Earthquake Engineering, Tenth World Conference © 1992 Balkema, Rotterdam. ISBN 90 5410 060 5

Frequency domain analysis of undamped systems using the exponential window method

E. Kausel
Massachusetts Institute of Technology, Mass., USA

J. M. Roësset
University of Texas at Austin, Tex., USA

ABSTRACT: A numerical tool commonly used in digital signal processing, the exponential window method, is briefly reviewed in this paper an applied to the analysis of a cantilever shear beam subjected to a dynamic load at the free end. This method allows carrying out analyses of undamped structures in the frequency domain, and yields highly accurate results for both discrete and continuous systems.

1 INTRODUCTION

It is well known that conventional numerical methods based on the fast Fourier transform (FFT) algorithm, cannot be applied to the analysis of undamped systems for dynamic loads because of the singularities at the resonant frequencies of the system. While such singularities do not exist in lightly damped systems, it is still necessary to include a sufficient number of points so as to resolve accurately the transfer functions in the neighborhood of the natural frequencies. Also, it is necessary to add at the end of the force time history a quiet zone of trailing zeroes of sufficient duration to damp out the free vibration terms. This duration is thus a function of the fundamental period of the system and the amount of damping, and can be very large for lightly damped systems. For undamped systems, the free vibration terms will never decay and, therefore, the standard application of the FFT algorithm is no longer possible.

Two novel approaches to eliminate this problem for single degree of freedom systems have been suggested by Meek and Veletsos (1972), and by Veletsos and Kumar (1982). While these are interesting schemes, their application is limited to one degree of freedom systems, as they cannot be easily extended to multi-degree of freedom systems, or to continuous systems, which are of primary interest in practice.

A more powerful and general approach to obtain solutions with the FFT method for undamped, or lightly damped systems is provided by the exponential window method described in this paper. Although this method has been used in signal processing and in seismology, its application in structural dynamics has been lacking.

The paper illustrates the application of the method to a continuous cantilever shear beam, subjected at its free end to a dynamic load of finite duration having a triangular variation with time. It is shown that the method can provide excellent results without the need for any trailing zeroes, if the time step is small enough and the number of points at which the transfer functions are evaluated is sufficient to reproduce properly all frequencies of interest. One cannot , however, use interpolation schemes in the computation of the transfer functions.

2 EXPONENTIAL WINDOW METHOD

The response of a lightly damped (or undamped) multi-degree of freedom (or continuous) system follows from the inverse Fourier transform

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)P(\omega)e^{i\omega t}d\omega \tag{1}$$

in which $H(\omega)$ is the transfer function at an arbitrary elevation due to a unitary harmonic excitation, and $P(\omega)$ is the Fourier transform of the excitation $p(t)$. A formal analytical evaluation of this integral can be accomplished by contour integration in the complex frequency plane, with the choice of integration path depending on the sign of t. For positive times, the exponential term is bounded in the upper half-plane, while for negative times, it is bounded in the lower. Since both the excitation and the vibrating system are causal, it follows that the lower half-plane cannot contain any poles. On the other hand, the value of the contour integral depends only on the poles enclosed by (or lying on) the integration path; hence, for positive times the contour can be taken along a path that runs parallel to the real axis at some arbitrary distance η below it, and is closed in the upper half-plane with a circle of infinite radius. Invoking standard arguments of contour integration, it can be shown that the integral along the infinite circle vanishes. Hence, equation 1 is equivalent to:

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega - i\eta)P(\omega - i\eta)e^{i(\omega - i\eta)t}d\omega \tag{2}$$

Since η does not depend on ω, it follows that the response is given by

$$u(t) = e^{\eta t}\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega - i\eta)P(\omega - i\eta)e^{i\omega t}d\omega \tag{3}$$

with

$$P(\omega - i\eta) = \int_0^{t_d} [e^{-\eta t}p(t)]e^{-i\omega t}dt \tag{4}$$

The transfer function H for complex frequency $z = \omega - i\eta$ is just one of the components of the vector $\mathbf{U}$ obtained from the solution of the well known equilibrium equation in the frequency domain:

$$(\mathbf{K} + iz\mathbf{C} - z^2\mathbf{M})\mathbf{U} = \mathbf{Q} \tag{5}$$

This equation differs from the classical equation in structural dynamics only in that ω is replaced by z. This system will not exhibit singularities along the axis of integration, even if $\mathbf{C}$ vanishes (i.e. for undamped systems). Hence, to compute the response in the frequency domain, it suffices to: a) compute the FFT of the excitation, modified by a decaying exponential window; b) evaluate the transfer functions for complex frequency; c) compute the inverse FFT of the product; and d) modify the result by a rising exponential window.

While in theory any arbitrary factor η could be used, in practice the choice of this number is limited by the finite precision with which the computations are made. Indeed, the value of the rising exponential term at the end of the window is $w=\exp(\eta T)$, where $T = N\Delta t$, with N being the number of points in the FFT. If this value exceeds some four orders of magnitude, numerical error develops, particularly for large times. Numerical experiments indicate that good results are obtained if w does not exceed a value of 10^3. A simple rule of thumb is the choice $\eta = 2\pi\Delta f = 2\pi / T$ (i.e. the imaginary component equals the frequency step); this implies $w = \exp(2\pi)=535=10^{2.73}$.

EXAMPLE OF APPLICATION

While the method could be demonstrated by means of a discrete systems with a finite number of degrees of freedom, a more interesting application can be accomplished with a continuous system, because such a system has infinitely many resonant frequencies.

Consider a homogeneous cantilever shear beam of length L having uniform cross-section A, shear modulus G, mass density ρ, and subjected to a concentrated load $p(t)=p_0 f(t)$ at the tip. From a solution of the differential equation for this problem by the method of characteristics, it is known that the response velocity $\dot{u}(t,x)$ in the beam consist of a pulse with the same shape as $f(t)$ that travels along the beam with velocity $c = \sqrt{G/\rho}$. This pulse repeatedly reflects at the two extremes of the shear beam, changing polarity every time it impinges on the fixed end. On the other hand, the solution to this problem in the frequency domain is given by the equation

$$H(\omega, x) = \frac{P \sin \alpha x}{\alpha G A \cos \alpha L} \tag{6}$$

in which $\alpha = \omega / c$. The solution of this equation with the exponential window method requires replacing ω by $z = \omega - i\eta$; also, the transfer function for velocities is obtained by multiplying the equation by iz.

Consider a beam with unit properties (length, modulus, mass, etc.), which has resonant periods of T_1=4 s, T_2=4/3 s, T_3=4/5 s, etc. The beam is subjected to a load with time variation $f(t)$ given by a triangular pulse and having a duration t_d=0.10 s. Choosing N=512 points for the FFT and a time increment of 0.01 s, it follows that the length of the Fourier window is T=5.12 s , which is only slightly larger than the fundamental period. Thus, the load is sampled at 11 points, the Nyquist frequency is 50 Hz, and the sampling rate in the frequency domain is 0.1953 Hz. Application of the method using $w=\exp(\eta T)$=1,000., that is η=1.349 (i.e. 0.2147 Hz) yields the results shown in figures 1 through 4. The first two figures depict the velocity and displacement at the top of the beam, while the other two show the response at the center of the beam. The computed results are excellent, to the point that they cannot be distinguished from the exact analytical solution. Rather remarkable is the faithful reproduction of the sharp temporal discontinuities, and in the case of the displacement time histories, the large differences between initial and final values (it should be remembered that in a conventional implementation of the FFT method, the response is periodic, so that initial and final values agree). This shows also that trailing zeroes are not necessary in this method, as confirmed by the authors by means of separate analyses not shown here.

CONCLUSIONS

On the basis of the results presented, and additional computations for discrete and continuous systems carried out by the authors, it was verified that the exponential window method described herein provides accurate results when applied to the analysis of dynamic systems in the frequency domain. The imaginary component of frequency should be chosen in such way that excessively small /large values at the end of the exponential windows are avoided. A simple choice is an imaginary component of frequency equal to the sampling rate in the frequency domain. Because the method is very sensitive to inaccuracies in the computation of the transfer functions, it follows that interpolation schemes cannot be used in the evaluation of these functions. However, this shortcoming is compensated by the fact that no trailing zéroes are needed, so that the number of points in the FFT can be kept to a minimum.

REFERENCES

Meek, J.W. and Veletsos, A.S. (1972): "Dynamic analysis by extra fast Fourier transform", *Journal of Engineering Mechanics*, ASCE, Vol 98, pp 367-384

Veletsos, A.S. and Kumar, A. (1982): "Steady state and transient responses of linear structures", research report Nr. 25, Department of Civil Engineering, Rice University, August

Fig. 1

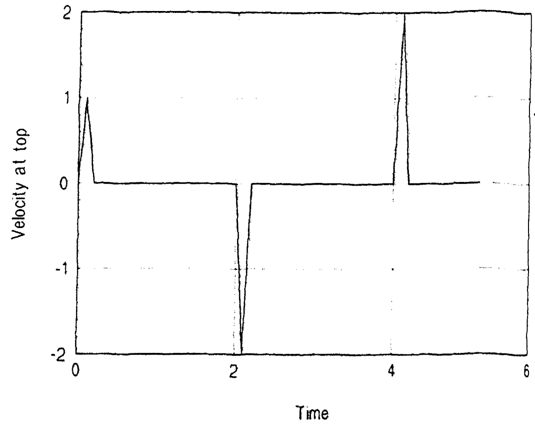

Fig. 2

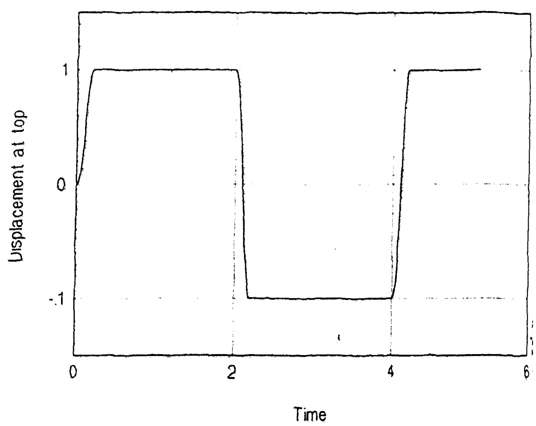

Fig. 3

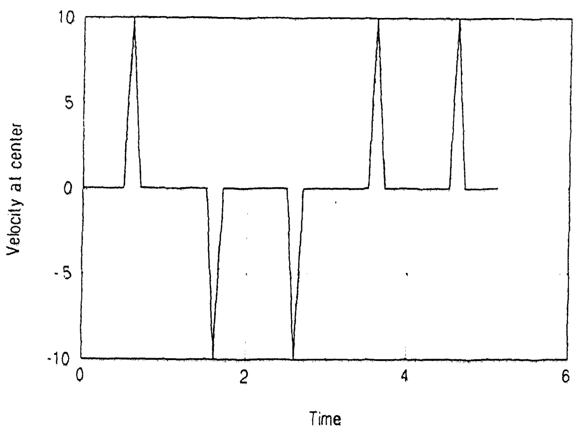

Fig. 4