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On the dynamics of rigid block motion under harmonic forcing

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ABSTRACT: The response of a free-standing rigid block to horizontal simple harmonic forcing is examined in detail using the model of Housner. It is shown that (i) all types of subharmonic response are possible, (ii) different responses can occur at one point in parameter space, (iii) exact expressions are found for stability boundaries in parameter space, (iv) asymmetric solutions exist just outside the upper boundaries of symmetric solutions, (v) period- and impact-doubling cascades occur as parameter values are varied even further outside the boundaries, (iv) aperiodic (or chaotic) responses are possible, (vii) periodic responses can occur which appear to violate West's formula and (viii) steady state responses of the forced system can be so large as to produce toppling of the block if the system were unforced. These results are used to explain and provide quantitative agreement with experiments of Wong and Tso.

1 INTRODUCTION

The effects of an earthquake on man-made structures have been the subject of many articles in recent years. For a simple object like a rigid block, a great deal of progress has been made since the seminal paper by Housner (1963.). This type of model is applicable to things such as concrete shields around radiation equipment, prefabricated buildings, free standing equipment (e.g. power transformers) and even to the inside of a nuclear reactor core made form graphite blocks.

More recent experimental and computational work has revealed intriguing patterns of response as the earthquake amplitude or frequency is varied. These observations encouraged the present author to embark on a detailed mathematical study of the steady state responses (Hogan 1989, 1990). At the same time, a combined theoretical and experimental study of the steady state solutions was published by Wong and Tso (1989). The point of the current study is to reconcile the apparent differences between the two theoretical approaches and to demonstrate the quantitative agreement between the theory of Hogan (1989, 1990) and the experiments of Wong and Tso (1989). We shall not discuss the transient responses in this paper.

In section 2, we introduce the governing equations. The necessary stability analysis is outlined in section 3 and extended numerically in section 4. The experimental work of Wong and Tso (1989) is outlined in section 5 and shown to be completely explained by our theory. Section 6 contains our conclusions.

2. THE GOVERNING EQUATIONS

We consider the simplest model of a rocking block, sketched in Figure 1. The two-dimensional block is taken to be rigid and uniform with breadth B and height H. The block angle α is given by

$$\alpha = \tan^{-1}(B/H) \tag{1}$$

The harmonic excitation, α_H, is only in the horizontal direction. We take it to have the form

$$\alpha_H = \beta\alpha g \cos(\Omega\tau) \tag{2}$$

The block has mass M and the moment of inertia of the block about O or O' is given by I_o. The system frequency ϱ is given by

$$\varrho^2 = \frac{MgR}{I_o} \tag{3}$$

The governing equations then become

$$\alpha\ddot{x} + \sin[\alpha(1 - x)] = -\alpha\beta\cos wt \cos[\alpha(1 - x)] \tag{4}$$

for x > 0 and

$$\alpha\ddot{x} + \sin[\alpha(1 - x)] = -\alpha\beta\cos wt \cos[\alpha(1 + x)] \tag{5}$$

for x > 0, where $t = \varrho\tau$ and $w = \Omega/\varrho$.

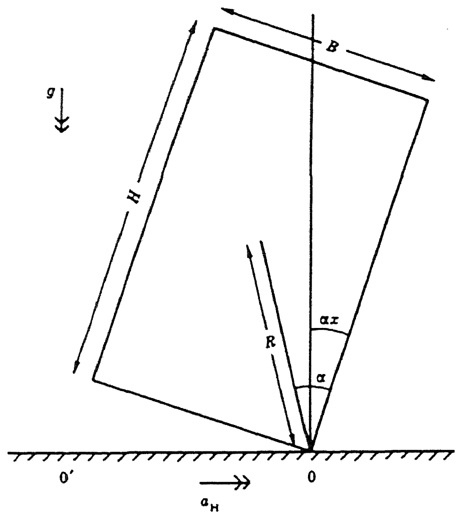

Figure 1. Definition sketch for rocking block

The dots denote differentiation with respect to t.

For slender blocks we can linearize these equations (but not the rocking system itself) to give the following:

$$\ddot{x} - x + 1 = -\beta \cos wt \quad (x > 0) \tag{6}$$

$$\ddot{x} - x - 1 = -\beta \cos wt \quad (x > 0) \tag{7}$$

At impact we take

$$\dot{x}(t_A) = r\dot{x}(t_B) \tag{8}$$

where r is the coefficient of restitution such that $0 \leq r \leq 1$, t_A, t_B = time just after and before impact. Throughout this paper we take $t_A = t_B$.

West's formula (Milne 1885) corresponds to

$$\beta > 1 \tag{9}$$

This condition is satisfied when the overturning moment of the excitation about one corner exceeds the restoring moment due to gravity. The block may rock or even topple.

In what follows, we set $y(t) = \dot{x}(t)$, the scaled angular velocity. Note that the unforced block ($\beta = 0$) would topple if $|x| > 1$.

3. STABILITY ANALYSIS

Direct numerical integration of equations (4) - (8) for different values of β, w and r from quiescent initial conditions reveals several types of steady state solution, described in Spanos and Koh (1984) as type (m,n) reponses. These correspond to 2m impacts in an orbit which repeats itself every n periods of the drive. 'Single' impact orbits are said to occur when m = 1. Despite numerous attempts, the use of numerical integration has failed to produce an established pattern for these responses. In addition, if initial conditions are varied at one point in parameter space, several solutions are possible (Hogan 1989, 1990).

An analytic approach was therefore adopted in order to provide a test for the numerical simulations and to provide the framework in which the responses could be interpreted.

The details of the method are to be found in Hogan (1989, 1990). The symmetric single impact type (1,n) orbits occur widely throughout numerical and laboratory experiments and hence are singled out for analysis. The approach can be summarised as follows. Firstly areas of (β, w, r) parameter space are found in which the orbits can exist. It is easy to show that no symmetric orbits exist for n even. Secondly, using techniques of orbital stability analysis, these orbits are tested for stability to small parameter changes. No assumptions are made about initial conditions and the method is valid for all values of the parameters, provided the block angle α is small. Thus the block is slender and the exact solutions to equations (6) and (7) form the basis of the approach.

Results from the analysis are given in Figure 2 for r = 0.925, a value corresponding to a concrete block impacting an aluminium base. For each value of n in Figure 2, there are two curves, in between which symmetric single impact responses are stable. For example, at the cross (X), only the steady state solutions corresponding to n = 1, 3, 5 and 7 are stable. Curves for $n \geq 9$ pass above this point but are not shown for clarity. No symmetric responses at all are stable outside the n = 1 curves, although transients may be significant both in amplitude and duration. If β_- denotes the upper boundary and β_+ the lower, then explicit expressions for $\beta_\pm$ for arbitrary n, w and r are given by

$$\beta_\pm = (1 + \omega^2)\left[K_\pm^2 + \frac{X^2}{W^2}\right]^{1/2} \tag{11}$$

where

$$X = (1 - r)\left(1 - \cosh \frac{n\pi}{\omega}\right) \tag{12}$$

$$W^2 = Y^2 + Z^2 \tag{13}$$

$$Y = -\omega \sinh \frac{n\pi}{\omega}[1 - r] \tag{14}$$

$$Z = (1 + r)\left(1 + \cosh \frac{n\pi}{\omega}\right) \tag{15}$$

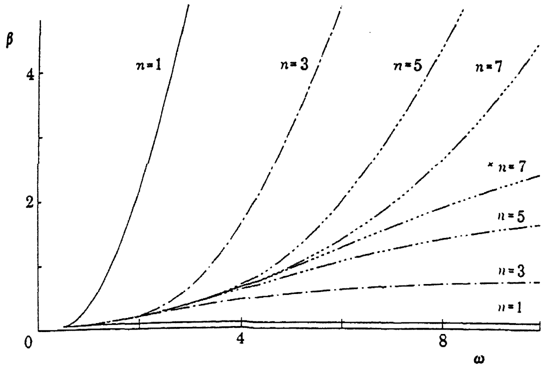

Figure 2. Stability boundaries for symmetric type (1,n orbits with r = 0.925.

$$K_\pm = \frac{(N_\pm Y + M_\pm Z)X - L_\pm W^2}{W(N_\pm Z - M_\pm Y)} \tag{16}$$

$$L_\pm = \left(1 - \cosh\frac{n\pi}{\omega}\right)\left(1 + r^2 \pm r\left(\cosh\frac{n\pi}{\omega} - r\right)\right) \pm r\sinh^2\frac{n\pi}{\omega} \tag{17}$$

$$M_\pm = (1 + r^2)\left(1 + \cosh\frac{n\pi}{\omega}\right) \pm r(1 - r)\left(1 + \cosh\frac{n\pi}{\omega} - \omega^2\sinh^2\frac{n\pi}{\omega}\right) \tag{18}$$

$$N_\pm = -\,\omega\sinh\frac{n\pi}{\omega}\left(1 + r^2 \pm r(1 + r)\cosh\frac{n\pi}{\omega}\right) \tag{19}$$

It is straightforward to show that $K_+ \equiv 0$.

It is reasonable to ask how the different steady state responses can be obtained. The answer lies with the initial conditions, where in certain parts of parameter space, a tiny change in the starting position or velocity can lead to a completely different solution whose amplitude may be markedly different from the original. For example, at the cross in Figure 2, the n = 7 solution has a maximum amplitude seven times greater than the harmonic response (Hogan 1990). This unpredictability is shown in Figure 4 of Hogan (1990) for a wide range of initial conditions.

Note that from figure 2, orbits are possible for $\beta < 1$. This observation would appear to violate West's criterion. But that criterion takes no account of initial conditions and actually merely implies that if $\beta > 1$, rocking will occur. It is silent on the position when $\beta < 1$. In fact rocking can occur in this part of parameter space, provided the initial conditions are correct.

We conclude this section with a discussion of the theoretical approach of Wong and Tso (1989). They concentrated on finding sufficient conditions for the existence of steady state solutions to the problem. This approach led them to find differences between their experiments and theory. The problem lies in the fact that some of the solutions may exist but are unstable, and hence cannot be found in practice. In addition, a robust numerical method is required to explore parameter space in regions untouched by the analysis, as shall be demonstrated in the next section.

4. EXTENSION OF STABILITY ANALYSIS

It is natural to enquire what happens outside the stability boundaries. Several possibilities are discussed in Hogan (1989,1990). In this paper we shall focus on the changes which occur as the upper (β_-) boundary is crossed by for example an increase in forcing amplitude at fixed frequency. A good example is shown in Figure 3 where an arbitrary value of block angle is chosen. (This demonstrates that our conclusions do not only apply for small α. The exact solutions are only vaid then but similar $\beta_\pm$ curves can be calculated numerically for all α such that $0 < \alpha < \pi/2$). In Figure 3(a), the symmetric type (1,1) orbit at $\beta = 2.0$ is shown. Then as β is increased, a symmetry breaking bifurcation occurs at $\beta = \beta_-$. A solution on one (of the two) branches at $\beta = 3.1$ is shown in Figure 3(b). It is still type (1,1) but is now clearly asymmetric. Note also that x > 1. Therefore the block is rocking during forced motion beyond the point at which it would topple if it were not being forced. (A highly counter-intuitive observation!). Further increases in β cross a series of period doubling bifurcations to produce the asymmetric type (2,2) orbit at $\beta = 3.2$ in Figure 3(c), the asymmetric type (4,4) orbit at $\beta = 3.21$ in Figure 3(d) and the asymmetric type (8,8) orbit at $\beta = 3.23$ in Figure 3(e). At $\beta = 3.25$, there is a chaotic response (not shown). The first return map of an attractor at another point in parameter space is shown in Figure 20 of Hogan (1989).

The lower stability boundary corresponds to a saddle-node bifurcation. All the results in Figure 3 were obtained using a numerical program described in the Appendix of Hogan (1989). It has proved an invaluable complement to the analytic work of section 3. Needless to say, in areas where agreement was to be expected between the two approaches, it was found.

5. EXPERIMENTAL WORK

This analysis can be used to explain the experimental observations of Wong and Tso (1989). Full details of the comparison are given in Hogan (1990). Briefly, the authors conducted experiments on a slender block with $\alpha = 0.18$ and r = 0.88. Amongst other things, they found a symmetric type (1,1) orbit at $\beta = 2$, $w = 3.07$ (Figure 4, point A). As they reduced the frequency to $w = 1.9$, keeping the amplitude constant, the response became asymmetric. This corresponds to

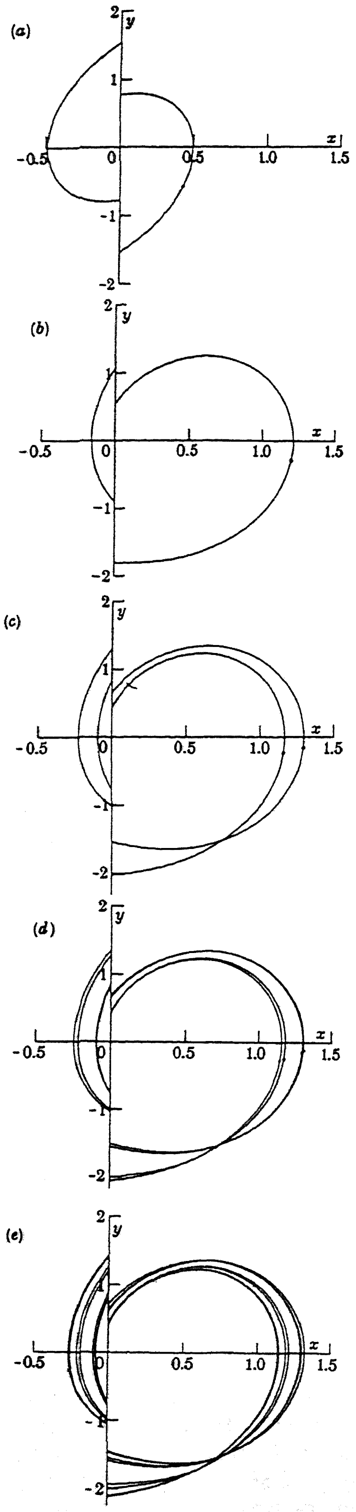

Figure 3. Excerpts from the period - and impact-doubling cascade at α = arctan ½, r = 0.5 and $w = 2\pi/3$.

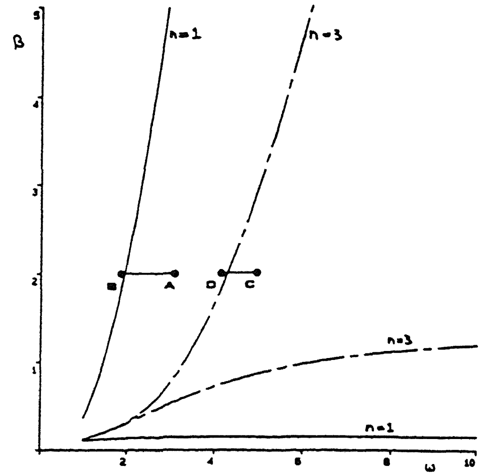

Figure 4. Stability boundaries for r = 0.88. The points A, B, C and D correspond to experimental points chosen by Wong and Tso (1989).

Table 1. Comparison of experimental results of Wong and Tso (1989) with present theoretical work.

Point	x	Experiment	Theory	Error
A	x > 0	3.3°	3.2°	3%
	x < 0	3.3°	3.2°	3%
B	x > 0	7.6°	8.2°	6%
	x < 0	6.5°	6.0°	8%
C	x > 0	3.9°	3.8°	3%
	x < 0	3.9°	3.8°	3%
D	x > 0	5.8°	5.6°	3%
	x < 0	3.8°	4.0°	5%

point B of our Figure 4. As can be seen, this coincides exactly with our predictions for a symmetry breaking bifurcation, using equation (11). Similarly, Wong and Tso (1989) observed a symmetric type (1,3) response at $\beta = 2$, $w = 4.9$ (point C) and an asymmetric response at $\beta = 2$, $w = 4.14$ (point D). Our analysis agrees very well indeed. These authors also observed many other features all of which can be satisfactorily explained using the present theory. The details of the responses are also captured by our numerical method, as can be seen from a comparison of response maxima given in Table I (see Hogan (1990) for further detailed comparison of results).

6. CONCLUSIONS AND FURTHER WORK

The steady state responses outlined here are a necessary prelude to understanding the effect of a real earthquake on any free-standing structure. The simple model of Housner (1963) is shown to possess extremely complicated dynamics, including chaos. The experimental work of Wong and Tso (1989) is successfully explained, confirming the existence and the form of subharmonic and asymmetric responses.

Bruhn & Koch (1991) have shown that heteroclinic orbits are possible in this problem, leading to horseshoe dynamics. They give an exact formula for the tangencies and explicit examples of manifold intersections.

Hogan (1992a) has added a damping term $k\dot{x}$ to the left hand side of equations (6) and (7). It is shown that despite this extra constraint the full range of responses is still possible, albeit in a smaller part of parameter space. Heteroclinic tangencies also occur in this problem (Hogan 1992b).

In the case of a block tethered at one corner, the problem can be reduced to that of an inverted pendulum impacting asymmetrically placed side-walls (Hogan 1992c). The analysis proceeds essentially as before, except that all orbits are necessarily asymmetric and hence solutions with n even are possible. The bifurcations obtained in the stability analysis are saddle-node and period-doubling respectively.

It is hoped that presentation of this work at IOWCEE will give the above method greater publicity as some authors (e.g. Yim and Lin 1991a, b) appear unaware of its existence.

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