Earthquake Engineering, Tenth World Conference © 1992 Balkema, Rotterdam. ISBN 90 5410 060 5

On the use of residual shapes in modal analysis

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ABTRACT: Some conclusions about the use of residual shapes in modal analysis are presented in this paper. The main idea is to demostrate how the method becomes useful only in those cases in which it is formulated as the static correction methods are.

1 METHOD DESCRIPTION

The dynamic equilibrium equation, for a linear multi-deqree of freedom system is expresed as :

$$M\ddot{X}+C\dot{X}+KX=-MJ\ddot{U}_S \qquad (1)$$

Where M, C, and K are the nxn mass, damping and stiffness matrices. J is a vector whose components are the displacements at each degree of freedom when a unit displacement is applied to the supports in the direction of the earthquake. X is the vector of relative displacements and $\ddot{u}_S$ is the ground acceleration.

As usual, damping matrix is supposed to be able to transform into a diagonal one when projected over the modal basis.

When the change to normal coordinates is performed, the system becomes:

$$\ddot{\xi}_i+2\zeta_i\omega_i\dot{\xi}_i+\omega_i^2\xi_i=-\frac{L_i}{m_i}\ddot{u}_S(t) \qquad (2)$$

ζ_i : Damping coefficient.
m_i : Modal Mass
L_i : Participation Factor

It is usual to formulate system (2) only for a few degrees of freedom "m", (m<n). In the seismic case these correspond to the lowest modes.

The main idea in the method treated herein is to add to the truncated modal basis a new vector which takes into account the effect of neglected modes. This new vector, called "residual", is defined simply as:

$$\Phi_R=\sum_{i=m+1}^{n}\frac{L_i}{m_i}\Phi_i \qquad (3)$$

in which Φ_i are the neglected eigenvectors.

This formula is not valid for practical use, as it is based upon the modes whose computation is intended to avoid. Instead of that, the following expression is preferable:

$$\Phi_R=J-\sum_{i=1}^{m}\frac{L_i}{m_i}\Phi_i \qquad (4)$$

derived from :

$$J=\sum_{i=1}^{n}\frac{L_i}{m_i}\Phi_i \qquad (5)$$

easy to be demostrated. Developing the right-hand side :

$$J=\sum_{i=1}^{m}\frac{L_i}{m_i}\Phi_i+\sum_{i=m+1}^{n}\frac{L_i}{m_i}\Phi_i= \\ =\sum_{i=1}^{m}\frac{L_i}{m_i}\Phi_i+\Phi_R \qquad (6)$$

expression (4) is obtained.

And so, the ease with which the new vector can be computed becomes clear : it is only needed to substract from J vector each computed eigenvector multiplied by its participation factor.

Also it is simple to demonstrate the orthogonality of the residual vector in respect to the truncated basis :

$$\Phi_i^t M \Phi_R = \Phi_i^t M [\sum_{j=m+1}^{n} \frac{L_j}{m_j} \Phi_j] = 0 \; ; \; i=1,2,..m \tag{7}$$

This is why the inclusion of this new vector results in a new uncoupled equation :

$$\ddot{\xi}_R + 2\zeta_R \omega_R \dot{\xi}_R + \omega_R^2 \xi_R = -\frac{L_R}{m_R} \ddot{u}_S(t) \tag{8}$$

Where :

$\omega_R = \dfrac{\Phi_R^t K \Phi_R}{\Phi_R^t M \Phi_R}$; residual "frequency"

ζ_R ; residual damping coefficient
L_R ; residual participation factor
m_R ; residual mass coefficient

This new equation represents the work done by the part of the load normal to the truncated basis.

2 INTEGRATION

The total response of the system is

$$X = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \Phi_i \xi_i(t) \tag{9}$$

In general, modal displacements $\xi_i(t)$ are :

$$\xi_i = -\frac{L_i}{m_i} \frac{V_i(t)}{\omega_i} \tag{10}$$

Where $V_i(t)$ is the response of a single degree of freedom system with frequency ω_i and damping ratio ζ_i to the considered earthquake.

If the frequency of the excitation is much lower than that of a particular mode, then it is possible to ignore inertial and damping forces. In that case, and making zero the firts two terms in (2), a simple expression for equation (10) is obtained:

$$\xi_i(t) = -\frac{L_i}{m_i} \frac{\ddot{u}_S(t)}{\omega_i^2} \tag{11}$$

Thus, for modes having the largest frequencies :

$$X_i(t) = -\Phi_i \frac{L_i}{m_i} \frac{\ddot{u}_S(t)}{\omega_i^2} = -K^{-1} M \Phi_i \frac{L_i}{m_i} \ddot{u}_S(t) = -K^{-1} Q_i \tag{12}$$

in which the vector :

$$Q_i = M \, \Phi_i \, \frac{L_i}{m_i} \ddot{u}_S(t) \tag{13}$$

is called "equivalent load".

If it is admitted that the frequencies of the n-m modes included in the residual shape are much higher than that of the excitation, then it is possible to write :

$$\sum_{i=m+1}^{n} X_i(t) = -K^{-1} M [\sum_{i=m+1}^{n} \Phi_i \frac{L_i}{m_i}] \ddot{u}_S(t) = -K^{-1} M \Phi_R \, \ddot{u}_S(t) \tag{14}$$

If the integration is performed by the response spectrum method, a parallel formulation can be developed :

$$[\, X_i \,]_{\max} = \Phi_i \; [\, \xi \,]_{\max} \tag{15}$$

In general, the maximum modal displacement is :

$$[\xi_i]_{\max} = -\frac{L_i}{m_i} \frac{S_a(\omega_i)}{\omega_i^2} \tag{16}$$

Where S_a the spectral pseudo-acceleration. Again, for the a highest frequencies we get :

$$[\xi_i]_{\max} = -\frac{L_i}{m_i} \frac{[\ddot{u}_S]_{\max}}{\omega_i^2} \tag{17}$$

The maximun value of ground acceleration is usually called Zero Period Acceleration (ZPA).Thus :

$$[X_i]_{\max} = -\Phi_i \frac{L_i}{m_i} \frac{[ZPA]}{\omega_i^2} = -K^{-1} M \Phi_i \frac{L_i}{m_i} [ZPA] = -K^{-1} Q_i \tag{18}$$

and the expression for the equivalent load is now :

$$Q_i = M\Phi_i \frac{L_i}{m_i} [ZPA] \tag{19}$$

If, as in the former case, the frequencies of the "m-n" modes included in the residual shape are much higher than that at which the spectrum reaches the Zero Period Acceleration, it is possible to write:

$$\sum_{i=m+1}^{n} [X_i]_{\max} = \\ = K^{-1}M[\sum_{i=m+1}^{n} \Phi_i \frac{L_i}{m_i}] [ZPA] = \\ = K^{-1}M\Phi_R [ZPA] \tag{20}$$

It must be pointed out that the results given by equation (20) are theoretically exact. In this case no assumption is needed regarding inertial or damping forces.

If any low frequency mode is included in the residual shape, equations (14),(20) can not be written. In fact, those equations are based upon the perfect correlation existing between the highest modes, but this correlation is lost when a low frequency mode is included.

Thus there are no arguments to support the method in that case. Only in a few special cases (when simply one mode between those included in the residual shape are significant, or when this shape is composed of several local modes) the method becomes useful. This was pointed out in [1].

Unfortunatly, the analyst can not recognize those cases "a priori".

3 EXAMPLES

In this section an example similar to those proposed in [1] is presented.

The idea in doing this is to analyse the reason for such good results as those reported there.

Thus a typical building structure is studied. Its main characteristics are collected in table 1.

The values of elastic and Poison modulus are :

$E = 2.06$ E10 N/m^2

$\nu = 0.3$

Columns are modelled as bars of unit area and zero density. Only horizontal translations are considered.

In order to obtain base shear and moment easily, a very stiff spring is used to simulate the fixed-end condition.

Table 1. Characteristics of building structure.

FLOOR	MASS (ton)	HEIGHT (m)	INERTIA (m^4)
MAT	800.	0.00	----
1	472.1	3.5	0.278
2	472.1	6.45	0.217
3	472.1	9.40	0.195
4	472.1	12.35	0.154
5	472.1	15.30	0.130
6	472.1	18.25	0.108
7	472.1	21.20	0.083
8	472.1	24.15	0.071
9	472.1	27.10	0.071
10	472.1	30.05	0.047
11	472.1	33.00	0.037
12	472.1	35.95	0.029
13	472.1	38.90	0.025
14	472.1	41.85	0.025
15	472.1	44.80	0.004

Earthquake loads are specified by means of Newmark`s design spectrum, scaled to a maximum ground acceleration of 10% of g.

Figure 1 shows the distribution of equivalent loads. This figure allows for a fast understanding of the relative importance of each mode in relation to total response.

Some ideas can be extracted :

1 The relative importance of each mode diminishes as the mode number increases, even when their associated frequencies remain still below the ZPA point. This seems to be a very general trend in building structures. (in fact, that is why the use of only a few of the lowest modes renders very good results).

2 This trend is interrupted at the last mode. This is an "artificial" mode in the sense that it is the result of a modelitation process and has no physical meaning. The base

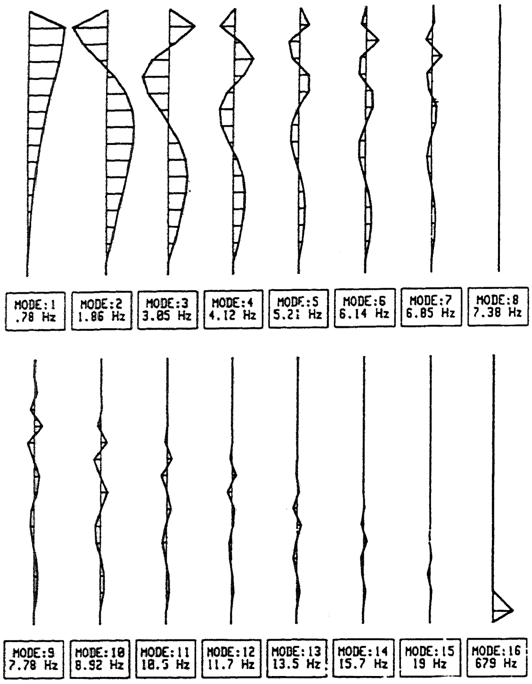

Figure 1.

shear value of this last mode represents more than 10% of the total base shear.

3 The frequency associated with the artifical mode is, by definition, very high. Its response can then be considered static and computed from :

$$X_{16}(t) = -K^{-1}M\Phi_{16}\frac{L_{16}}{m_{16}}\ddot{u}_S(t) = -K^{-1}Q_{16} \quad (21)$$

Due to the special shape of this artificial mode, (it only has a no null component: the one associated to the node adjacent to the support) the equivalent load vector formulates very easily : all its components are zero except for the first one, which has a value:

$$q(t) = m\ddot{u}_S(t) \quad (22)$$

Where "m" is the mass of the foundation. When integration is performed by the response spectrum method, we have :

$$q(t) = m[ZPA] \quad (23)$$

4 If a residual shape is used in this case, it could include not only the artificial mode, but also the seven or eight last modes. There would not be any apreciable difference. In fact, almost all of its response would be due to the artificial mode and could be obtained simply by formula (23).

4 CONCLUSIONS

From this analysis it is possible to conclude:

1 When the residual shape includes only high-frequency modes, it becomes a feasible formulation for static correction methods. Then all the advantages of these methods are directly applicable.

For building structures under earthquake loads, those kinds of methods are, in general, useless (due to the fact that only the lowest modes are excited).

2 The inclusion of low frequency modes in the residual shape will, generally, produce erroneous results. Some exceptions to this rule can be postulated but, as they are based on the knowledge of the neglected modes, become useless.

3 When the supports are simulated by means of placing stiff springs, the effect of the mass of the structure attached to the spring on the total reactive boundary force can be computed by performing very simple calculations: there is no need to use any residual shape.

REFERENCES

Salmonte, A.J.1982. Considerations on the residual contribution in modal analysis. Earthquake Engineering and Structural Dynamics. 10: 295-304

Cornwell, R.E; Craig, R.R; Johnson, C.P. 1983. On the application of the mode-acceleration method to structural engineering problems. Earthquake Engineering and Structural Dynamics. 11:679-688.

Hansteen, O.E; Bell,K. 1979. On the accuracy of mode superposition analysis in structural dynamics. Earthquake Engineering and Structural Dynamics. 7: 405-411.

Gupta, A.K; Jaw, J.W. 1986. Modal combination in response spectrum analysis of piping systems. Journal of pressure vessel technology. 108: 73-77.

Camarda, C.J; Haftka, R.T; Riley, M.F. 1987. An evaluation of

higher-order modal methods for calculating transient structural response. Computers and Structures. 27: 89-101.
Leger, J; Wilson, E.L. 1988. Modal summation methods for structural dynamic computations. Earthquake Engineering and Structural Dynamics. 16: