Earthquake Engineering, Tenth World Conference © 1992 Balkema, Rotterdam. ISBN 90 5410 060 5

A mode superposition procedure for seismic analysis of nonlinear base-isolated structures

B. Mohraz & Y.C. Jian
Southern Methodist University, Dallas, Tex., USA

C.J. Chang
SSC Laboratory, Dallas, Tex., USA

ABSTRACT: An alternative procedure to complex mode method for solving non-classically damped nonlinear structures is given. This procedure uses undamped modes to transform the equations of motion from geometric coordinates to modal coordinates by including the nonlinearity in the stiffness and the coupling in the modal damping matrix as a pseudo force. The method is used to compute the response of a five story base-isolated frame to the NS component of the accelerogram from El Centro, the Imperial Valley earthquake of May 18, 1940. Comparisons of the solution with the direct integration and with the complex mode show that the responses from the three methods are in excellent agreement while the procedure is simpler and requires less computational time than the other two.

1. INTRODUCTION

Structures subjected to severe earthquake motion experience excessive and often nonlinear deformations. Isolators may be used to reduce the response and control the nonlinear deformations in the structure during earthquakes. Elastomeric pads with large damping have been used as base isolators. Although the isolators experience large and at times nonlinear deformations, the motion in the structure is reduced significantly, and in most case is confined to the elastic range.

The large damping of the isolators compared to the small damping of the structure often results in a non-proportionally or non-classically damped structure. In addition, even if classical damping can be assumed in the analysis, the presence of the nonlinear deformations in the isolators would make it difficult to use the conventional mode superposition unless the nonlinearity is treated as a piece-wise linearization by using a truncated series such as the Taylor series approximation (Chang and Mohraz (1990)).

In general, the response of non-classically damped structures can be obtained using direct integration of the coupled equations of motion. In the direct integration all equations must be included in the analysis which increases the computational effort. An alternative to direct integration is the complex mode method. Although the computation of the response involves arithmetic in the complex domain, the procedure insures that the modal equations are uncoupled without approximating the damping matrix. The use of the complex mode method, however, necessitates the solution of 2N (N being the number of degrees of freedom) first order differential equations.

In this paper an undamped mode superposition procedure for nonlinear analysis of base-isolated structures is presented. Undamped modes are used to transform the equations of motion to modal coordinates. The coupling of the modal equations caused by the off-diagonal elements of the modal damping matrix as well as the nonlinearity in the structural stiffness and the isolators are treated as a pseudo force. The method of undetermined coefficients (Nigam and Jennings (1969)) is used to solve the modal equations. Solutions from the proposed method are compared with those from the direct integration of the coupled equations of motion and with those from the complex mode method.

2. FORMULATION

The equations of motion of a multi-story structure with nonlinear resisting elements which is subjected to a base excitation $\ddot{x}_g(t)$ can be written as

$$[m]\{\ddot{x}(t)\} + [c]\{\dot{x}(t)\} + \{r(t)\} = -[m]\{\rho\}\ddot{x}_g(t) \quad (1)$$

in which $[m]$ is the mass matrix, $[c]$ is the damping matrix, $\{r(t)\}$ is the nonlinear restoring force which varies with deformation, $\{x(t)\}$ is the displacement vector of the stories relative to the base, and $\{\rho\}$ is the excitation participation vector. The restoring force $\{r(t)\}$ may be divided into

$$\{r(t)\} = [k]\{x(t)\} + \{r_n(t)\} \quad (2)$$

where $[k]\{x(t)\}$ and $\{r_n(t)\}$ are the linear and nonlinear parts, respectively. Substituting equation (2) into equation (1) and transferring the nonlinear term to the right hand side, equation (1) becomes

$$[m]\{\ddot{x}(t)\} + [c]\{\dot{x}(t)\} + [k]\{x(t)\} = -[m]\{\rho\}\ddot{x}_g(t) - \{r_n(t)\} \quad (3)$$

Figure 1 shows a structure with its base supported on an isolation system. Assuming x_s denotes the displacement of the floors relative to the base and x_b

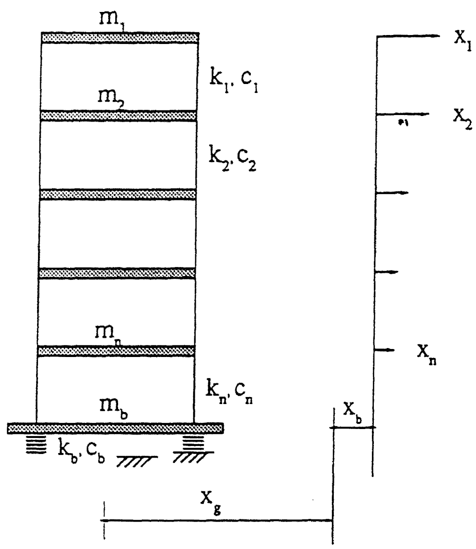

Fig. 1 Base isolated frame

refers to the displacement of the base relative to the ground, the equations of motion of the structure subjected to ground movement $\ddot{x}_g(t)$ can be written (Tsai and Kelly (1988), Novak and Henderson (1989)) as

$$[m_s]\{\ddot{x}_s(t)\} + [m_s]\{\rho_s\}\ddot{x}_b(t) + [c_s]\{\dot{x}_s(t)\} + [k_s]\{x_s(t)\} = -[m_s]\{\rho_s\}\ddot{x}_g(t) - \{r_{ns}(t)\} \tag{4.a}$$

$$\{[m_s]\{\rho_s\}\}^T\{\ddot{x}_s(t)\} + (m_b+\Sigma m_i)\ddot{x}_b(t) + c_b\dot{x}_b(t) + k_b x_b(t) = -(m_b+\Sigma m_i)\ddot{x}_g(t) - r_{nb}(t) \tag{4.b}$$

In the above equations the subscripts s refers to the structure, b to the base, and i to the story. These two equations may be combined into a single equation as

$$[m]\{\ddot{x}(t)\} + [c]\{\dot{x}(t)\} + [k]\{x(t)\} = -[m_r]\{\rho_r\}\ddot{x}_g(t) - \{r_n(t)\} \tag{5}$$

in which

$$[m] = \begin{bmatrix} [m_s] & [m_s]\{\rho_s\} \\ \{[m_s]\{\rho_s\}\}^T & m_b+\Sigma m_i \end{bmatrix}$$

$$[c] = \begin{bmatrix} [c_s] & \{0\} \\ \{0\}^T & c_b \end{bmatrix}$$

$$[k] = \begin{bmatrix} [k_s] & \{0\} \\ \{0\}^T & k_b \end{bmatrix}$$

$$[m_r] = \begin{bmatrix} [m_s] & \{0\} \\ \{0\}^T & m_b+\Sigma m_i \end{bmatrix}$$

$$\{\rho_r\} = \begin{Bmatrix} \{\rho_s\} \\ 1 \end{Bmatrix}$$

$$\{r_n\} = \begin{Bmatrix} \{r_{ns}\} \\ r_{nb} \end{Bmatrix}$$

$$\{x(t)\} = \begin{Bmatrix} \{x_s(t)\} \\ x_b(t) \end{Bmatrix}$$

The solution of the undamped eigenvalue problem corresponding to the linearized left-hand side of equation (5) yields frequencies ω_n and mode shape matrix $[\phi]$. Introducing the transformation

$$\{x(t)\} = [\phi]\{Y(t)\} \tag{6}$$

where $\{Y(t)\}$ is the vector of modal coordinates, substituting it into equation (5) and then premultipling by the transpose of the mode shape matrix, one obtains

$$[\phi]^T[m][\phi]\{\ddot{Y}(t)\} + [\phi]^T[c][\phi]\{\dot{Y}(t)\} + [\phi]^T[k][\phi]\{Y(t)\} = -[\phi]^T[m_r]\{\rho_r\}\ddot{x}_g(t) - [\phi]^T\{r_n(t)\} \tag{7}$$

Since orthogonality conditions do not uncouple a non-proportional damping matrix, equation (7) represents a set of coupled equations. The modal damping matrix in equation (7) contains diagonal and off-diagonal elements. By transferring the off-diagonal elements to the right-hand side of the equations, the n-th modal equation may be written as

$$M_n\ddot{Y}_n(t) + C_{nn}\dot{Y}_n(t) + K_nY_n(t) = -\left(L_n\ddot{x}_g(t) + R_n(t) + \sum_{j=1}^{N} C_{nj}\dot{Y}_j(t)\right) \qquad j \neq n \tag{8}$$

where C_{nn} and C_{nj} are the diagonal and off-diagonal elements of the damping matrix, R_n is the nonlinear contribution of the restoring force, and L_n is he modal participation factor. The right-hand side in equation (8) represents a pseudo force which includes the ground excitation, nonlinearities in the structural stiffness and the isolators, and the coupling of the modal damping matrix. Equation (8) may be written in the familiar form

$$\ddot{Y}_n(t) + 2\xi_n\omega_n\dot{Y}_n(t) + \omega_n^2Y_n(t) = -P_nF_n(t) \tag{9}$$

where $P_n = L_n/M_n$, ξ_n and ω_n are modal dampings and frequencies, and $F_n(t)$ is the pseudo force given by

$$F_n(t) = \ddot{x}_g(t) + \frac{1}{L_n}\left(R_n(t) + \sum_{j=1}^{N} C_{nj}\dot{Y}_j(t)\right) \qquad j \neq n \tag{10}$$

3. SOLUTION

Equation (9) may be solved using Duhamel integral, Wilson theta method (Wilson et al. (1973)), or

Newmark beta method (Newmark (1959)). In this paper the method of undetermined coefficients (Nigam and Jennings (1969), Kreyszig (1983)) was used to solve the equations and to evaluate the response at specified time intervals. Assuming that the time interval used in the solution is Δt and that $F_n(t)$ varies linearly within this time interval, one may write

$$F_n(\tau) = F_n(t - \Delta t) + \frac{\tau}{\Delta t}(F_n(t) + F_n(t - \Delta t)) \tag{11}$$

where τ is measured from the beginning of the interval. Substituting equation (11) into equation (9) gives

$$\ddot{Y}_n(\tau) + 2\xi_n\omega_n\dot{Y}_n(\tau) + \omega_n^2 Y_n(\tau) = - P_n(\alpha_n + \beta_n\tau) \tag{12}$$

where

$$\alpha_n = F_n(t - \Delta t) \tag{13a}$$

$$\beta_n = \frac{\tau}{\Delta t}(F_n(t) + F_n(t - \Delta t)) \tag{13b}$$

The solution of equation (12) may be expressed as

$$Y_n(\tau) = e^{-\xi_n\omega_n\tau}((A_n\cos(\omega_{nd}\tau) + B_n\sin(\omega_{nd}\tau)) + \frac{P_n}{\omega_n^2}((\frac{2\xi_n}{\omega_n} - \tau)\beta_n - \alpha_n) \tag{14}$$

where the first two terms are the homogeneous solution, the last three are the particular solution, and ω_{nd} is the damped modal frequency. Differentiating equation (14) with respect to time and solving it together with $Y_n(\tau)$ at $\tau = 0$ (i.e. the beginning of the time interval), the coefficients A_n and B_n can be determined. Thus,

$$A_n = Y_n(t - \Delta t) + \frac{P_n}{\omega_n^2}(\alpha_n - \frac{2\xi_n}{\omega_n}\beta_n) \tag{15a}$$

$$B_n = \frac{1}{\omega_{nd}}(\dot{Y}_n(t - \Delta t) + \xi_n\omega_n Y_n(t - \Delta t) + \frac{P_n}{\omega_n^2}(\xi_n\omega_n\alpha_n - 2(\xi_n^2 - 1)\beta_n)) \tag{15b}$$

Substituting equation (15) into equation (14) and regrouping the terms, the modal displacements and velocities may be expressed as

$$\begin{Bmatrix} Y_n(t) \\ \dot{Y}_n(t) \end{Bmatrix} = [A_n]\begin{Bmatrix} Y_n(t-\Delta t) \\ \dot{Y}_n(t-\Delta t) \end{Bmatrix} + P_n[B_n]\begin{Bmatrix} \ddot{x}_g(t - \Delta t) \\ \ddot{x}_g(t) \end{Bmatrix} + \frac{P_n}{L_n}[B_n]\begin{Bmatrix} R_n(t-\Delta t) + \sum_{j=1}^{N} C_{nj}\dot{Y}_j(t-\Delta t) \\ R_n(t) + \sum_{j=1}^{N} C_{nj}\dot{Y}_j(t) \end{Bmatrix} \tag{16}$$

where the 2x2 matrices of coefficients A_n and B_n are functions of ξ_n, ω_n, ω_{nd}, and Δt. The elements of these two matrices are given in the Appendix A.

The terms $R_n(t)$ and $\Sigma C_{nj}\dot{Y}_j(t)$ in equation (16) are an *a priori* function. If the time interval in the solution is Δt, the expansion of these terms in a Taylor series about t-Δt would result in

$$R_n(t) + \sum_{j=1}^{N} C_{nj}\dot{Y}_j(t) = \sum_{k=0}^{\infty}\frac{(\Delta t)^k}{k!}\frac{d^k}{dt^k}[R_n(t-\Delta t) + \sum_{j=1}^{N} C_{nj}\dot{Y}_j(t-\Delta t)] \qquad j \neq n \tag{17}$$

Equation (16) and equation (17) can be evaluated without numerical integration to obtain the modal response.

4. RESULTS

The formulation presented here was used to compute the response of a five story shear type frame with its base supported on a set of soft elastomeric bearings. The mass and stiffness properties of the frame (Karasudshi, et al. (1975), Koh and Balendra (1989)) are given in Table 1. A damping ratio of five percent of critical was assumed for each mode and a ratio of 20 percent for the isolation bearings. The modal damping ratios for the frame with base-isolation were computed using the method suggested by Wilson and Penzien (1972) and Clough and Penzien (1974). The modal frequencies of the frame without the base-isolation range from 1.51 *cps* to 16.46 *cps*. With the base-isolation the frequencies range from 0.49 *cps* to 16.84 *cps* and the modal damping ratios from 5.47 percent to 19.26 percent. Two frames - one with elastic-plastic properties and the other with bilinear properties with a primary to secondary stiffness ratio of 2 - were considered. A yield displacement of 2.0 *in.* was

Table 1. Properties of the frame.

story	mass *(k-sec²/in)*	stiffness *(k/in)*
Top	0.0423	69.9
2	0.0423	93.2
3	0.0423	116.5
4	0.0423	139.8
5	0.0423	163.1
base	*0.0635*	
bearings		2.7156

Table 2. Comparison of maximum story displacements (*in.*) for elastic - plastic case.

story	proposed method	complex mode	"direct integration
Top	0.199	0.202	0.203
2	0.173	0.177	0.177
3	0.136	0.138	0.138
4	0.093	0.094	0.094
5	0.047	0.048	0.047
Base	3.469	3.457	3.459
CPU(*sec*)	8.84	13.39	9.31

Displacements were computed at 0.01 *sec* intervals.

Table 3. Comparison of maximum story displacements (*in.*) for bilinear case with a primary to secondary stiffness ratio of 2.

story	proposed method	complex mode	direct integration
Top	0.203	0.199	0.199
2	0.178	0.176	0.177
3	0.142	0.142	0.143
4	0.099	0.099	0.100
5	0.051	0.051	0.051
Base	4.230	4.229	4.228
CPU(*sec*)	8.93	13.47	9.51

Displacements were computed at 0.01 *sec* intervals.

assumed for the columns and the bearings for both cases. The first three terms in the Taylor series expansion (k=0, 1, 2 in equation (17)) were used in the solutions. For the direct integration, an integration time step of 0.02 *sec* which corresponds to the intervals of the recorded accelerogram resulted in an unstable solution because of the shortest period of vibration of 0.06 *sec*. Consequently, the response computations were carried out at 0.01 *sec* intervals for all three methods.

Tables 2 and 3 show the maximum displacement of each story when the frame is subjected to the NS component of the accelerogram from El Centro, the Imperial Valley earthquake of May 18, 1940. Also presented in the tables are the results from the the complex mode and the direct integration, and the CPU times for each case. The results in Tables 2 and 3 show that the displacements computed by the method presented herein are in excellent agreement with those computed by both the direct integration and the complex mode method.

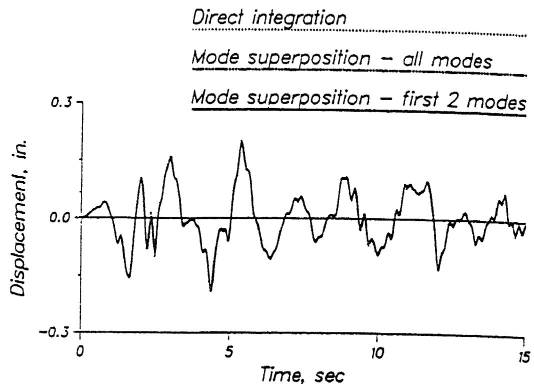

Fig. 2 Displacement time-history for the top floor

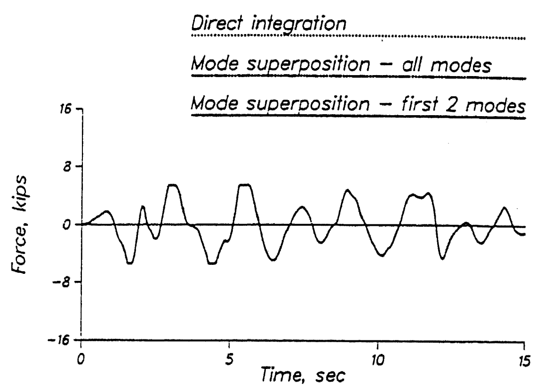

Fig. 3 Force time-history for the bearings

One advantage of using mode superposition is that fewer modes may be used in computing the response. Figure 2 shows the displacement response-time history of the top floor for the elastic-plastic case when all modes and then when the first two modes were used in computing the response. The displacements are in such excellent agreement that the differences between the curves are not detectable. Figure 3 shows a similar comparison for the force in the isolation bearings. It should be noted that when the force in the bearings reaches 5.43 *kips* yielding initiates and the force remains constant until unloading takes place (Figure 3). An examination of the forces in the columns indicated that they all remained within the elastic range during the excitation.

It is interesting to note that neglecting the coupling (the off-diagonal elements in equation (8)) in the damping matrix results in significant error in the solution. Figure 4 shows the displacement response-time history of the top floor of the frame with elastic-plastic column properties when the coupling is neglected. Figure 5 shows the comparison for the force in the bearings. These figures indicate that neglecting the coupling can result in significant errors in the response. The error is even more pronounced when computing the forces. Similar observations have been made by Warburton and Soni (1977).

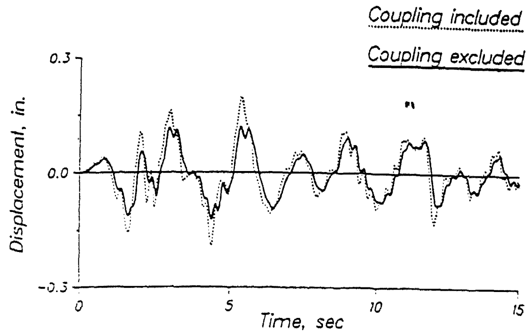

Fig. 4 Displacement time-history for the top floor

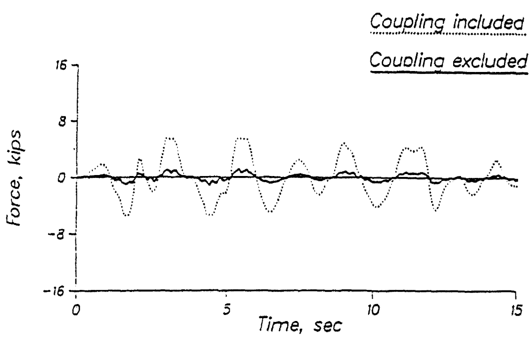

Fig. 5 Force time-history for the bearings

5. CONCLUSIONS

The dynamic response of non-classically damped nonlinear systems including base-isolated structures may be computed using a mode superposition procedure. The nonlinearity in the stiffness and the coupling in the modal damping matrix resulting from non-classical damping may be treated as a pseudo force in the solution. The application of the method to a frame supported on soft bearings indicates that the computed response is in excellent agreement with those from the direct integration of the equations of motion and the complex mode; moreover, it requires less computational time. The advantage of the proposed method over the direct integration is the simplicity in solving the modal equations and the flexibility in using fewer modes in computing the response with the same degree of accuracy. When compared to the complex mode, the formulation is less complicated, the solution requires half as many equations, and the computations are performed in real rather than complex domain.

6. ACKNOWLEDGMENT

This study was supported by the Civil and Mechanical Engineering Department, School of Engineering and Applied Science, Southern Methodist University. The numerical solutions were performed on the IBM 3081D of the SMU Computer Center.

REFERENCES

Chang, C. J. and Mohraz, B. 1990. Modal analysis of nonlinear systems with classical and nonclassical damping. *Computers & Structures*, Vol. 36, No. 6, 1067-1080.

Clough, R. W. and Penzien, J. 1975. *Dynamics of Structures*. McGraw-Hill, Inc., New York.

Karasudshi, P., Balendra, T. and Lee S. L. 1975. An efficient method of seismic analysis of structure-foundation systems. *Geotechnical Engineering*, Vol. 6, 133-154.

Koh, C. G. and Balendra, T. 1989. Seismic response of base isolated buildings including P-Δ effects of isolation bearings. *Earthquake Engineering and Structural Dynamics*, Vol. 18, 461-473.

Kreyszing, E. 1983. *Advanced Engineering Mathematics*, John Wiley & Sons, New York.

Newmark, N.M. 1959. A method of computation for structural dynamics. *J. of the Engineering Mechanics Division*, ASCE, Vol. 85, 67-94.

Nigam, N. C. and Jennings, P. C. 1969. Calculation of response spectra from strong-motion earthquake records. *Bulletin of the Seismological Society of America*, Vol. 59, No. 2, 909-922.

Novak, M. and Henderson, P. 1989. Based-isolated buildings with soil-structure interaction. *Earthquake Engineering and Structural Dynamics*, Vol. 18, 751-765.

Tsai, H. C. and Kelly, J. M. 1988. Non-classical damping in dynamic analysis of base-isolated structures with internal equipment. *Earthquake Engineering and Structural Dynamics*, Vol. 16, 29-43.

Tsai, H. C. and Kelly, J. M. 1988. Non-classical damping in dynamic analysis of base-isolated structures with internal equipment. *Earthquake Engineering and Structural Dynamics*, Vol. 16, 29-43.

Warburton, G. B., and Soni, S. R. 1977. Errors in response calculations for non-classically damped structures. *Earthquake Engineering and Structural Dynamics*, Vol. 5, 365-376.

Wilson, E. L. and Penzien, J. 1972. Evaluation of orthogonal damping matrices. *Earthquake Engineering and Structural Dynamics*, Vol. 4, 5-10.

APPENDIX A

Elements of matrices $[A_n]$ and $[B_n]$ in equation (16) are as follows:

$$a_{11} = e^{-\xi_n\omega_n\Delta t}\left[\frac{\xi_n}{\sqrt{1-\xi_n^2}}\sin(\omega_{nd}\Delta t) + \cos(\omega_{nd}\Delta t)\right] \quad (A.1)$$

$$a_{12} = \frac{e^{-\xi_n\omega_n\Delta t}}{\omega_{nd}}\sin(\omega_{nd}\Delta t) \quad (A.2)$$

$$a_{21} = -\frac{\omega_n}{\sqrt{1-\xi_n^2}}e^{-\xi_n\omega_n\Delta t}\sin(\omega_{nd}\Delta t) \quad (A.3)$$

$$a_{22} = e^{-\xi_n\omega_n\Delta t}\left[\cos(\omega_{nd}\Delta t) - \frac{\xi_n}{\sqrt{1-\xi_n^2}}\sin(\omega_{nd}\Delta t)\right] \quad (A.4)$$

$$b_{11} = e^{-\xi_n\omega_n\Delta t}\left[\left(\frac{2\xi_n^2-1}{\omega_n^2\Delta t}+\frac{\xi_n}{\omega_n}\right)\frac{\sin(\omega_{nd}\Delta t)}{\omega_{nd}} + \left(\frac{2\xi_n}{\omega_n^3\Delta t}+\frac{1}{\omega_n^2}\right)\cos(\omega_{nd}\Delta t)\right] - \frac{2\xi_n}{\omega_n^3\Delta t} \quad (A.5)$$

$$b_{12} = e^{-\xi_n\omega_n\Delta t}\left[\left(\frac{2\xi_n^2-1}{\omega_n^2\Delta t}\right)\frac{\sin(\omega_{nd}\Delta t)}{\omega_{nd}} + \left(\frac{2\xi_n}{\omega_n^3\Delta t}\right)\cos(\omega_{nd}\Delta t)\right] - \frac{1}{\omega_n^2} + \frac{2\xi_n}{\omega_n^3\Delta t} \quad (A.6)$$

$$b_{21} = e^{-\xi_n\omega_n\Delta t}\left[\left(\frac{2\xi_n^2-1}{\omega_n^2\Delta t}+\frac{\xi_n}{\omega_n}\right)\left(\cos(\omega_{nd}\Delta t) - \frac{\xi_n}{\sqrt{1-\xi_n^2}}\sin(\omega_{nd}\Delta t)\right) - \left(\frac{2\xi_n}{\omega_n^3\Delta t}+\frac{1}{\omega_n^2}\right)\left(\omega_{nd}\sin(\omega_{nd}\Delta t)+\xi_n\omega_n\cos(\omega_{nd}\Delta t)\right)\right] + \frac{1}{\omega_n^2\Delta t} \quad (A.7)$$

$$b_{22} = e^{-\xi_n\omega_n\Delta t}\left[\left(\frac{2\xi_n^2-1}{\omega_n^2\Delta t}\right)\left(\cos(\omega_{nd}\Delta t) - \frac{\xi_n}{\sqrt{1-\xi_n^2}}\sin(\omega_{nd}\Delta t)\right) - \left(\frac{2\xi_n}{\omega_n^3\Delta t}\right)\left(\omega_{nd}\sin(\omega_{nd}\Delta t)+\xi_n\omega_n\cos(\omega_{nd}\Delta t)\right)\right] - \frac{1}{\omega_n^2\Delta t} \quad (A.8)$$