optimal design of axisymmetric plates, *Int. J. of Solids and Structures*, **18** (2), 305-323.

Chieu, T.C. (1989), Sensitivity analysis and shape optimization of axisymmetric structures, *Int. J. for Numerical Methods in Engineering*, **28**, 95-108.

Dems, K. and Mors, Z. (1984), Variational approach by means of an adjoint system to structural optimization and sensitivity analysis, *Int. J. of Solids and Structures*, **20** (6), 527-552.

Dopker, B., Choi, K.K., and Benedtict, L. (1988), Shape design sensitivity analysis of structures containing arches, *Computers and Structures*, **28** (1), 1-13.

Frank, P.M. (1978), *Introduction to System Sensitivity Theory*, Academic Press, New York.

Haftka, R.T. and Morz, Z. (1986) , First- and second-order sensitivity analysis of nonlinear structures, *AIAA Journal*, **24**, 1187-1192.

Meric, R.A. (1988), Shape design sensitivity analysis of dynamic structures, *AIAA Journal*, **26**, 206-261.

Mota, C.A., Rodrigues, H.C., and Choi KK (1984), Shape optimal design usign boundary elements and minimum compliance techniques, *Journal of Mechanisms, Transmissions, and Automation in Engineering*, **106**, 518-523.

Mroz, Z. , Kamat, M.P., and Plaut, R.H. (1985), Sensitivity analysis and optimal design of nonlinear beams and plates, *AIAA Journal*, **24**, 1187-1192.

Nelson, R.B. (1976), Simplified calculation of eigenvector derivatives, *AIAA Journal*, **14** 1201-1205.

Ojalvo, I.U. (1986), Efficient computation of mode-shape derivatives, *27 th Structures Structural Dynamics and Materials Conference*, San Antonio, TX, 242-247.

Pedersen, P. (1987), On sensitivity analysis and optimal design of specially orthotropic laminates, *Engineering Optimization*, **11**, 305-316.

Schmit, L.A. and Farshi, B. (1974), Some approximations concepts for efficient structural synthesis, *AIAA Journal* , **12**, 692-699.

Sepulveda, A.E. and Schmit, L.A., (1993) Approximation based global optimization strategy for structural synthesis, *AIAA Journal*, **31**, 180-188.

Tortorelli, D.A., Lu, C.Y., and Haber RB (1990), Design sensitivity analysis for elastodynamic systems, *Mechanics of Structures and Machines* **18** (1), 77-105.